Matrix Inequalities of Cubic Type

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Abstract. Let $A = (a_{ij})$ be an $m \times n$ nonnegative matrix, with row-sums r_i and column-sums c_j. We show that

$$mn \sum_{i,j} a_{ij} f(r_i) f(c_j) \geq \sum_{i,j} a_{ij} \sum_{i} f(r_i) \sum_{j} f(c_j)$$

providing the function f meets certain conditions. When f is the identity function this inequality is one proven by Atkinson, Watterson and Moran in 1960. We also prove another inequality, of similar type, that refines a result of Ajtai, Komlós and Szemerédi (1981).

1 Introduction

Let $A = (a_{ij})$ be an $m \times n$ matrix with nonnegative real entries. In what follows, r_i will always denote the ith row-sum of A, and c_j and the jth column-sum. Also $\sigma(A)$ is the sum of all the entries of A. Please note that $\sigma(A) = \sum_{i,j} a_{ij} = \sum_{i} r_i = \sum_{j} c_j$.

Atkinson, Watterson and Moran [2] have proved that

$$mn \sum_{i,j} a_{ij} r_i c_j \geq \sigma(A)^3. \tag{1}$$

For a recent application of inequality (1), see [3]. Here we give a generalization of (1), as well as a variant of it. The generalization is as follows.

Theorem 1. *Let $f : [0, \infty) \to [0, \infty)$ be a differentiable function such that $f(0) = 0$, f is strictly increasing and concave, and $\frac{xf'(x)}{f(x)}$ is increasing.*

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Then for every $m \times n$ nonnegative matrix A,

$$mn \sum_{i,j} a_{ij} f(r_i) f(c_j) \geq \sigma(A) \sum_i f(r_i) \sum_j f(c_j). \tag{2}$$

A class of functions satisfying all of the hypotheses of Theorem 1 is $f(x) = x^t$, where $0 < t \leq 1$. When $t = 1$ we recover (1). More generally, positive linear combination of the preceding functions, for example $f(x) = 2x^s + x^t$ with $0 < s \leq 1$ and $0 < t \leq 1$, are admissible. On the other hand, the restrictions imposed on f by Theorem 1 seem quite severe. Is it possible to classify such functions f?

It is of interest to look for conditions on f that reverse the inequality in (2). (See below for motivation.) One such set of conditions is given by the following theorem, which deals only with the case of symmetric matrices.

Theorem 2. *Let $f : [0, \infty) \to [0, \infty)$ be a differentiable functions such that f is decreasing and convex, and $xf(x)f'(x)$ is strictly decreasing. Then for any $n \times n$ symmetric nonnegative matrix A ,*

$$n^2 \sum_{i,j} a_{ij} f(r_i) f(r_j) \leq \sigma(A) (\sum_i f(r_i))^2. \tag{3}$$

Our motivation for Theorem 2 is the following. Let G be an undirected graph on the vertex-set V and edge-set E. Ajtai, Komlós and Szemerédi ([1], Lemma 5) have shown that

$$\frac{1}{|E|} \sum_{ij \in E} e^{-t(d_i + d_j)} \leq \frac{1}{n^2} (\sum_{i \in V} e^{-td_i})^2 \tag{4}$$

providing $0 \leq t \leq \frac{1}{10\Delta}$, where $|V| = n$, d_i is the degree of vertex i (number of edges incident to i) and Δ is the maximum degree of G. Now let A be the adjacency matrix of G : $a_{ij} = 1$ if $ij \in E$, $a_{ij} = 0$ otherwise. Since $\sigma(A) = 2 |E|$, one easily sees that (4) is equivalent to

$$n^2 \sum_{i,j} a_{ij} e^{-td_i} e^{-td_j} \leq \sigma(A) (\sum_i e^{-td_i})^2. \tag{5}$$

Now (5) follows from (3), by taking $f(x) = e^{-tx}$. A simple calculation shows that in fact (4) is valid (i.e. the hypotheses of Theorem 2 are satisfied) for $0 \leq t \leq \frac{1}{2\Delta}$, which slightly improves the result of Ajtai et al. A more general example of a function satisfying the hypotheses of Theorem 2 is $f(x) = exp(-tx^r)$ where $t > 0, 0 < r \leq 1$ and $0 \leq x \leq (2t)^{-\frac{1}{r}}$. But as in Theorem 1, the restrictions on f given by Theorem 2 are severe. Is it possible to classify such functions?

2 The perturbation argument

We will prove Theorem 2 by adapting the method of Atkinson et al. [1]. Since the proof of Theorem 1 is similar, we will omit it.

First of all, note that (3) is true (with equality) if all of the r_i are equal. Now define $\Phi := \sum_{i,j} a_{ij} f(r_i) f(r_j)$.

Our objective is to show that Φ is a maximum, for $\sigma(A)$ fixed, when all of the r_i are equal. If this is true, then (3) follows easily. Indeed, given A let B be the diagonal matrix with each diagonal entry equal to $n^{-1}\sigma(A)$. Then

$$\begin{aligned} n^2 \sum_{i,j} a_{ij} f(r_i) f(r_j) &\leq n^2 \sum_{i,j} b_{ij} f(\frac{\sigma(A)}{n})^2 \\ &= \sigma(A) n^2 f(\frac{\sigma(A)}{n})^2 \\ &\leq \sigma(A)(\sum_i f(r_i))^2 \end{aligned}$$

by the convexity of f, which gives (3).

Since the case $n = 1$ is trivial, we will suppose that $n > 2$. Without loss of generality, we may take r_1 to be the minimum row-sum, r_n the maximum row- sum. We will assume that $r_1 < r_n$, and show that A can be transformed, keeping $\sigma(A)$ fixed, in such a way that Φ is increased. This will prove our contention concerning the maximum value of Φ.

We first observe that without loss, $a_{nn} > 0$. For suppose $a_{nn} = 0$. Then $a_{ni} = a_{in} > 0$ for some i, because $r_n > 0$. Now replace $a_{ii}, a_{in}, a_{ni}, a_{nn}$ by $a_{ii} + a_{in}, 0, 0$ and $a_{nn} + a_{in}$, respectively. The net change in Φ is $a_{in}(f(r_n) - f(r_1))^2 \geq 0$. So this transformation has made a_{nn} positive and not decreased Φ. Thus for A with maximum Φ we can assume $a_{nn} > 0$, as desired.

Now change a_{n1}, a_{1n}, a_{nn} to $a_{n1}+x, a_{1n}+x, a_{mn}-2x$, respectively, where x is a real indeterminate; and let $\Phi(x)$ denote the Φ-function of the transformed matrix. We claim that $\Phi'(0)$, the derivative of $\Phi(x)$ at $x = 0$, is strictly positive. This clearly implies that we may choose x positive and sufficiently small so that the transformed matrix $\hat{A}$ is still nonnegative and symmetric, has $\sigma(\hat{A}) = \sigma(A)$ and larger Φ; and this is sufficient to complete the proof of our theorem. Now it is straightforward to calculate that

$$\frac{1}{2}\Phi'(0) = f(r_n)[f(r_1) - f(r_n)] + f'(r_1)\sum_i a_{i1} f(r_i) - f'(r_n)\sum_i a_{in} f(r_i) \quad (6)$$

Since f is decreasing, we have $f(r_i) \leq f(r_1)$ for all i and $f'(r_1) \leq 0$; hence

$f'(r_1)\sum_i a_{i1} f(r_i) \geq r_1 f(r_1) f'(r_1)$. We can similarly bound the last term in (6) to get $\frac{1}{2}\Phi'(0) \geq f(r_n)[f(r_1) - f(r_n)] + r_1 f(r_1) f'(r_1) - r_n f(r_n) f'(r_n)$.

Our assumptions on f immediately yield that $\Phi'(0) > 0$, as desired.

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