Some Group Divisible Design Constructions

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ABSTRACT. The main object of this paper is the construction of BIBD's with $6 \leq k \leq 11$ and $\lambda = 1$. These balanced incomplete block designs can be simply constructed from some associated group divisible designs with the number of groups being a prime power, and it is these group divisible designs that are constructed directly. Other related designs are discussed.

1 Notation

The notation is largely taken from Hanani [12].

A design is a pair $(X, \mathcal{B})$ where X denotes a set of points of finite cardinality, v, and $\mathcal{B}$ is a family of (not neccessarily distinct) subsets of X. The cardinality of the subsets are called the block sizes.

A group divisible design (a GDD) is a design with a partition of X. X is partitioned into groups with cardinalities in M. The block sizes have cardinalities in K. The design also satisfies the condition that every pair of points from distinct groups is contained in λ blocks, whilst no block contains a pair of points from the same group. The design is denoted by $GD[K, \lambda, M; v]$, and the set of v for which such designs exist is denoted by $GD(K, \lambda, M)$.

A pairwise balanced block design is a GDD with group size of 1, and is denoted by $B[K, \lambda; v]$. A $B[\{K, k^*\}, \lambda; v]$ denotes that a block of size k is present (if $k \in K$, then more may be present). A balanced incomplete block design (a BIBD) is a GDD with all groups of size 1, and a constant block size of k, and is denoted by $B[k, \lambda; v]$.

A resolvable design is a design that admits a partition of the blocks of $\mathcal{B}$ into resolution sets. Each resolution set is a partition of X. Resolvable designs are denoted by the prefix R.

Let $Z(p)$ denote the cycle of residua mod p, and let $Z(p, x)$ denote $Z(p)$ with the additional information that x is the primitive element used.

Let $GF(q)$ denote the Galois field of order q, and $GF(q, f(x) = 0)$ denote $GF(q)$ with the additional information that x is the primitive element used.

The design will be given in the form of base blocks of exponents of the primitive element. For the residuum of 0 the symbol $\emptyset$ is used.

In the case of group divisible designs, $X = Y \times Z$, and Y denotes the set of points in a group and Z the set of groups. It will be convenient to refer to the value of Y as the membership number, and to the value of Z as the group number.

Let the index function be defined for $GF(q, f(x) = 0)$ by

$$ind(x^t) = s \qquad \text{if} \qquad s \equiv t \qquad \text{mod } (q-1).$$

The residue function is the inverse of the index function, and is defined by

$$res(t) \equiv x^t \qquad \text{mod } q.$$

2 Background

The parameters of a BIBD must obey the integrality conditions

$$\lambda(v-1) \equiv 0 \text{ mod } (k-1)$$

and

$$\lambda v(v-1) \equiv 0 \text{ mod } (k(k-1)).$$

The case of interest here is $\lambda = 1$. If k is a prime power, then either $v = k(k-1)t + 1$ or $v = k(k-1)t + k$. If k is not, then v can take on other residua mod$(k(k-1))$ (such as 15, 21 for $k = 6$). In [8] many designs were constructed with v a prime power. If $v = k(k-1)t + k$, then obviously k is a divisor of v, so alternative methods are needed. Although the constructions in this paper are given in terms of GDD's, the $GD(k, \lambda, k)$ are easily converted to BIBD's by just forming a (parallel) set of blocks on the groups λ times, and the $GD(k, \lambda, k-1)$ are converted to BIBD's by forming λ sets of blocks on the groups together with a point at infinity.

I have included the following account of the development of the paper as an indication of the sources of the ideas I have used. The starting point was my attempt to show the sufficiency of the neccessary conditions in the $B(7, \lambda)$ case. This study will be reported elsewhere [3], but it naturally caused me to consider the $GD[7, 3, 7; 7q]$ series with q an odd prime power and $q \geq 7$ given by Hanani [12]. Hanani also gives a $GD[7, 1, 7; 7 \cdot 13]$. This design can be obtained from the former series by a judicious choice of parameters and a judicious omission of base blocks. Could this feature be generalized? I first attempted to construct $GD[7,1,7;7q]$ designs with q a prime power and $q \equiv 1 \bmod 6$. I obtained 5 successes in the first 8 cases. Hanani [12] gives the generalization to the $GD[k,(k-1)/2,k;kq]$ series with $k = 4s+3$ a prime power. In a similar vein I tried the $GD[11,1,11;11q]$ series with $q \equiv 1 \bmod 10$, again with some success. A further reading of [12] yielded his $GD[6,1,5;125]$. There are two new features here: firstly the group size is $k-1$; secondly there are two base blocks in the underlying $B(3,3;5)$ design. The first feature yielded some $GD[8,1,7;7q]$ designs; the second gave some $GD[9,1,9;9q]$ designs; both gave some $GD[6,1,5;5q]$ and $GD[10,1,9;9q]$ designs. With $q = kt+1$, for these designs with $\lambda = 1$ there are $t \cdot v$ blocks. To accommodate the two base blocks in the membership design, I needed t even. A related construction gives the $GD[k,k-1,k;kq]$ series with q and k odd prime powers and $q \geq k$ and $k = 4s+1$. Could this be improved to $GD[k,(k-1)/2,k;kq]$? When I looked at cases of $k < 6$, there was a hint that it could. Bose [6] gives a series (G_2) with no restriction that t be even. A hand check of prime $k < 128$ convinced me that the restriction that t be even could be removed from my basic construction of $GD[k,1,k;kq]$ designs for k odd. A proof of this was garnered from Hall [11]. Its application to these sort of designs seemed to be new. I then returned to the basic membership design to see whether I could exploit these in other ways when I allowed $\lambda > 1$, or considered other group sizes. The final facet of this paper is that the completion of my $GD[2s,1,2s-1;v]$ designs to $B[2s,1;v+1]$ designs actually yields resolvable BIBD designs. This resolvability result closely parallels Hanani et al.'s construction of $RB[4,1;3q+1]$ designs with q a prime power and $q \equiv 1 \bmod 4$ [14]. Other authors have published some related constructions for odd k. Mathon [16] has what amounts to a special case of the constructions I give. He imposes more structure on his base blocks (i.e. on the γ's) than I do. This restriction limits the cases where he obtains solutions, but greatly simplifies the searching, and he obtains some solutions for the case $k = 13$ which I did not even attempt. For $GD[7,1,7;7q]$ with $q \equiv 1 \bmod 6$, $q \leq 97$ and q a prime power, Hanani [13] presents solutions in the same cases that I give later. His solutions are very similar to mine, which is hardly surprising given the heritage of my ideas. Bagchi and Bagchi [5] deal with $k \leq 11$, k odd, and present number-theoretic arguments showing the existence of designs in a large number of cases.

3 Basic Designs

The starting point of this exposition is with four well known BIBD's.

Theorem 3.1 *Let p be an odd prime power. If $p = 4s+1$ then $p \in B(2s,2s-1)$ and $p \in B(2s+1,2s+1)$. If $p = 4s+3$ then $p \in B(2s+1,s)$ and $p \in B(2s+2,s+1)$.*

Proof: See [12, Lemmas 4.1–4.4]. Let $X = GF(p, f(x) = 0)$ with p an odd prime power. Let

$$\mathcal{B}_a = \langle a, a+2, \ldots, a+p-3 \rangle \mod (p)$$

$$\mathcal{B}'_a = \langle \emptyset, a, a+2, \ldots, a+p-3 \rangle \mod (p).$$

Then the 4 designs are given by $\mathcal{B}_0 \cup \mathcal{B}_1$, $\mathcal{B}'_0 \cup \mathcal{B}'_1$, $\mathcal{B}_0$, and $\mathcal{B}'_0$. □

Remark 3.2 *Note the similarity between $\mathcal{B}_a$ and $\mathcal{B}'_a$. This will be exploited repeatedly.*

I now turn to the consideration of what differences arise from the base block of $\mathcal{B}_0$ in the $4s+1$ case. The differences are found to be

$$(x^{2i} - 1)x^{2j} \quad \text{for } i = 1, 2, \ldots, 2s-1; \quad j = 0, 1, \ldots, 2s-1.$$

Suppose I write $x^r - 1 = x^s$. Then how many values of s are even as r ranges from 2 through even values to $4s-2$? The cyclotomic numbers of order m are denoted by (i, j) and are defined as the number of solutions to

$$1 + x^s = x^r \quad \text{with} \quad s \equiv i \bmod m \quad \text{and} \quad r \equiv j \bmod m.$$

It is supposed that $p-1$ is divisible by m and that p is a prime power. For cyclotomic numbers of order 2 with $p = 4s+1$, Hall [11, Equation 11.6.42] gives

$$(0,0) = s-1 \qquad (1,0) = (0,1) = (1,1) = s.$$

In particular, $(1,0) - (0,0) = 1$ so there is exactly one more odd multiplier than there are even multipliers of x^{2j}. In other words, each quadratic residue is represented $s-1$ times, and each non-square s times. Superimposing two $\mathcal{B}_0$ blocks quadruples these numbers, and also gives $4s$ zero differences. Now super-imposing a $\mathcal{B}_0$ and a $\mathcal{B}'_0$ will give each element (including $\emptyset$) $4s$ ocurrences in the signed differences. This is exploited in the following theorem.

Theorem 3.3 *Let p and q be odd prime powers with $q \geq p$. Then $pq \in GD(p, (p-1)/2, p)$.*

Proof: Let $X = GF(p, f(x) = 0) \times GF(q, f(y) = 0)$ and $d = (q-1)/2$. Let

$$S_{ba} = \langle(\emptyset; \emptyset), \{(b+2i; am + \gamma_{bi} + jd) : i = 0, \ldots, (p-3)/2; \;\; j = 0, 1\}\rangle \mod (p, q).$$

Now take $m = 1$, $\gamma_{0i} = i$ and $\mathcal{B} = \{S_{0a} : a = 0, \ldots, d-1\}$. Then $(X, \mathcal{B})$ is the required design. □

Remark 3.4 *The case of $p = 4s+3$ is well known, but the case of $p = 4s+1$ seems to be new [12, Remark to Lemma 4.26].*

Theorem 3.5 *Let p and q be odd prime powers with $q > p$ and $p = 4s+1$. Then $pq \in GD(p+1, p+1, p)$.*

Proof: Let $X = GF(p, f(x) = 0) \times GF(q, f(y) = 0)$ and $d = (q-1)/2$. Let

$$S'_{ba} = \langle \{(\emptyset; am + \gamma_{b.} + jd) : j = 0, 1\}, \\ \{(b + 2i; am + \gamma_{bi} + jd) : i = 0, \ldots, (p-3)/2; \;\; j = 0, 1\} \rangle \\ \bmod (p, q).$$

Now take $m = 1$, $\gamma_{b.} = 0, \gamma_{bi} = i + 1$ and $\mathcal{B} = \{S'_{ba} : b = 0, 1; \;\; a = 0, \ldots, d-1\}$. Then $(X, \mathcal{B})$ is the required design. □

Theorem 3.6 *Let p and q be odd prime powers with $q > p$ and $p = 4s+3$. Then $pq \in GD(p+1, (p+1)/2, p)$.*

Proof: Let X and S'_{ba} be as defined in Theorem 3.5. Now take $m = 1$, $\gamma_{0.} = 0$, $\gamma_{0i} = i + 1$ and $\mathcal{B} = \{S'_{0a} : a = 0, \ldots, d-1\}$. Then $(X, \mathcal{B})$ is the required design. □

4 Designs with $\lambda = 1$

In the last section it was seen that Theorem 3.5 could be improved to Theorem 3.6 by dropping an appropriate collection of blocks when the conditions were right. Similar improvements might be made to Theorems 3.3, 3.5, and 3.6, especially if the γ's were nicely chosen. The best result one could obtain would be $\lambda = 1$, which in turn implies $q = (p-1)t + 1$ for Theorem 3.3, and $q = (p+1)t + 1$ in Theorems 3.5 and 3.6. The main objective in the rest of this paper is to find such designs for small p, (i.e. $5 \le p \le 11$). The designs will be given in terms of the S_{ba} and S'_{ba} for $0 \le b < B$ defined in the proofs of Theorems 3.3 and 3.5. For the other parameter, $0 \le a \le t/B - 1$. This will impose the restriction that t be even if $B = 2$.

Table 1.

Design	p	x	q	Blocks	B	m	Comments
$GD(6,1,5)$	5	2	$6t+1$	S'_{ba}	2	6	t even
$GD(7,1,7)$	7	3	$6t+1$	S_{ba}	1	3	
$GD(8,1,7)$	7	3	$8t+1$	S'_{ba}	1	4	
$GD(9,1,9)$	9	$x^2 = 2x+1$	$8t+1$	S_{ba}	1	4	
$GD(10,1,9)$	9	$x^2 = 2x+1$	$10t+1$	S'_{ba}	2	10	t even
$GD(11,1,11)$	11	2	$10t+1$	S_{ba}	1	5	

The next question is what conditions are needed on the γ's? The membership part of these designs in non-exponential form is:

on $Z(5,2)$	$(0,1,4)$ with $(0,2,3)$	mod (5)
on $Z(7,3)$	$(0,1,2,4)$	mod (7)
on $GF(9, x^2 = 2x+1)$	$(00,01,21,02,12)$	
	with $(00,10,22,20,11)$	mod (3, 3)
on $Z(11,2)$	$(0,1,4,5,9,3)$	mod (11).

The structure we have imposed now reduces the problem to one of searching for a suitable set of γ's, and checking that the pure and mixed differences are evenly spread amongst the m cyclotomic classes. Without loss of generality, we can take the first γ to be 0, and restrict the remaining γ's to $0 < \gamma < d$. Consideration of the pure differences shows that the γ's must be distinct modulo m. More structure on the γ's makes the searching easier. I initially used more structure in the case of $k = 7$, but my later choices of structure caused me to abandon this, except in the $B = 2$ cases. Here the structure I used caused the γ's to be distinct modulo m/B.

The extra structure imposed was as follows. For $k = 6$, I took $\gamma_{1i} = \gamma_{0i} + 3$; ($\gamma_{1.}$ was treated similarly). A similar simplification by adding 5 in the case of $k = 10$ is possible. However, there is also the choice of $\gamma_{1.} = \gamma_{0.} + 5$ and $\gamma_{1i} = \gamma_{0i} + 5$ for $i = 1, 3$, and $\gamma_{1i} = \gamma_{0,2-i} + 5$ for $i = 0, 2$; this was a much better choice, and the one finally used. In the case of $k = 7$, the simplification $\gamma_1 = 2\gamma_2$ was also used; γ_2 is given in the appendix. Although there will be some later improvements, the main results of this paper are summarized in the following theorems that are proved with the constructions in the appendix tables.

Theorem 4.1 *If $t \notin \{2,6,8,12\}$, and $t \leq 832$, and $6t+1$ is a prime power, and t is even, then $GD[6,1,5;30t+5]$ exists.*

Remark 4.2 *The four exceptions can be removed.*

Theorem 4.3 *If $t \notin \{3,4,6\}$, and $t \leq 512$, and $6t+1$ is a prime power, then $GD[7,1,7;42t+7]$ exists.*

Theorem 4.4 *If $t \notin \{1,2,3,11\}$ and $t \leq 512$, and $8t+1$ is a prime power, then $GD[8,1,7;56t+7]$ exists.*

Remark 4.5 *The first two of the four exceptions can be removed.*

Theorem 4.6 *If $t \notin \{2,3,5,10,12,14\}$, and $t \leq 729$, and $8t+1$ is a prime power, then $GD[9,1,9;72t+9]$ exists.*

Remark 4.7 *The $AG(3,9)$ design removes the $t = 10$ exception.*

Theorem 4.8 *If* $t \notin \{4, 6, 10, 12, 18, 24\}$, *and* $t \leq 729$, *and* $10t + 1$ *is a prime power, and* t *is even, then* $GD[10, 1, 9; 90t + 9]$ *exists.*

Theorem 4.9 *If* $t \notin \{3-19, 24-31, 49\}$, *and* $t \leq 121$, *and* $10t + 1$ *is a prime, then* $GD[11, 1, 11; 110t + 11]$ *exists.*

Remark 4.10 *The composite prime power cases in this range were not attempted for* $k = 11$ *(i.e.,* $t \in \{8, 12, 36, 84, 96\}$*).*

5 Resolvability

It is a simple matter to complete any of these designs to a BIBD. However, when the block size, k, is even, these completions yield resolvable designs. Consider the partial development of an S' type block developed $\mathrm{mod}(k-1, -)$, and augment this with a new base block:

$$\langle(\infty), (\emptyset; \emptyset), \{(i; \emptyset) : i = 0, \ldots, k-3\}\rangle.$$

These augmented base blocks form a resolution set. Clearly the development of this augmented set (developed $\mathrm{mod}(-, q)$) generates a resolvable BIBD.

Theorem 5.1 *If* $t \notin \{2, 6, 8, 12\}$, *and* $t \leq 832$, *and* $6t + 1$ *is a prime power, and* t *is even, then* $RB[6, 1; 30t + 6]$ *exists.*

Remark 5.2 *All but the first of the four exceptions can be removed.*

Theorem 5.3 *If* $t \notin \{1, 2, 3, 11\}$ *and* $t \leq 512$, *and* $8t+1$ *is a prime power, then* $RB[8, 1; 56t + 8]$ *exists.*

Remark 5.4 *The first two of the four exceptions can be removed.*

Theorem 5.5 *If* $t \notin \{4, 6, 10, 12, 18, 24\}$, *and* $t \leq 729$, *and* $10t + 1$ *is a prime power, and* t *is even, then* $RB[10, 1; 90t + 10]$ *exists.*

6 Other Designs

The focus of this section is on the small exceptions noted in the last two sections. Some of these missing designs can be constructed using a more flexible approach, and sometimes designs with $\lambda > 1$ can be constructed. These designs were constructed by hand in a rather ad hoc fashion. The basic approach is to look for β starter base blocks, instead of the original B, replace the old number of cyclotomic classes, m, by $m\beta/(B\lambda)$, and have each starter base block generate $t\lambda/\beta$ base blocks. The cyclotomic classes should have λ representatives each. The integrality of these numbers imposes some restriction on the choice of β. This approach was used to obtain Table 2, although it is partially hidden to compress the table. The notation of Theorems 3.3 and 3.5 is used for Table 2, with the γ's given in the order $\gamma_{b.}$, (if applicable), $\gamma_{b0}, \gamma_{b1}, \cdots, \gamma_{bn}$.

Table 2.

Design	t	q	x	Blocks	m	γ	γ	γ	γ	
$GD(6,1,5)$	8	49	$x^2=6x+4$	S'_{0a}	3	0	4	17		$a=0,1,4,5$
				S'_{1a}	3	22	11	18		$a=0,1,4,5$
$GD(6,1,5)$	12	73	5	S'_{0a}	12	0	11	19		$a=0,1,2$
					12	9	20	28		$a=0,1,2$
				S'_{1a}	12	3	14	22		$a=0,1,2$
					12	6	17	25		$a=0,1,2$
$GD(7,2,7)$	3	19	2	S_{0a}	3	0	6	8		$a=0,1,2$
					3	1	2	7		$a=0,1,2$
$GD(7,2,7)$	4	25	$x^2=4x+3$	S_{0a}	3	0	4	7		$a=0,\ldots,3$
					3	0	8	11		$a=0,\ldots,3$
$GD(7,2,7)$	6	37	2	S_{0a}	3	0	2	1		$a=0,\ldots,5$
					3	0	22	11		$a=0,\ldots,5$
$GD(8,2,7)$	3	25	$x^2=4x+3$	S'_{0a}	2	0	6	3	9	$a=0,\ldots,5$
$GD(9,2,9)$	2	17	3	S_{0a}	1	0	1	4	5	$a=0,\ldots,3$

The designs given above with k even can also be completed to yield resolvable BIBD's. In addition to the designs tabled above, the following are known: $B[6,1;66]$, $RB[6,1;186]$, $RB[8,1;64]$, and $RB[8,1;120]$. One is the $AG(2,8)$, and for the others, see [7, 15, 20].

Theorem 6.1 *If $q=6t+1$ is a prime power, then $RB[6,2;30t+6]$ exists.*

Proof: The above demonstration of resolvability also applies to Hanani's construction of $GD[6,2,5;5q]$ designs [12, Lemma 4.19]. □

7 Consequences

One use of these designs, in conjunction with those in [8], is the construction of $B(k,1)$ designs for $7 \leq k \leq 9$, and of $RB(8,1)$ designs which will be reported more fully elsewhere [9], but the initial findings were that there are constructions with just 40, 78, 157, and 95 exceptions, the largest of which were 4915, 12937, 32697, and 58192. There are some more immediate consequences, namely $\{246, 306, 486\} \subset RB(6,1)$. Two of these are new $B(6,1)$ designs; the value 306 is not new, (there is an unpublished construction by Hanani [17]). The smallest value also yields a construction of $B[6,1;5391]$ using a $T[22,1;49]$ design, and a $B[7,1;295]$ using the resolvability, with $AG(2,7)$ to fill in the resulting flat. The construction of $GD[8,1,7;287]$ enables a $B[7,3;575]$ to be constructed, eliminating one of the seven exceptions for this case [13, Theorem 103]; the $B[7,1;295]$ mentioned above can be used to construct a $B[7,3;323]$, thereby eliminating another of the seven exceptions. (Constructions for the remaining cases also exist; see [3] for details).

More recently, Abel has tackled these same problems [1], and with his constructions, the initial findings noted above can be improved upon (see [2] for a recent list). For the $RB(8,1)$ case there are now 66 exceptions, with 24480 being the largest [10].

8 Other B(6, 1) Designs

This section is quite independent of the rest of the paper, and is based on the constructions of Mullin et al. [19]. The objective is to remove two of the exceptional values given by Mullin [19, Table 1], in addition to the three given above. Actually, a construction for the value 1066 was known, and its inclusion in the exception list in [19] was an oversight; hence, the real interest in Corollary 8.3 lies in the other design constructed. These constructions, or alternatives, have been incorporated into recent lists, such as [4], but the techniques used could be of interest.

Lemma 8.1 *The incomplete transversal design* $T[6,1;155+a]-T[6,1;a]$ *exists for* $7 \leq a \leq 23$.

Proof: In the $T[24,1;23]$ design suppose the first and last blocks intersect in the first group. Retain the first 7 groups, delete all but the points of the last block in the next $a-7$ groups, delete all but the point of the first block in the next group, and delete all remaining groups, to give a $GD[\{7,8,9,8^*,a^*\},1,\{1,23\};155+a]$; filling in the groups, and removing the block of size a gives a $GD[\{7,8,9,23\},1,\{1,a^*\};155+a]$ and the result follows from [21]. □

Lemma 8.2 *If* $7 \leq a \leq 23$, *then a* $GD[6,1,\{31,(31+5a)^*\};961+5a]$ *exists.*

Proof: Use the blocks of $T(6,1;31)$ to fill in the groups of the incomplete transversal with $31-a$ points at infinity. These points and those of each missing subgroup are to lie in a flat of size 31. □

Corollary 8.3 *The designs* $B[6,1;1066]$ *and* $B[\{6,31^*\},1;996]$ *exist.*

Proof: Take $a=21$ and $a=7$, and note that $\{31,66,136\} \subset B(6,1)$ [19]. □

Corollary 8.4 *The design* $B[6,1;5901]$ *exists.*

Proof: We may apply [19, Theorem 1.1] with $v = 996$, $f = 31$, and $a = 16$. Since $8 \cdot 17 = 136 \in T(9,1)$, and $981 = 7 \cdot 136 + 13 + 16$, the needed incomplete transversal is easily constructed [21]. Note that $111 \in B(6,1)$ [19]. □

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[21] R.M. Wilson, Concerning the number of mutually orthogonal latin squares, *Discrete Math.* **9** (1974), 181–198.

A Appendix

The main purpose of the appendix is to give the basic constructions of BIBD's. Some variants were given in section 6. The first table gives the primitive polynomials for all odd prime powers ≤ 10000. The remaining tables give the parameters of successful constructions found. These tables give the primitive element, x, used to generate the $GF(v)$. In the case of prime powers this entry contains ** and the primitive element used is a root of the polynomial in the first table.

Table A.1.

Table of primitive polynomials of $GF(p^n)$ with p odd.
$f(x) = b_0x^0 + b_1x^1 + \cdots + b_nx^n$.

p^n	p	n	$b_0, \ldots, b_n$
9	3	2	2 1 1
27	3	3	1 0 2 1
81	3	4	2 0 0 1 1
243	3	5	1 0 1 0 1 1
729	3	6	2 0 0 0 0 1 1
2187	3	7	1 0 0 0 1 0 1 1
6561	3	8	2 0 0 0 0 1 0 0 1
25	5	2	2 1 1
125	5	3	2 0 1 1
625	5	4	3 1 0 1 1
3125	5	5	2 0 1 0 0 1
49	7	2	3 1 1
343	7	3	2 1 1 1
2401	7	4	3 0 1 1 1
121	11	2	7 1 1
1331	11	3	3 0 1 1
169	13	2	2 1 1
2197	13	3	2 0 1 1
289	17	2	3 1 1
4913	17	3	7 0 1 1
361	19	2	2 1 1
6859	19	3	6 0 1 1

p^n	p	n	b_0	b_1	b_2
529	23	2	7	1	1
841	29	2	3	1	1
961	31	2	12	1	1
1369	37	2	5	1	1
1681	41	2	12	1	1
1849	43	2	3	1	1
2209	47	2	13	1	1
2809	53	2	5	1	1
3481	59	2	2	1	1
3721	61	2	2	1	1
4489	67	2	12	1	1
5041	71	2	11	1	1
5329	73	2	11	1	1
6241	79	2	3	1	1
6889	83	2	2	1	1
7921	89	2	6	1	1
9409	97	2	5	1	1

Table A.2.

Table of γ's for $RB[6, 1; 5q + 1]$.

q	x	γ_1	γ_2	q	x	γ_1	γ_2	q	x	γ_1	γ_2
25	**	5	1	61	2	26	28	97	5	5	19
109	6	16	14	121	**	16	14	157	5	109	110
169	**	28	23	181	2	38	64	193	5	142	116
229	6	196	92	241	7	185	217	277	5	206	172
289	**	11	64	313	10	70	134	337	10	292	221
349	2	338	316	361	**	91	158	373	2	242	28
397	5	65	301	409	21	97	86	421	2	74	313
433	5	301	95	457	13	203	370	529	**	47	151
541	2	467	496	577	5	515	25	601	7	379	524
613	2	238	416	625	**	124	128	661	2	610	578

673	5	466	341	709	2	673	338	733	6	695	484
757	2	428	88	769	11	628	56	829	2	608	88
841	**	10	8	853	2	605	454	877	2	716	496
937	5	146	772	961	**	49	440	997	7	203	823
1009	11	380	430	1021	10	16	200	1033	5	794	43
1069	6	256	440	1093	5	835	494	1117	2	437	1096
1129	11	502	134	1153	5	208	164	1201	11	233	877
1213	2	665	793	1237	2	764	616	1249	7	487	1229
1297	10	938	1201	1321	13	1163	700	1369	**	8	169
1381	2	538	704	1429	6	278	463	1453	2	5	1318
1489	14	188	142	1549	2	49	215	1597	11	1375	1379
1609	7	1358	1051	1621	2	1612	1082	1657	11	1052	868
1669	2	134	655	1681	**	35	547	1693	2	1174	1322
1741	2	862	1202	1753	7	373	833	1777	5	940	788
1789	6	1742	1324	1801	11	734	1507	1849	**	185	235
1861	2	1081	206	1873	10	1724	919	1933	5	82	1502
1993	5	703	1298	2017	5	230	1549	2029	2	11	688
2053	2	1472	1300	2089	7	1216	500	2113	5	1708	392
2137	10	1202	1312	2161	23	1475	667	2197	**	190	308
2209	**	95	28	2221	2	1520	154	2269	2	1759	1145
2281	7	2008	956	2293	2	1418	1987	2341	7	758	862
2377	5	235	2333	2389	2	1958	400	2401	**	47	28
2437	2	326	37	2473	5	2321	1864	2521	17	154	2015
2557	2	2080	2273	2593	7	1948	1700	2617	5	1844	2173
2677	2	2534	2488	2689	19	164	226	2713	5	1916	526
2749	6	1382	1459	2797	2	1190	2509	2809	**	2	64
2833	5	853	1358	2857	11	1213	1907	2917	5	1	2147
2953	13	2488	2900	3001	14	1070	1885	3037	2	1741	2954
3049	11	958	1691	3061	6	2246	850	3109	6	2423	2269
3121	7	1318	74	3169	7	2104	2966	3181	7	2	2098
3217	5	760	1673	3229	6	3050	2590	3253	2	1814	2188
3301	6	740	2416	3313	10	683	3142	3361	22	2126	433
3373	5	2681	445	3433	5	2255	634	3457	7	907	2510
3469	2	2350	1406	3481	**	80	694	3517	2	2	1609
3529	17	908	85	3541	7	1829	751	3613	2	1511	652
3637	2	1577	2446	3673	5	2824	1697	3697	5	536	2428
3709	2	1157	2251	3721	**	7	1103	3733	2	1538	2203
3769	7	1384	1853	3793	5	197	1714	3853	2	2833	320
3877	2	413	415	3889	11	1046	370	4021	2	2986	2723
4057	5	1264	3692	4093	2	3104	1408	4129	13	2404	1160
4153	5	1664	1591	4177	5	353	2779	4201	11	2512	3188
4261	2	718	1538	4273	5	2620	1523	4297	5	3875	1207
4357	2	2566	2528	4441	21	1	4106	4489	**	1	890
4513	7	1921	500	4549	6	1	164	4561	11	2654	3424
4597	5	1	2987	4621	2	935	3463	4657	15	2140	2660
4729	17	4256	2548	4789	2	5	4375	4801	7	2156	1147
4813	2	2762	3127	4861	11	538	2327	4909	6	3304	3206
4933	2	2	358	4957	2	2096	4486	4969	11	4775	1456
4993	5	1648	3281								

Table A.3.

Table of γ_2 for $GD[7,1,7;7q]$.

q	x	γ_2	q	x	γ_2	q	x	γ_2	q	x	γ_2
7	3	2	13	2	2	31	3	5	43	3	14
49	**	16	61	2	2	67	2	7	73	5	2
79	3	26	97	5	25	103	5	4	109	6	17
121	**	14	127	3	7	139	2	46	151	6	13
157	5	23	163	2	46	169	**	23	181	2	5
193	5	2	199	3	8	211	2	28	223	3	20
229	6	31	241	7	31	271	6	5	277	5	32
283	3	5	289	**	7	307	5	26	313	10	16
331	3	23	337	10	40	343	**	8	349	2	37
361	**	22	367	6	8	373	2	4	379	2	29
397	5	13	409	21	11	421	2	17	433	5	23
439	15	31	457	13	5	463	3	59	487	3	34
499	7	17	523	2	70	529	**	25	541	2	1
547	2	5	571	3	10	577	5	52	601	7	8
607	3	16	613	2	26	619	2	20	625	**	32
631	3	10	643	11	37	661	2	65	673	5	10
691	3	32	709	2	17	727	5	38	733	6	47
739	3	67	751	3	4	757	2	2	769	11	8
787	2	13	811	3	61	823	3	16	829	2	5
841	**	22	853	2	4	859	2	26	877	2	1
883	2	20	907	2	4	919	7	2	937	5	4
961	**	13	967	5	8	991	6	44	997	7	16
1009	11	7	1021	10	2	1033	5	13	1039	3	17
1051	7	106	1063	3	11	1069	6	22	1087	3	11
1093	5	2	1117	2	4	1123	2	31	1129	11	1
1153	5	4	1171	2	1	1201	11	8	1213	2	59
1231	3	38	1237	2	19	1249	7	62	1279	3	4
1291	2	44	1297	10	20	1303	6	5	1321	13	8
1327	3	2	1369	**	8	1381	2	37	1399	13	29
1423	3	11	1429	6	11	1447	3	14	1453	2	1
1459	3	14	1471	6	2	1483	2	19	1489	14	47
1531	2	14	1543	5	52	1549	2	91	1567	3	5
1579	3	4	1597	11	37	1609	7	80	1621	2	2
1627	3	11	1657	11	10	1663	3	8	1669	2	10
1681	**	25	1693	2	1	1699	3	25	1723	3	19
1741	2	1	1747	2	32	1753	7	5	1759	6	19
1777	5	1	1783	10	13	1789	6	7	1801	11	37
1831	3	13	1849	**	8	1861	2	47	1867	2	29
1873	10	19	1879	6	32	1933	5	28	1951	3	13
1987	2	8	1993	5	17	1999	3	25	2011	3	55
2017	5	16	2029	2	2	2053	2	4	2083	2	16
2089	7	1	2113	5	8	2131	2	31	2137	10	2
2143	3	8	2161	23	47	2179	7	14	2197	**	44
2203	5	11	2209	**	28	2221	2	35	2239	3	4
2251	7	2	2269	2	41	2281	7	1	2287	19	7
2293	2	4	2311	3	2	2341	7	29	2347	3	8
2371	2	1	2377	5	13	2383	5	5	2389	2	56
2401	**	5	2437	2	29	2467	2	8	2473	5	1
2503	3	5	2521	17	13	2539	2	1	2551	6	7
2557	2	11	2593	7	1	2617	5	17	2647	3	28
2659	2	16	2671	7	7	2677	2	22	2683	2	32

2689	19	2	2707	2	23	2713	5	11	2719	3	7
2731	3	2	2749	6	5	2767	3	22	2791	6	59
2797	2	13	2803	2	13	2809	**	2	2833	5	20
2851	2	4	2857	11	10	2887	5	5	2917	5	41
2953	13	13	2971	10	23	3001	14	10	3019	2	13
3037	2	22	3049	11	29	3061	6	20	3067	2	23

Table A.4.

Table of γ's for $RB[8,1;7q+1]$.

q	x	γ_0	γ_1	γ_2	q	x	γ_0	γ_1	γ_2
41	6	15	25	30	49	**	14	9	11
73	5	29	39	2	81	**	1	59	78
97	5	5	14	31	113	3	75	21	14
121	**	71	46	105	137	3	10	35	105
169	**	94	93	103	193	5	118	69	11
233	3	165	211	226	241	7	90	53	139
257	3	55	201	110	281	3	186	239	189
289	**	262	257	223	313	10	214	125	167
337	10	109	262	127	353	3	309	27	330
361	**	185	359	74	401	3	26	241	215
409	21	106	391	97	433	5	234	327	145
449	3	254	257	351	457	13	51	138	397
521	3	318	103	37	529	**	287	298	297
569	3	510	249	483	577	5	102	205	43
593	3	535	590	437	601	7	382	419	461
617	3	321	339	134	625	**	486	289	379
641	3	470	571	101	673	5	487	602	297
729	**	723	301	718	761	6	215	10	61
769	11	487	321	494	809	3	138	253	19
841	**	411	129	430	857	3	62	291	109
881	3	62	151	369	929	3	810	299	333
937	5	905	806	279	953	3	126	275	517
961	**	511	649	162	977	3	50	265	231
1009	11	666	271	797	1033	5	365	166	367
1049	3	98	59	565	1097	3	1003	790	1045
1129	11	1070	881	211	1153	5	155	1042	189
1193	3	699	1126	941	1201	11	774	235	61
1217	3	819	1169	1086	1249	7	347	717	946
1289	6	790	795	949	1297	10	627	293	246
1321	13	726	1163	1097	1361	3	126	371	241
1369	**	1	774	607	1409	3	58	131	53
1433	3	511	237	186	1481	3	1118	759	561
1489	14	930	821	1087	1553	3	489	715	1474
1601	3	587	50	1121	1609	7	1358	379	1453
1657	11	1026	1231	1245	1681	**	235	1110	413
1697	3	370	1065	1311	1721	3	1227	394	33
1753	7	373	402	1679	1777	5	1177	71	1630
1801	11	926	1077	711	1849	**	1479	202	13
1873	10	1674	1423	1037	1889	3	1685	1110	15
1913	3	1077	1899	274	1993	5	1203	1086	1021
2017	5	61	703	1150	2081	3	1358	99	1273
2089	7	1286	707	1857	2113	5	673	1659	250
2129	3	114	155	965	2137	10	1202	1481	735

2153	3	1910	1475	1489	2161	23	834	13	1135
2209	**	2183	766	1001	2273	3	745	2070	2063
2281	7	1394	675	1365	2297	5	310	225	1551
2377	5	2	647	69	2393	3	181	1178	199
2401	**	1531	1673	654	2417	3	310	1601	115
2441	6	1	486	1975	2473	5	298	559	261
2521	17	2126	1557	327	2593	7	1753	2158	995
2609	3	930	245	1811	2617	5	1458	1443	1805
2633	3	1914	545	531	2657	3	545	711	2314
2689	19	2386	1383	1117	2713	5	959	313	338
2729	3	2062	1685	2523	2753	3	1074	165	543
2777	3	2398	581	1087	2801	3	58	2447	177
2809	**	7	1046	69	2833	5	1993	1471	1226
2857	11	2471	110	2613	2897	3	2274	1477	1391
2953	13	1	2778	1259	2969	3	2242	2449	635
3001	14	1070	2353	839	3041	3	2186	527	2189
3049	11	990	201	1207	3089	3	1122	1617	999
3121	7	1142	1701	2539	3137	3	1365	1134	791
3169	7	1578	321	1911	3209	3	514	243	5
3217	5	710	593	3	3257	3	2695	774	1465
3313	10	1490	197	2243	3329	3	1134	1105	543
3361	22	1466	269	2467	3433	5	2070	893	1031
3449	3	1219	2518	1605	3457	7	126	3403	993
3481	**	3151	973	802	3529	17	738	1859	1001
3593	3	2970	1677	3	3617	3	18	1689	1171
3673	5	2083	3313	782	3697	5	94	1629	615
3721	**	1865	1603	1486	3761	3	622	1939	3409
3769	7	10	3217	199	3793	5	3714	3215	2337
3833	3	3749	538	3759	3881	13	710	2395	1085
3889	11	1046	1763	333	3929	3	1114	3387	757
4001	3	1425	3363	870	4049	3	2410	3007	3441
4057	5	902	2827	3049	4073	3	706	3479	481

Table A.5.

Table of γ's for $GD[9,1,9;9q]$.

q	x	γ_1	γ_2	γ_3	q	x	γ_1	γ_2	γ_3
9	**	2	3	1	49	**	10	3	1
73	5	27	18	9	89	3	27	22	5
121	**	18	39	9	137	3	42	31	33
169	**	30	11	5	193	5	86	75	65
233	3	87	58	29	241	7	94	23	49
257	3	30	3	105	281	3	70	3	73
289	**	98	3	49	313	10	82	31	25
337	10	78	3	41	353	3	42	27	173
361	**	2	3	1	401	3	54	3	45
409	21	134	19	89	433	5	86	11	197
449	3	110	11	85	457	13	122	3	29
521	3	162	35	133	529	**	170	47	149
569	3	102	79	181	577	5	66	31	33
593	3	90	27	233	601	7	254	19	197
617	3	242	3	213	625	**	178	159	25
641	3	34	67	281	673	5	166	55	141
729	**	254	11	217	761	6	306	7	213

769	11	322	31	301	809	3	106	15	89
841	**	78	7	41	857	3	298	55	289
881	3	186	23	17	929	3	250	11	209
937	5	154	39	137	953	3	442	7	369
961	**	242	47	225	977	3	278	3	213
1009	11	174	3	121	1033	5	22	23	481
1049	3	254	15	181	1097	3	526	91	497
1129	11	282	11	277	1153	5	330	15	321
1193	3	390	23	357	1201	11	162	3	57
1217	3	450	11	389	1249	7	494	19	477
1289	6	106	27	73	1297	10	362	95	349
1321	13	102	27	65	1361	3	530	59	493
1369	**	482	31	461	1409	3	542	7	533
1433	3	566	11	473	1481	3	390	23	373
1489	14	226	19	217	1553	3	490	51	485
1601	3	418	15	377	1609	7	110	27	85
1657	11	474	3	433	1681	**	610	23	581
1697	3	594	55	505	1721	3	454	51	413
1753	7	346	3	321	1777	5	546	19	425
1801	11	542	67	425	1849	**	410	63	369
1873	10	314	139	269	1889	3	162	3	157
1913	3	222	43	181	1993	5	150	23	117
2017	5	206	39	133	2081	3	762	99	729
2089	7	478	59	445	2113	5	742	83	693
2129	3	110	59	105	2137	10	138	87	125
2153	3	34	7	1029	2161	23	534	15	465
2209	**	166	23	97	2273	3	546	15	521
2281	7	682	131	649	2297	5	294	71	225
2377	5	278	47	269	2393	3	170	23	121
2401	**	158	103	97	2417	3	482	23	473
2441	6	622	3	601	2473	5	798	3	765
2521	17	18	39	13	2593	7	194	15	173
2609	3	34	11	25	2617	5	286	27	229
2633	3	550	7	545	2657	3	90	27	25
2689	19	618	43	613	2713	5	650	55	641
2729	3	30	23	21	2753	3	202	27	161
2777	3	218	23	193	2801	3	1038	75	1013
2809	**	630	3	605	2833	5	546	19	541
2857	11	718	7	717	2897	3	362	75	357
2953	13	30	59	25	2969	3	170	11	109
3001	14	54	23	45	3041	3	502	75	493
3049	11	1422	3	1417	3089	3	214	15	125
3121	7	410	103	397	3137	3	1054	51	1045
3169	7	738	15	729	3209	3	358	11	289
3217	5	174	15	109	3257	3	66	19	5
3313	10	1478	43	1469	3329	3	106	39	65
3361	22	210	19	197	3433	5	262	15	241
3449	3	898	7	877	3457	7	1446	3	1425
3481	**	2	3	1	3529	17	30	7	13
3593	3	290	39	221	3617	3	898	23	893
3673	5	1302	67	1293	3697	5	130	35	97
3721	**	86	23	57	3761	3	546	31	537
3769	7	42	55	13	3793	5	238	39	193
3833	3	682	31	653	3881	13	778	75	761
3889	11	730	3	709	3929	3	882	15	829

4001	3	930	51	893	4049	3	590	7	569
4057	5	458	11	453	4073	3	754	7	717
4129	13	1954	11	1949	4153	5	534	11	513
4177	5	658	15	577	4201	11	142	11	89
4217	3	314	7	305	4241	3	770	35	761
4273	5	302	27	289	4289	3	878	31	865
4297	5	358	7	317	4337	3	1618	23	1565
4409	3	406	43	389	4441	21	198	31	133
4457	3	198	59	169	4481	3	146	7	125
4489	**	462	23	457	4513	7	186	11	169
4561	11	302	19	293	4649	3	98	3	13
4657	15	610	7	593	4673	3	258	35	253
4721	6	722	63	717	4729	17	1182	15	1129
4793	3	218	19	205	4801	7	298	23	241
4817	3	226	7	221	4889	3	154	27	73
4913	**	30	47	17	4937	3	362	167	333
4969	11	674	15	581	4993	5	126	15	93
5009	3	2410	31	2401	5041	**	338	3	265
5081	3	406	7	293	5113	19	222	23	217
5153	5	154	47	133	5209	17	1074	3	1049
5233	10	630	19	577	5273	3	62	31	33
5281	7	354	43	333	5297	3	22	39	17
5329	**	518	35	409	5393	3	230	7	149
5417	3	234	55	197	5441	3	302	3	269
5449	7	1082	39	1037	5521	11	170	23	145
5569	13	770	35	749	5641	14	490	27	485
5657	3	994	3	901	5689	11	638	79	609
5737	5	446	43	393	5801	3	466	23	453

Table A.6.

Table of γ's for $RB[10, 1; 9q + 1]$.

q	x	γ_0	γ_1	γ_2	γ_3	q	x	γ_0	γ_1	γ_2	γ_3
81	**	16	32	24	8	281	3	71	77	3	134
361	**	86	167	73	9	401	3	116	2	148	99
421	2	126	192	78	59	461	2	66	72	74	128
521	3	121	247	73	14	541	2	11	142	13	54
601	7	121	297	133	139	641	3	96	22	208	189
661	2	151	222	33	279	701	2	96	62	288	34
761	6	56	82	78	289	821	2	11	247	88	334
841	**	16	307	368	139	881	3	1	172	63	129
941	2	46	317	208	39	961	**	321	62	13	384
1021	10	41	242	453	164	1061	2	61	197	143	14
1181	7	36	542	393	209	1201	11	6	42	8	284
1301	2	16	387	498	444	1321	13	16	62	188	429
1361	3	11	32	153	649	1381	2	31	482	528	144
1481	3	86	142	238	704	1601	3	1	37	418	224
1621	2	6	452	393	134	1681	**	1	562	208	549
1721	3	6	857	343	19	1741	2	1	732	538	354
1801	11	1	187	188	384	1861	2	6	872	148	449
1901	2	1	312	183	4	2081	3	1	282	283	479
2141	2	1	337	1038	739	2161	23	1	922	293	479
2221	2	1	437	933	984	2281	7	1	612	23	944
2341	7	1	272	3	289	2381	3	1	407	698	749

2401	**	1	852	263	124	2441	6	1	362	233	429
2521	17	1	1232	338	399	2621	2	1	1147	153	1304
2741	2	1	142	323	659	2801	3	1	682	243	1364
2861	2	6	767	133	1134	3001	14	1	592	603	739
3041	3	1	1057	983	374	3061	6	1	612	238	389
3121	7	1	857	433	229	3181	7	1	387	48	14
3221	10	1	1147	688	409	3301	6	1	1372	48	889
3361	22	1	707	128	1544	3461	2	1	577	8	1039
3481	**	1	1007	593	1594	3541	7	1	1547	298	1294
3581	2	1	817	43	349	3701	2	1	337	233	789
3721	**	1	887	13	1154	3761	3	21	862	333	1689
3821	3	1	1512	408	894	3881	13	6	597	63	489
4001	3	1	507	1303	859	4021	2	1	957	73	1259
4201	11	1	1602	218	1519	4241	3	1	1092	8	334
4261	2	1	732	3	19	4421	3	1	572	78	1659
4441	21	1	407	148	204	4481	3	1	1527	208	1104
4561	11	1	562	113	1869	4621	2	1	1407	13	1989
4721	6	1	2332	268	1359	4801	7	1	1232	238	2299
4861	11	1	777	198	2304	5021	3	1	1452	403	2174
5041	**	1	97	43	624	5081	3	1	1067	483	1649
5101	6	1	617	43	844	5261	2	1	1237	83	764
5281	7	1	637	8	1359	5381	3	1	2302	48	2054
5441	3	11	2317	28	1684	5501	2	1	2127	58	1544
5521	11	1	1052	18	594	5581	6	6	1242	358	1229
5641	14	1	47	13	949	5701	2	1	1752	208	2704
5741	2	1	632	413	2229	5801	3	1	1992	3	1309
5821	6	1	1262	13	1659	5861	3	1	1067	18	2634
5881	31	1	2147	43	1629	5981	3	1	497	593	304
6101	2	1	1057	158	369	6121	7	1	912	318	1094
6221	3	1	2867	28	2289	6241	**	1	1597	38	1599
6301	10	1	2607	43	1399	6361	19	1	342	328	1509
6421	6	1	622	198	1834	6481	7	1	1912	63	2159
6521	6	1	1187	38	1859	6561	**	1	2922	98	2869
6581	14	1	2647	3	2159	6661	6	1	132	3	349
6701	2	1	772	78	984	6761	3	1	142	18	1479
6781	2	1	2642	208	1709	6841	22	1	1667	18	139
6961	13	1	602	228	2529	7001	3	1	272	78	2809
7121	3	1	22	73	2749						

Table A.7

Table of γ's for $GD[11,1,11;11q]$.

q	x	γ_1	γ_2	γ_3	γ_4	q	x	γ_1	γ_2	γ_3	γ_4
11	2	1	2	8	4	211	2	84	203	182	91
331	3	121	34	318	87	401	3	26	309	182	73
421	2	202	248	296	254	431	7	338	271	192	299
461	2	108	407	166	439	521	3	304	13	381	92
541	2	431	107	208	309	571	3	231	252	153	234
601	7	33	97	36	189	631	3	427	186	428	74
641	3	606	312	228	94	661	2	543	257	186	259
691	3	12	573	391	264	701	2	3	72	506	149
751	3	433	257	121	164	761	6	66	13	147	699
811	3	118	36	252	534	821	2	339	182	231	458
881	3	416	577	838	124	911	17	111	497	373	759

941	2	323	227	776	294	971	6	461	222	308	279
991	6	768	841	987	409	1021	10	53	766	2	334
1031	14	473	712	946	749	1051	7	81	137	98	224
1061	2	477	381	513	109	1091	2	807	656	523	1059
1151	17	858	187	766	219	1171	2	902	108	71	1069
1181	7	1091	392	198	1014	1201	11	996	513	402	434