Existence of (3, 1, 2)-HCOLS and (3, 1, 2)-ICOILS

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ABSTRACT. A Latin square $(S, \cdot)$ is said to be (3, 1, 2)-*conjugate-orthogonal* if $x \cdot y = z \cdot w$, $x \cdot_{312} y = z \cdot_{312} w$ imply $x = z$ and $y = w$, for all $x, y, z, w \in S$, where $x_3 \cdot_{312} x_1 = x_2$ if and only if $x_1 \cdot x_2 = x_3$. Such a Latin square is said to be *holey* ((3, 1, 2)–HCOLS for short) if it has disjoint and spanning holes corresponding to missing sub-Latin squares. Let (3, 1, 2)–HCOLS(h^n) denote a (3, 1, 2)–HCOLS of order hn with n holes of equal size h. We show that, for any $h \geq 1$, a (3, 1, 2)–HCOLS(h^n) exists if and only if $n \geq 4$, except $(n, h) = (6, 1)$, and except possibly $(n, h) = (10, 1)$ and $(4, 2t + 1)$ for $t \geq 1$. Let (3, 1, 2)–ICOILS(v, k) denote an idempotent (3, 1, 2)-COLS of order v with a hole of size k. We prove that a (3, 1, 2)–ICOILS(v, k) exists for all $v \geq 3k + 1$ and $1 \leq k \leq 5$, except possibly $k = 4$ and $v \in \{35, 38\}$.

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1 Introduction

Let $(S, \cdot)$ be a quasigroup where the multiplication table of $\cdot$ forms a Latin square indexed by S. The (i, j, k)-conjugate of $(S, \cdot)$ is $(S, \cdot_{ijk})$, where (i, j, k) is a permutation of $(1, 2, 3)$ and $x_i \cdot_{ijk} x_j = x_k$ if and only if $x_1 \cdot x_2 = x_3$. Following the convention (see [5]), we call $(S, \cdot)$ a Latin square (i.e., the multiplication table of $\cdot$ is a Latin square indexed by S). A Latin square is said to be (i, j, k)-*conjugate-orthogonal* ((i, j, k)-COLS for short) if $x \cdot y = z \cdot w$ and $x \cdot_{ijk} y = z \cdot_{ijk} w$ imply $x = z$ and $y = w$, where $x \cdot y$ denotes the entry in the cell (x, y) of the square. We will use (i, j, k)–HCOLS($h_1^{n_1} \cdots h_k^{n_k}$) to denote the type of holey (i, j, k)-COLS of order $\sum_{i=1}^{k} h_i n_i$, which has n_i holes of size h_i, $1 \leq i \leq k$, and all the holes are assumed to be mutually disjoint, and each of them corresponds to a missing sub-Latin square. It is well-known that there does not exist any $(1, 2, 3)$-HCOLS(h^n) for $n > 1$; a $(3, 2, 1)$-HCOLS(h^n) exists if and only if a $(1, 3, 2)$-HCOLS(h^n) exists; a $(3, 1, 2)$-HCOLS(h^n) exists if and only if a $(2, 3, 1)$-HCOLS(h^n) exists.

The existence of $(2, 1, 3)$-HCOLS(h^n) has been completely settled [5, 9]. The existence of $(3, 2, 1)$-HCOLS(h^n) has also been settled [10], with the only possible exception of $(h, n) = (13, 6)$. In this paper, we investigate the existence of $(3, 1, 2)$–HCOLS using a similar approach. As mentioned in [5], the nonexistence of a $(3, 1, 2)$–COILS(4) has made the investigation of $(3, 1, 2)$–HCOLS(h^n) considerably more difficult than that carried out for the other conjugates. Despite this difficulty, we are still able to provide an almost conclusive result to the existence of $(3, 1, 2)$–HCOLS(h^n).

Note that an idempotent $(3, 1, 2)$-COLS of order v can be written as a $(3, 1, 2)$–HCOLS(1^v). An incomplete idempotent $(3, 1, 2)$-COLS of order v with a hole of size k, denoted by $(3, 1, 2)$–ICOILS(v, k), exists if and only if a $(3, 1, 2)$–HCOLS($1^{v-k} k^1$) exists. The previous results concerning the existence of $(3, 1, 2)$–HCOLS(h^n) are summarized in the following theorem of the survey paper [5]:

Theorem 1.1 ([5]) *There exists a* $(3, 1, 2)$*–HCOLS*(h^n) *if and only if* $h \geq 1$ *and* $n \geq 4$*, except* $(n, h) = (6, 1)$*, and except possibly*

1. *when* $n \in \{10, 12, 14, 15\}$ *and* $h = 1$*;*
2. *when* $n \in \{4, 6\}$ *and* h *is odd;*
3. *when* $n = 15$ *and* $h \equiv 1$ *or* $5 \pmod 6$*;*
4. *when* $(n, h) \in \{(6, 2), (6, 6), (6, 10), (6, 14), (6, 18), (6, 22), (6, 26), (6, 30), (6, 34), (6, 38), (6, 42), (8, 2), (9, 2), (10, 2), (10, 3), (10, 34), (10, 38), (12, 2), (12, 3), (12, 4), (12, 14), (14, 3), (30, 2)\}$.

In this paper, we remove most possible exceptions in Theorem 1.1 and thus obtain the following one:

Theorem 1.2 *There exists a* $(3,1,2)$*-HCOLS*(h^n) *if and only if* $h \geq 1$ *and* $n \geq 4$*, except* $(n,h) = (6,1)$*, and except possibly* $(n,h) = (10,1)$ *and* $(4, 2t+1)$ *for* $t \geq 1$*.*

The previous result regarding the existence of $(3,1,2)$–ICOILS(v,k) is summarized in the following theorem.

Theorem 1.3 ([5]) *For any integer* $v \geq 1$*, a* $(3,1,2)$*-ICOILS*(v,k) *exists if* $v \geq (10/3)k + 68$*. For* $2 \leq k \leq 5$*, a* $(3,1,2)$*-ICOILS*(v,k) *exists if* $v \geq 3k+1$ *except possibly when*

$k = 2$, $v \in \{8, 12, 14, 16, 17, 18, 20, 21\}$;
$k = 3$, $v \in \{15, 17, 19, 20, 21, 23, 24, 25, 27, 28, 29, 30\}$;
$k = 4$, $v \in \{18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32, 34, 35, 36, 37, 38\}$;
$k = 5$, $v \in \{23, 24, 26, 28, 30, 31, 32, 34, 38, 39\}$.

We are able to solve all the open cases when $2 \leq k \leq 5$ except $k = 4$ and $v \in \{35, 38\}$. That is, we have the following:

Theorem 1.4 *For* $2 \leq k \leq 5$*, a* $(3,1,2)$*-ICOILS*(v,k) *exists if and only if* $v \geq 3k+1$*, except possibly when* $k = 4$ *and* $v \in \{35, 38\}$*.*

2 Construction Techniques

The construction techniques that we used are conventional (such as the cyclic group construction, the "fill-in-holes" construction and the group-divisible designs) and can be found in the survey paper [5]. The use of these techniques is similar to that of [9] where the existence of $(2,1,3)$-HCOLS(2^n3^1) is established.

2.1 Direct Constructions

Our most direct constructions use a starter-adder type construction, called the cyclic group construction, which constructs a $(3,1,2)$–HCOLS of type (h^nk^1) from its first row and first column using an Abelian group of order hn. In [5], this technique is described using the Abelian group Z_{hn}. In [10], the construction using an arbitrary Abelian group of order hn is presented:
The Cyclic Group Construction Let $(G,+)$ be an Abelian group of order hn and H a subgroup of order h. In general, we assume $G = \{0, 1, ..., hn-1\}$ and $H = \{i.n : 0 \leq i < h\}$. Let $X = \{x_1, ..., x_k\} = \{hn, ..., hn+k-1\}$ and $S = G \cup X$.

Let $\mathbf{e} \in (S \cup \{\emptyset\})^{hn}$ be a vector of length hn, where $\emptyset$ denotes that a cell is empty. Let $\mathbf{f}, \mathbf{g} \in G^k$ be two vectors of length k. A $(hn+k) \times (hn+k)$ square L can be constructed from $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{g}$, as follows: Let $(i \cdot j)$ denote the entry in the cell (i, j) of L. The first row is filled by the two vectors $\mathbf{e}$ and $\mathbf{f}$, i.e.,

$$\mathbf{e} + \mathbf{f} = (0 \cdot 0, ..., 0 \cdot (hn-1), 0 \cdot hn, ..., 0 \cdot (hn+k-1)),$$

and the last k elements of the first column are filled by $\mathbf{g}$, i.e.,

$$\mathbf{g} = (hn \cdot 0, ..., (hn+k-1) \cdot 0).$$

The entire L is constructed from $\mathbf{e}, \mathbf{f}$ and $\mathbf{g}$ as follows:

1. For $s \in G$ and $t \in G$,

$$s \cdot (s+t) = \begin{cases} (0 \cdot t) + s & \textit{if } (0 \cdot t) \in G \\ (0 \cdot t) & \textit{otherwise.} \end{cases}$$

2. For $s \in G$, $t \in X$, $s \cdot t = (0 \cdot t) + s$.

3. For $s \in X$, $t \in G$, $s \cdot t = (s \cdot 0) + t$.

Note that $+$ is the one in the Abelian group $(G, +)$.

There are obviously conditions that the vectors $\mathbf{e}, \mathbf{f}$ and g must satisfy in order to produce a (3,1,2)–HCOLS($h^n k^1$) and they are given in the following lemma.

Lemma 2.1 *Let L be a square generated by the cyclic construction using the Abelian group $(G, +)$. L is a $(3,1,2)$–HCOLS($h^n k^1$) if and only if*

1. *for any $x \in G$, $0 \cdot x = \emptyset$ if and only if $x \in H$;*

2. *for any $x \notin H$, $0 \cdot x \notin H$, and either $0 \cdot x \in G$ or $x \cdot 0 \in G$;*

3. *the following difference conditions hold:*

$$\begin{aligned} & \{(0 \cdot x) + -(0 \cdot_{312} x) \mid 0 \cdot x \in G, 0 \cdot_{312} x \in G, x \in S \setminus H\} \cup \\ & \{(x \cdot 0) + -(x \cdot_{312} 0) \mid x \in X\} \\ = \;& G \setminus H, \end{aligned}$$

where $-(x)$ is the inverse of x in the Abelian group $(G, +)$.

Example 2.2 *Let $G = Z_6, H = \{0,3\}, X = \{x, y\}$. $\mathbf{e} = (\emptyset, 2, x, \emptyset, y, 4)$, $\mathbf{f} = (1,5)$ and $\mathbf{g} = (4,2)$. The square constructed by the cyclic group construction is given below, where the elements of* $\mathbf{e}$ *are underlined and those of* $\mathbf{f}$ *and* $\mathbf{g}$ *are in bold type.*

·	0	1	2	3	4	5	x	y
0		2	x		y	4	1	5
1	5		3	x		y	2	0
2	y	0		4	x		3	1
3		y	1		5	x	4	2
4	x		y	2		0	5	3
5	1	x		y	3		0	4
x	4	5	0	1	2	3		
y	2	3	4	5	0	1		

It is easy to check that all the conditions are satisfied for the square. □

Note that the hole of size k of L in the above lemma is indexed by $X \times X$, and the n holes of size h are indexed by $(g+H) \times (g+H)$, where $g+H$ runs over all cosets of H in G.

In the following, unless specified explicitly, we always use the Abelian group Z_{hn} in the cyclic group construction to construct $(3,1,2)$–HCOLS$(h^n k^1)$.

Lemma 2.3 (Product construction) *Suppose there exists a $(3,1,2)$–HCOLS $(h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k})$, then there exists a $(3,1,2)$–HCOLS$((mh_1)^{n_1}(mh_2)^{n_2} \cdots (mh_k)^{n_k})$, where $m \neq 2, 6$.*

Lemma 2.4 (Filling in holes)

(1) *Suppose there exists a $(3,1,2)$–HCOLS of type $\{s_i : 1 \leq i \leq n\}$. Let $a \geq 0$ be an integer. For each i, $1 \leq i \leq n-1$, suppose there exists a $(3,1,2)$–HCOLS of type $\{s_{ij} : 1 \leq j \leq k(i)\} \cup \{a\}$, where $s_i = \sum_{j=1}^{k(i)} s_{ij}$. Then there exists a $(3,1,2)$–HCOLS of type $\{s_{ij} : 1 \leq j \leq k(i), 1 \leq i \leq n-1\} \cup \{a + s_n\}$.*

(2) *Suppose there exists a $(3,1,2)$–HCOLS of type $\{s_i : 1 \leq i \leq n\}$. Suppose there is also a $(3,1,2)$–HCOLS of type $\{t_j : 1 \leq j \leq k)\}$, where $s_n = \sum_{j=1}^{k} t_j$. Then there exists a $(3,1,2)$–HCOLS of type $\{s_i : 1 \leq i \leq n-1\} \cup \{t_j : 1 \leq j \leq k\}$.*

2.2 Stein's Third Law

Let $(S, \cdot)$ be a quasigroup. It is well-known that the Stein's third law $(y \cdot x) \cdot (x \cdot y) = x$ is conjugate-equivalent to the identity $(y \cdot (x \cdot y)) \cdot x = y$, using $(1,3,2)$–conjugate operation. This means that the $(1,3,2)$–conjugate of a quasigroup satisfying Stein's third law satisfies the identity $(y \cdot (x \cdot y)) \cdot x = y$. It is not difficult to check that an idempotent quasigroup satisfying $(y \cdot (x \cdot y)) \cdot x = y$ is orthogonal to its $(2,3,1)$–conjugate. As mentioned in Section 1, the existence of $(2,3,1)$–COILS implies the existence of $(3,1,2)$–COILS and vice versa. So, the existence of a quasigroup satisfying Stein's third law implies the existence of $(3,1,2)$–COILS of the same type.

Let us remark that an idempotent quasigroup $(Q,\cdot)$ of order v satisfying the identity $(y\cdot(x\cdot y))\cdot x = y$ is equivalent to a $(v,4,1)$–perfect Mendelsohn design, where the cyclically ordered blocks of size four are given by $\{(x,y,x\cdot y, y\cdot(x\cdot y)) : x, yx \neq y\}$ [2, 3, 4, 5]. That is, let v, k be positive integers. A $(v,k,1)$–*Mendelsohn design*, briefly $(v,k,1)$–MD, is a pair $(X,\mathcal{B})$, where X is a v–set (of *points*) and $\mathcal{B}$ is a collection of cyclically ordered k–subsets of X (called *blocks*) such that every ordered pair of points of X are consecutive in exactly one block of $\mathcal{B}$, where a cyclically ordered block $(a_1, a_2, \cdots, a_k)$ means $a_1 \le a_2 \le \cdots \le a_k \le a_1$. If for all $t = 1, 2, \cdots, k-1$, every ordered pair of points of X are t–apart in exactly one block of $\mathcal{B}$, then the $(v,k,1)$–MD is called *perfect* and is denoted by $(v,k,1)$–PMD.

In [4], the following result is essentially established using $(v,4,1)$–HPMDs and Stein's third law:

Lemma 2.5 *A $(3,1,2)$–HCOLS(h^n) exists if and only if $n \ge 4$ and $n(n-1)h^2 \equiv 0 \pmod 4$, except $(n,h) = (4,1),(4,2),(8,1)$ and except possibly $(n,h) = (4, 2t+1)$ for $t \ge 1$.*

Note that this lemma removes all the cases in Theorem 1.1 where h is even. It also removes the cases where $(n,h) = (12,1),(12,3)$.

Using the same technique, the following result is established in [3].

Lemma 2.6 *A $(3,1,2)$–HCOLS($2^n 3^1$) exists if and only if $n \ge 4$.*

2.3 Recursive Constructions

The weighting construction uses group divisible designs [7, 6, 9]. A *group divisible design* (GDD) is a triple $(X,\mathcal{G},\mathcal{B})$, which satisfies the following properties:

1. $\mathcal{G}$ is a partition of X into subsets called *groups*.
2. $\mathcal{B}$ is a set of subsets of X, called *blocks*, such that a group and a block contain at most one common point.
3. Every pair of points from distinct groups occurs in a unique block.

The following construction is used in [7]; see also [5, 9].

Lemma 2.7 *(Weighting) Let $(X,\mathcal{G},\mathcal{B})$ be a GDD and let $w : X \to Z^+ \cup \{0\}$ be a weighting. Suppose that there exists a $(3,1,2)$–HCOLS of type $w(B)$ for every $B \in \mathcal{B}$. Then there exists a $(3,1,2)$–HCOLS of type $\{\sum_{x\in G} w(x) : G \in \mathcal{G}\}$.*

For our recursive constructions, we will make use of transversal designs. A *transversal design* TD(k,n) is a GDD with kn points, k groups of size n, and n^2 blocks of size k. It is well known that a TD(k,n) is equivalent to $k-2$ MOLS of order n.

Lemma 2.8 ([1]) *There exists a TD$(6,m)$ for all $m \geq 5$, where $m \notin \{6,10,14,18,22\}$.*

3 $(3,1,2)$–HCOLS

3.1 $(3,1,2)$–HCOLS(h^n) for $h \leq 4$

Lemma 3.1 *There exists a $(3,1,2)$-HCOLS(1^n) for $n = 14, 15$.*

Proof: It is sufficient to give the vectors **e**, **f** and **g**, as shown below. □

type	e	f	g
1^{14}	($\emptyset$ 12 9 x 10 6 8 4 11 5 1 3 7)	(2)	(12)
1^{15}	($\emptyset$ 3 9 6 2 14 1 12 7 13 11 8 5 4 10)	( )	( )

Lemma 3.2 *There exists a $(3,1,2)$-HCOLS(3^n) for $n \in \{6,10,14\}$.*

Proof: For $n = 14$, we obtain it by the product construction from $(3,1,2)$–HCOLS(1^{14}) and $(3,1,2)$–COLS(3). For the other two cases, we give the vectors **e**, **f** and **g**, as shown in Table 1, which satisfy Lemma 2.1. For 3^{10}, the Abelian group $Z_3 \times Z_3 \times Z_3$ is used instead of Z_{27}. Each element $\langle i,j,k \rangle$ in $Z_3 \times Z_3 \times Z_3$ is encoded by $9i + 3j + k$. □

3.2 $(3,1,2)$–HCOLS(h^n) for $n = 6, 15$

The remaining outstanding cases for $n = 6$ are when h is odd.

Lemma 3.3 *There exists a $(3,1,2)$-HCOLS(h^6) for $h \in \{3,5,7,9\}$.*

Proof: The case of $h = 3$ is covered by Lemma 3.2. For $h = 9$, we obtain it by the product construction from $(3,1,2)$–HCOLS(3^6) in Lemma 3.2 and $(3,1,2)$–COLS(3).. We list the vectors **e**, **f** and **g** of the cyclic group construction for the other two cases in Table 1. □

Lemma 3.4 *(a) If there exists a TD$(6,m)$, then there exists a $(3,1,2)$-HCOLS$(2m+1)^6$.*

(b) If there exists a TD$(6,m)$ and $m \neq 5$, then there exists a $(3,1,2)$-HCOLS$(2m-1)^6$.

type	e, f, g
(5^6)	($\emptyset$ 23 x_4 19 x_1 $\emptyset$ 13 11 22 21 $\emptyset$ 17 6 14 12 $\emptyset$ x_2 9 4 3 $\emptyset$ x_3 18 x_5 2), (1 7 8 16 24), (18 2 13 8 24)
(7^6)	($\emptyset$ 32 21 1 7 $\emptyset$ 8 18 2 x_4 $\emptyset$ x_6 11 27 3 $\emptyset$ 22 x_3 26 6 $\emptyset$ 13 34 9 x_5 $\emptyset$ 17 x_2 x_7 12 $\emptyset$ 24 x_1 14 16), (4 19 23 28 29 31 33), (9 23 1 13 7 32 4)

type	e	f	g
3^6	($\emptyset$ 12 9 x_1 8 $\emptyset$ 7 4 11 2 $\emptyset$ x_2 3 x_3 1)	(13 6 14)	(9 13 14)
3^{10}	($\emptyset$ 8 x_1 17 6 2 10 x_2 7 $\emptyset$ 3 1 19 25 21 4 22 13 $\emptyset$ 14 16 24 x_3 12 5 20 11)	(23 15 26)	(24 11 1)

Table 1. Vectors for some $(3,1,2)$–HCOLS(h^6).

Proof: For (a), we select a block B of the TD$(6,m)$ and give every point of B weight three. We then give all the remaining points of the TD weight two and apply the Weighting Construction, using as input designs $(3,1,2)$–HCOLS of the types 2^5, 2^6, $(2^5 3^1)$ and 3^6, to obtain the desired $(3,1,2)$-HCOLS of type $(2m+1)^6$. For the proof of (b), we take a slightly different approach. Here we select two disjoint blocks B and B' of the TD$(6,m)$, which is possible since $m \neq 5$. We give each point of the block B weight zero and give each point of the block B' weight three. We then give all of the remaining points of the TD weight two and apply the Weighting Construction to get a $(3,1,2)$–HCOLS$(2m-1)^6$, using $(3,1,2)$–HCOLS of types 2^5, 2^6, $(2^4 3^1)$, $(2^5 3^1)$ and 3^6. This completes the proof of the lemma. □

We are now in a position to prove the following:

Lemma 3.5 *There exists a* $(3,1,2)$*–HCOLS*(h^6) *for all odd* $h \geq 3$.

Proof: For $h \leq 9$, the proof is given in Lemma 3.3. For all odd $h \geq 11$, we apply Lemmas 2.8 and 3.4 with the appropriate values of m to get the desired results. □

For $n = 15$, the remaining outstanding cases are when $h \equiv 1$ or $5 \ (mod\ 6)$.

Lemma 3.6 *There exists a* $(3,1,2)$*–HCOLS*(h^{15}) *for all odd* $h \geq 3$.

Proof: The lemma is easily established by the product construction, because of the existence of $(3,1,2)$–HCOLS(1^{15}) and $(3,1,2)$–COLS(h) for all odd $h \geq 3$. □

Combining Theorem 1.1 with Lemmas 2.5, 3.3–3.6, we have proved

Theorem 3.7 *There exists a* $(3,1,2)$*–HCOLS*(h^n) *if and only if* $h \geq 1$ *and* $n \geq 4$, *except* $(n,h) = (6,1)$, *and except possibly* $(n,h) = (10,1)$ *and* $(4, 2t+1)$ *for* $t \geq 1$.

4 (3, 1, 2)–ICOILS

Recall that (3, 1, 2)–ICOILS(v, k) denotes an idempotent (3, 1, 2)–COILS of order v with a hole of size k and is equivalent to a (3, 1, 2)–HCOLS($1^{v-k}k^1$). The nonexistence of (3, 1, 2)–ICOILS(8, 2) was confirmed by an exhaustive computer search [8].

By the property of Stein's third law, the following lemma can be easily established using the results provided in [2, 11]:

Lemma 4.1 *There exists a* (3, 1, 2)*–ICOILS*(v, k) *for*

$$\begin{array}{ll} k=2, & v \in \{14, 18\}, \\ k=3, & v \in \{15, 19, 23, 27\}, \\ k=4, & v \in \{20, 24, 25, 28, 32, 34, 36, 37\}, \\ k=5, & v \in \{24, 28, 32, 33\}. \end{array}$$

Using the cyclic group construction, we are able to prove the following lemma.

Lemma 4.2 *There exists a* (3, 1, 2)*–ICOILS*(v, k) *for*

$$\begin{array}{ll} k=2, & v \in \{12, 16, 17, 20, 21\}, \\ k=3, & v \in \{17, 20, 21, 24, 25, 29\}, \\ k=4, & v \in \{18, 19, 22, 23, 26, 27, 30, 31\}, \\ k=5, & v \in \{23, 26\}. \end{array}$$

Proof: We list in Tables 2 and 3 the vectors **e**, **f** and **g** for these cases. □

Using the fill-in-hole construction, we can establish the following lemma.

Lemma 4.3 *There exists a* (3, 1, 2)*–ICOILS*(v, k) *for* $(v, k) = (28, 3), (34, 4)$ *and*

$$k = 5, v \in \{30, 31, 34, 38, 39\}.$$

Proof: For $(v, k) = (28, 3)$, we fill (3, 1, 2)–COILS(5) into (3, 1, 2)–HCOLS ($5^5 3^1$).

For $(v, k) = (34, 4)$, we fill (3, 1, 2)–COILS(5) into (3, 1, 2)–HCOLS($5^6 4^1$).

We obtain (3, 1, 2)–ICOILS(31, 5) from (3, 1, 2)–HCOLS($4^6 6^1$) by adjoining one point to it and then filling it with (3, 1, 2)–COILS(5) and (3, 1, 2)–ICOILS(7, 1). Similarly, we obtain (3, 1, 2)–ICOILS(30, 5) from (3, 1, 2)–HCOLS(5^6); (3, 1, 2)–ICOILS(34, 5) from (3, 1, 2)–HCOLS($5^5 9^1$); (3, 1, 2)–ICOILS(38, 5) from (3, 1, 2)–HCOLS($5^6 8^1$); and (3, 1, 2)–ICOILS(39, 5) from (3, 1, 2)–HCOLS($5^6 9^1$).

The vectors **e**, **f** and **g** of the required designs for the fill-in-hole constructions are listed in Table 4. □

type	e	f	g
(12, 2)	(0 x_2 3 6 1 10 x_1 2 11 5)	(4 7)	(4 9)
(16, 2)	(0 8 x_1 5 13 10 12 11 9 6 4 7 x_2 2)	(1 3)	(12 13)
(17, 2)	(0 7 12 6 9 13 x_1 1 4 11 2 8 x_2 14 3)	(5 10)	(13 14)
(20, 2)	(0 16 9 4 8 10 17 3 x_1 15 13 2 x_2 7 6 5 11 1)	(14 12)	(16 17)
(21, 2)	(0 x_1 12 10 9 14 8 15 x_2 1 7 5 16 6 17 11 3 18 13)	(4 2)	(17 18)
(17, 3)	(0 7 12 11 x_3 x_2 8 10 9 13 1 6 5 x_1)	(2 3 4)	(11 12 13)
(20, 3)	(0 8 x_1 13 15 10 9 11 x_2 4 2 x_3 1 14 16 6 12)	(7 5 3)	(14 15 16)
(21, 3)	(0 5 4 11 7 15 17 x_1 13 3 x_2 6 1 9 2 16 x_3 8)	(10 14 12)	(15 16 17)
(24, 3)	(0 4 11 x_1 6 19 13 20 12 x_2 18 7 1 8 5 9 17 2 x_3 3 10)	(15 16 14)	(18 19 20)
(25, 3)	(0 18 11 4 20 17 16 x_1 15 13 x_2 14 5 x_3 19 6 12 1 10 21 9 7)	(2 8 3)	(19 20 21)
(29, 3)	(0 x_1 13 25 21 10 x_3 19 23 4 17 1 15 7 22 2 8 18 20 3 24 14 x_2 6 12 5)	(9 11 16)	(23 24 25)

Table 2. Vectors for some (3, 1, 2)–ICOILS(v, 2) and (3, 1, 2)–ICOILS(v, 3).

type	e	f	g
(18, 4)	(0 x_2 x_1 5 11 x_3 10 12 9 x_4 13 3 6 8)	(1 2 4 7)	(10 11 12 13)
(19, 4)	(0 8 x_1 17 14 13 x_2 12 x_3 2 11 1 16 15 6 18 10 x_4)	(5 3 9 7 4)	(15 16 17 18)
(22, 4)	(0 10 x_1 4 7 15 17 x_2 14 11 5 16 x_3 3 8 x_4 2 6)	(1 13 9 12)	(14 15 16 17)
(23, 4)	(0 8 x_1 17 14 13 x_2 12 x_3 2 11 1 16 15 6 18 10 x_4 5)	(3 9 7 4)	(15 16 17 18)
(26, 4)	(0 2 13 15 17 10 16 9 12 1 x_2 14 x_1 8 7 x_3 3 11 x_4 4 6 5)	(18 19 20 21)	(18 19 20 21)
(27, 4)	(0 19 15 6 11 x_4 10 13 x_3 18 22 16 20 x_2 8 17 7 x_1 5 12 21 9 14)	(1 2 3 4)	(19 20 21 22)
(30, 4)	(0 10 9 16 12 11 x_2 8 19 1 x_1 4 15 17 2 5 21 6 20 3 14 7 13 18 x_4 x_3)	(22 23 24 25)	(22 23 24 25)
(31, 4)	(0 12 3 6 22 21 18 x_1 13 17 20 x_2 16 15 1 x_3 9 7 10 5 14 x_4 2 11 19 4 8)	(23 24 25 26)	(23 24 25 26)
(23, 5)	(0 13 11 x_1 10 x_2 17 x_3 9 x_4 14 16 4 15 3 5 x_5 2)	(8 6 7 1 12)	(13 14 15 16 17)
(26, 5)	(0 5 x_1 13 12 8 18 20 9 x_2 4 x_3 19 6 16 3 1 x_4 2 x_5 10)	(11 7 15 17 14)	(16 17 18 19 20)

Table 3. Vectors for some (3, 1, 2)–ICOILS(v, 4) and (3, 1, 2)–ICOILS(v, 5).

type	e, f, g
$(5^5 3^1)$	(∅ 23 x_1 x_2 6 ∅ 19 14 16 21 ∅ 7 1 x_3 17 ∅ 9 3 12 11 ∅ 22 13 4 8), (18 2 24), (23 4 24)
$(5^6 4^1)$	(∅ x_2 25 4 x_1 2 ∅ x_4 23 x_3 26 1 ∅ 20 13 19 27 15 ∅ 10 29 17 14 7 ∅ 28 21 16 11 9), (3 5 8 22), (8 2 5 17)
$(4^6 6^1)$	(∅ 15 9 7 x_4 x_3 ∅ 4 x_5 x_2 20 22 ∅ 5 17 8 x_1 19 ∅ 10 16 2 x_6 21), (1 3 11 13 14 23), (9 13 8 1 19 23)
$(5^5 9^1)$	(∅ 3 x_8 x_5 11 ∅ 19 x_7 x_1 x_2 ∅ x_3 18 14 x_4 ∅ 8 x_6 2 21 ∅ 17 16 x_9 22), (1 3 4 6 7 9 12 13 24), (16 11 14 18 12 3 4 8 24)
$(5^6 8^1)$	(∅ 28 23 17 20 x_4 ∅ x_3 x_2 x_5 25 15 ∅ 21 13 10 x_6 22 ∅ 2 x_1 1 14 26 ∅ 27 19 16 x_7 x_8), (3 4 5 7 8 9 11 29), (7 11 20 9 1 17 26 28)
$(5^6 9^1)$	(∅ 28 x_3 x_8 17 x_1 ∅ 16 10 x_5 x_4 22 ∅ x_7 x_6 20 x_2 25 ∅ 14 11 1 23 26 ∅ 21 19 13 27 x_9), (2 3 4 5 7 8 9 15 29), (14 19 4 20 7 15 17 22 28)

Table 4. Vectors for some fill-in-hole constructions.

Combining Theorem 1.3 with Lemmas 4.1–4.3, we have essentially proved the following result.

Theorem 4.4 *For $2 \le k \le 5$, a $(3,1,2)$-ICOILS(v,k) exists if and only if $v \ge 3k+1$, except possibly when $k = 4$ and $v \in \{35, 38\}$.*

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