Some New Difference Triangle Sets

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ABSTRACT. In [5] Kløve gave tables of the best bounds known on the size of optimal difference triangle sets. In this note we give examples of difference triangle sets found by computer search which improve on the upper bounds in [5]. In four cases these examples are proved to be optimal.

In [5] Kløve defined an (I, J) difference triangle set, Δ, as a set of integers $\{a_{ij} | 1 \leq i \leq I, 0 \leq j \leq J\}$ such that all the differences $a_{ij} - a_{ik}$, $1 \leq i \leq I$, $0 \leq k \neq j \leq J$ are distinct. Let $m = m(\Delta)$ be the maximum difference. Kløve defined $M(I, J) = \min\{m(\Delta) | \Delta \text{ is an } (I, J) \text{ difference triangle set}\}$ and gave tables of bounds for M. In this note we present difference triangle sets found by computer search which improve these bounds.

Table 1 shows the improved upper bounds. The bounds marked with an $*$ are exact. Examples achieving these bounds are given in Table 2. In [5] Kløve stated $M(3, 6) = 77$, $M(4, 5) = 65$ and $M(5, 5) = 83$. These values apparently should have been upper bounds.

The computer program used was a modified version of the author's Golomb ruler finding program ([8], see also [9]). Golomb rulers are (I, J) difference triangle sets with $I = 1$. The Golomb ruler search program implements a straightforward depth first backtrack search. It attempts to build up a Golomb ruler with k marks and length B or less by picking $0 = a_0 < a_1 < a_2 < \cdots a_k \leq B$ in order (a_j being picked at level j of the search tree). At each node of the search tree the program keeps track of which differences have been used (i.e. are formed by pairs of the elements which have already been picked) and of which integers can be added to the current partial ruler without violating the distinct difference condition. The sons of a node at level j are formed by setting a_{j+1} to each of the elements in the level j eligibility list in turn. The search tree is pruned (i.e. the program backtracks) when too few integers remain eligible to allow completion of the ruler, when not enough small differences remain unused to allow completion of the ruler or when allowed by a symmetry condition. (Since the mirror image of a Golomb ruler is also a Golomb ruler we may require for example that $a_{k/2} < B/2$.)

We modified the program to search for (I, J) difference triangle sets with $I > 1$ as follows. Such difference triangle sets may be thought of as sets of I Golomb rulers of $J+1$ marks each, such that no difference occurs in more than one ruler. We may assume $a_{10} = a_{20} = \cdots = a_{I0} = 0$. The modified program attempts to build a (I, J) difference triangle set, Δ, with $m(\Delta) \leq B$ (for some fixed bound B) by picking $a_{11}, a_{12}, \ldots, a_{1J}$, $a_{21}, a_{22}, \ldots, a_{2J}$, $\ldots$, $a_{I1}, a_{I2}, \ldots, a_{IJ}$ in order ($0 < a_{h_1} < a_{h_2} < \cdots < a_{h_J} \leq B$, $1 \leq h \leq I$). While building up the hth ruler the program, at each node of the search tree, keeps track of which differences have been used (in the current and earlier rulers) and of which integers can be added to the current partial ruler without violating the distinct difference condition. The program backtracks when too few integers remain eligible to allow completion of the current ruler, when not enough small differences remain unused to allow completion of the current and any remaining rulers or when allowed by symmetry. Since we may permute the order of the rulers in a difference triangle set as well as reversing rulers we may require $a_{11} > a_{21} > \cdots > a_{I1}$ as well as $a_{hJ/2} < B/2$ (or if J is odd $a_{hJ/2} + a_{hJ/2+1} < B$) $1 \leq h \leq I$.

In most cases it was not feasible to perform a complete search. To handle these cases we modified the program to allow it to search a fraction of the complete search tree by arbitrarily pruning many nodes in the search tree. The complete program amounts to about 150 lines of VS Fortran code. The program was run on an IBM 3090 mainframe. Each partial search was run for at least 5000 seconds of cpu time. For the four cases where a complete search was feasible we give all the examples which achieve the bound. Otherwise we give a single example even if several were found.

This work was done in 1990-1991. Since then some other work has improved on Kløve's bounds ([2], [3] and [1]). However, except for $M(2,7) = 70$ the results in Table 1 remain the best known. Chee and Colbourn [1] independently showed by computer search that $M(2,7) = 70$. Their program ran for a week on a network of 30 machines (and did not show uniqueness). The author's program ran for about 3000 seconds (with $B = 70$) on a 3090.

Table 1

$J\backslash I$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
4								93		116	127	138	150	160
5			*64	81	100	118	137	157	170	191	209			
6		*72	98	125	153	178	204	232	264					
7	*70	104	144	182	218	256	302	334						
8	*94	145	199	251	302	353	405							
9	126	196	269	329	405									
10		254												

Table 2

I	J	Difference Triangle Set									
2	7	0	6	11	18	28	37	62	70		
		0	1	4	24	40	54	67	69		
2	8	0	10	15	29	45	51	77	85	94	
		0	1	3	24	28	61	74	81	92	
2	8	0	14	16	22	34	43	62	79	94	
		0	4	7	37	42	68	81	91	92	
2	9	0	7	10	22	52	68	99	100	118	123
		0	4	17	38	44	73	87	98	124	126
3	6	0	5	19	21	50	57	70			
		0	4	10	32	43	58	66			
		0	3	12	30	47	71	72			
3	6	0	7	22	30	62	66	71			
		0	3	19	20	48	54	72			
		0	2	13	27	39	60	70			
3	6	0	10	15	27	61	70	72			
		0	7	13	33	49	63	71			
		0	1	25	29	48	66	69			
3	6	0	13	16	31	56	67	68			
		0	7	21	26	48	65	71			
		0	2	10	30	34	63	72			
3	7	0	17	31	36	57	65	102	103		
		0	4	16	27	55	79	97	104		
		0	2	22	32	35	76	82	91		
3	8	0	6	40	50	54	110	111	126	135	
		0	3	21	38	66	103	115	134	145	
		0	2	22	29	55	91	130	138	143	

3	9	0	3	20	26	86	95	117	162	173	194	
		0	2	36	55	63	120	134	167	182	192	
		0	1	39	44	74	90	103	144	184	196	
3	10	0	16	43	90	94	116	154	231	236	242	251
		0	8	21	31	79	107	170	200	202	237	254
		0	3	42	56	75	125	143	187	228	252	253
4	5	0	12	13	31	58	60					
		0	8	24	34	57	62					
		0	4	21	41	56	63					
		0	3	9	39	53	64					
4	6	0	5	6	25	59	72	89				
		0	4	12	48	77	91	98				
		0	3	26	35	63	81	96				
		0	2	24	40	51	82	92				
4	7	0	21	43	59	69	125	143	144			
		0	6	30	50	77	102	111	142			
		0	5	7	46	60	83	95	140			
		0	4	17	32	68	130	138	141			
4	8	0	21	62	73	77	149	159	193	199		
		0	9	23	28	88	112	155	180	192		
		0	7	20	38	46	101	167	194	196		
		0	1	17	49	71	107	140	182	185		
4	9	0	20	75	76	118	140	157	207	234	269	
		0	11	83	95	113	119	219	222	247	263	
		0	8	10	48	69	101	155	186	243	266	
		0	5	19	34	71	97	201	246	250	259	
5	5	0	16	23	43	68	81					
		0	10	31	39	73	79					
		0	9	14	50	76	80					
		0	3	15	47	64	75					
		0	1	19	54	56	78					
5	6	0	18	28	48	92	111	118				
		0	14	55	57	72	108	124				
		0	13	38	42	88	119	120				
		0	9	49	54	114	122	125				
		0	6	27	39	62	86	123				
5	7	0	24	43	69	103	149	179	181			
		0	23	65	81	82	120	170	182			
		0	8	35	48	92	156	166	177			
		0	5	41	61	114	128	165	180			
		0	3	9	31	99	146	153	171			

5	8	0	26	36	74	96	125	175	220	244	
		0	12	42	88	109	144	171	215	249	
		0	8	25	31	111	176	229	231	242	
		0	7	39	54	82	167	216	219	235	
		0	1	59	64	73	157	214	247	251	
5	9	0	32	82	119	152	163	289	306	310	324
		0	30	71	78	129	174	175	267	296	323
		0	22	42	61	116	150	202	262	275	329
		0	8	31	57	140	164	254	279	317	326
		0	3	5	69	105	193	203	288	304	316
6	5	0	14	35	54	92	96				
		0	11	26	48	71	99				
		0	9	29	41	84	85				
		0	8	25	58	89	94				
		0	7	13	66	90	100				
		0	2	18	67	70	97				
6	6	0	18	45	59	123	146	153			
		0	10	22	70	110	149	152			
		0	5	21	34	102	119	145			
		0	4	50	61	117	126	141			
		0	2	49	55	86	118	138			
		0	1	36	74	107	132	151			
6	7	0	60	61	89	111	163	197	216		
		0	30	71	73	140	150	212	218		
		0	27	38	58	134	200	204	217		
		0	15	64	97	100	187	195	213		
		0	9	32	44	124	161	208	215		
		0	5	21	45	133	175	189	214		
6	8	0	38	46	90	110	222	250	287	300	
		0	16	45	47	134	174	215	283	298	
		0	14	71	75	145	196	280	289	299	
		0	7	32	33	137	179	233	239	302	
		0	5	27	106	141	194	261	279	291	
		0	3	39	95	119	181	247	258	281	
6	9	0	28	49	83	102	144	269	286	336	361
		0	26	70	103	190	201	316	353	375	398
		0	10	46	62	64	199	205	212	335	403
		0	9	40	119	133	233	265	306	382	387
		0	8	38	86	129	176	247	348	404	405
		0	4	24	84	113	218	284	380	383	395

7	5	0	23	28	65	85	115			
		0	13	32	66	91	113			
		0	11	46	60	104	114			
		0	8	39	75	102	118			
		0	6	21	76	105	117			
		0	4	7	52	90	116			
		0	1	18	74	98	107			
7	6	0	30	36	84	106	172	173		
		0	27	47	82	108	160	176		
		0	24	58	77	150	163	178		
		0	18	39	56	135	146	177		
		0	10	50	75	124	168	175		
		0	8	12	41	110	153	156		
		0	5	14	37	109	169	171		
7	7	0	37	71	82	118	207	242	256	
		0	23	77	108	127	156	217	250	
		0	16	41	96	111	203	209	255	
		0	9	51	68	73	199	220	252	
		0	4	12	30	105	132	208	248	
		0	3	63	87	154	216	226	254	
		0	2	99	112	119	177	233	234	
7	8	0	31	68	88	147	217	279	302	348
		0	28	49	78	130	236	281	313	335
		0	27	67	92	122	233	237	319	337
		0	24	80	121	174	265	301	339	352
		0	11	71	146	154	255	327	344	353
		0	10	15	58	148	182	305	308	324
		0	2	35	119	163	224	331	332	338
8	5	0	28	48	67	111	129			
		0	26	34	61	99	136			
		0	17	24	70	121	124			
		0	6	31	80	95	125			
		0	5	21	76	108	117			
		0	4	47	60	126	137			
		0	2	42	52	120	134			
		0	1	23	59	116	128			

8	6	0	33	58	87	155	191	202		
		0	26	64	85	150	167	201		
		0	24	37	90	118	163	190		
		0	18	60	95	156	188	196		
		0	14	30	71	159	179	198		
		0	7	22	70	113	114	193		
		0	5	9	83	192	194	204		
		0	3	49	55	105	180	203		
8	7	0	47	60	68	142	257	259	294	
		0	26	66	123	156	204	287	297	
		0	20	69	85	139	238	296	302	
		0	12	53	132	147	166	268	293	
		0	11	18	91	114	220	263	291	
		0	9	39	101	137	179	225	288	
		0	5	50	77	193	244	248	300	
		0	3	87	111	125	269	270	301	
8	8	0	44	62	90	162	195	258	371	405
		0	40	55	115	142	308	333	362	374
		0	24	94	131	136	240	316	325	396
		0	23	30	91	188	236	358	389	403
		0	21	22	73	138	263	310	345	377
		0	17	20	101	160	199	255	385	404
		0	4	10	103	181	287	300	336	379
		0	2	88	141	152	275	344	352	402
9	4	0	27	34	86	91				
		0	21	40	77	93				
		0	20	38	87	88				
		0	15	44	80	90				
		0	14	31	55	85				
		0	12	25	47	73				
		0	6	39	82	84				
		0	4	32	74	83				
		0	3	11	69	92				
9	5	0	45	62	83	125	157			
		0	23	25	64	111	154			
		0	20	34	85	137	142			
		0	18	49	58	118	145			
		0	15	19	92	121	147			
		0	12	71	78	150	153			
		0	11	44	81	105	135			
		0	10	16	114	136	149			
		0	8	36	84	151	152			

9	6	0	38	49	112	148	225	232	
		0	29	72	86	161	195	217	
		0	18	53	55	159	210	218	
		0	17	64	79	144	186	231	
		0	13	41	67	181	197	221	
		0	12	32	100	202	205	211	
		0	5	44	71	90	206	229	
		0	4	52	73	133	216	226	
		0	1	31	92	125	150	220	
9	7	0	33	58	137	141	215	320	329
		0	27	71	84	118	238	274	334
		0	22	73	134	196	206	283	331
		0	16	54	86	175	218	294	318
		0	15	23	65	193	242	245	308
		0	7	35	116	162	255	256	323
		0	6	17	37	231	276	306	332
		0	5	69	122	151	282	303	322
		0	2	14	99	138	268	286	327
10	5	0	57	66	73	152	160		
		0	37	48	107	156	158		
		0	34	39	97	123	164		
		0	29	72	93	155	168		
		0	19	50	90	118	170		
		0	18	54	74	165	166		
		0	12	35	81	136	169		
		0	10	14	116	141	163		
		0	6	38	82	143	167		
		0	3	30	45	145	162		
10	6	0	29	49	113	149	221	244	
		0	28	59	83	193	239	240	
		0	18	44	106	169	243	260	
		0	14	66	85	141	184	245	
		0	12	15	60	167	206	264	
		0	8	73	89	166	176	217	
		0	7	57	124	220	253	262	
		0	5	27	119	188	228	241	
		0	4	34	102	182	207	261	
		0	2	37	123	224	235	256	

11	4	0	38	53	88	112	
		0	27	40	89	98	
		0	25	42	73	107	
		0	23	52	55	116	
		0	18	37	84	104	
		0	11	54	110	114	
		0	10	36	87	115	
		0	7	21	90	102	
		0	6	8	78	100	
		0	5	44	85	101	
		0	1	46	76	109	
11	5	0	31	41	91	180	189
		0	30	42	97	156	185
		0	21	57	58	167	173
		0	18	25	53	136	188
		0	14	65	84	159	164
		0	13	33	137	160	175
		0	11	74	106	123	187
		0	8	62	101	169	191
		0	4	73	100	144	178
		0	3	46	128	168	184
		0	2	26	87	134	179
12	4	0	55	59	121	127	
		0	31	34	94	126	
		0	27	38	111	116	
		0	20	21	96	122	
		0	19	42	90	119	
		0	18	53	103	117	
		0	16	52	97	109	
		0	15	58	98	123	
		0	13	46	74	125	
		0	10	17	54	124	
		0	8	49	88	118	
		0	2	24	106	115	

12	5	0	43	97	105	180	209
		0	39	50	63	142	201
		0	36	66	123	179	194
		0	32	41	102	157	197
		0	23	33	129	173	207
		0	16	38	98	206	208
		0	14	49	107	135	183
		0	7	27	101	118	182
		0	6	37	159	178	204
		0	5	51	131	196	200
		0	3	21	88	130	202
		0	1	53	100	177	189
13	4	0	43	59	117	130	
		0	29	54	106	115	
		0	22	30	120	138	
		0	15	62	103	137	
		0	14	56	80	135	
		0	12	50	96	123	
		0	10	45	102	128	
		0	7	51	100	119	
		0	6	17	95	131	
		0	4	32	85	133	
		0	3	63	94	127	
		0	2	23	99	136	
		0	1	40	105	110	
14	4	0	31	35	126	134	
		0	25	65	97	150	
		0	24	27	101	139	
		0	21	57	79	130	
		0	20	69	128	144	
		0	18	28	84	145	
		0	15	67	137	148	
		0	13	30	113	136	
		0	12	41	88	143	
		0	7	44	87	149	
		0	6	45	116	135	
		0	5	14	68	146	
		0	2	48	98	140	
		0	1	34	94	120	

15	4	0	57	74	156	157
		0	39	53	98	149
		0	29	60	115	150
		0	28	70	141	159
		0	27	32	105	143
		0	24	61	67	130
		0	23	33	152	155
		0	21	47	112	148
		0	20	66	124	154
		0	16	50	118	158
		0	13	25	120	139
		0	11	52	144	146
		0	8	64	136	145
		0	7	22	84	160
		0	4	48	97	151

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