# Equitable Colorings of Planar Graphs

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ABSTRACT. In this paper we prove that the Equitable $\Delta$-Coloring Conjecture holds for planar graphs with maximum degree $\Delta \geq 13$.

## 1 Introduction

The graphs we consider are finite, simple and undirected. For a graph $G$, we use $V(G)$, $|G|$, $E(G)$, $e(G)$, $\Delta(G)$ and $\delta(G)$ to denote respectively the vertex set, order, edge set, size, maximum (vertex) degree and minimum (vertex) degree of $G$. If $U \subseteq V(G)$, then the subgraph of $G$ induced by $U$ is denoted by $G[U]$. However, if $u \in V(G)$, then we use $G - u$ to denote the subgraph of $G$ induced by $V(G) \setminus \{u\}$. We write $xy \in E(G)$ if the two vertices $x$ and $y$ of $G$ are adjacent in $G$. For an edge $xy$ of $G$, we use $G - xy$ to denote the graph obtained from $G$ by deleting $xy$. We use $d(v)$ to denote the degree of $v$ in $G$ and $N(v)$ to denote the set of vertices of $G$ adjacent to $v$ in $G$. For any $U \subset V(G)$ and $W \subset V(G)$, we let $e(U, W)$ to denote the number of edges joining vertices of $U$ to vertices of $W$. In particular, we use $e(u, W)$ to denote $e(\{u\}, W)$. The complete bipartite graph with bipartition $(X, Y)$, where $|X| = m$ and $|Y| = n$, is denoted by $K_{m,n}$. Let $G \cup H$ denote the union of two vertex-disjoint graphs $G$ and $H$.

We call a (proper) vertex-coloring $\phi$ of a graph $G$ an *equitable coloring* of $G$ if the number of vertices in any two color classes differ by at most one. If $\phi$ is an equitable coloring of $G$ using $k$ colors, then we say that $\phi$ is an *equitable k-coloring* of $G$. The least integer $k$ for which $G$ has an equitable $k$-coloring is defined to be the *equitable chromatic number* of $G$ and is denoted by $\chi_e(G)$. The least integer $k$ for which $G$ has an equitable $k'$-coloring of $G$ for every $k' \geq k$ is denoted by $\chi^*(G)$. A set $U \subseteq V(G)$ is called an independent $t$-set if $|U| = t$ and no two vertices of $U$ are adjacent in $G$.

Hajnal and Szemerédi [2] proved that every graph $G$ has an equitable $k$-coloring for any $k \geq \Delta(G)+1$. Meyer [4] proved that any tree $T$ has an equitable $(\lceil \Delta(T)/2 \rceil + 1)$-coloring and he made the following conjecture:

**The Equitable Coloring Conjecture (ECC):** For any connected graph $G$, except the complete graph and the odd cycle, $\chi_e(G) \leq \Delta(G)$.

(A survey on equitable colorings of graphs can be found in the book *Graph Coloring Problems* [3].)

In [1], Chen, Lih and Wu proved that if $G$ is a connected graph with $\Delta(G) \geq \frac{|G|}{2}$, and $G$ is not the complete graph, or the odd cycle, or the complete bipartite graph $K_{2m+1,2m+1}$, then $G$ is equitably $\Delta$-colorable. Based on this result, Chen, Lih and Wu put forth the following conjecture:

**The Equitable $\Delta$-Coloring Conjecture:** Let $G$ be a connected graph with maximum degree $\Delta$. Suppose $G$ is not the complete graph, or the odd cycle, or the complete bipartite graph $K_{2m+1,2m+1}$. Then $G$ is equitably $\Delta$-colorable.

(Note that if a graph satisfies the Equitable $\Delta$-Coloring Conjecture, then it also satisfies the ECC. Hence the Equitable $\Delta$-Coloring Conjecture is stronger than the ECC.)

In [5], we proved that if $G$ is a connected graph of order $n$ with $\frac{n}{3}+1 \leq \Delta(G) \leq \frac{n}{2}$, then $\chi^*(G) \leq r+s+t \leq \Delta(G)$ for some parameters $r$, $s$ and $t$ of $G$. Thus the Equitable $\Delta$-Coloring Conjecture is true for graphs $G$ of order $n$ with $\Delta(G) \geq \frac{n}{3}+1$. In [6], we proved that the Equitable $\Delta$-Coloring Conjecture holds for outerplanar graphs.

## 2 Some Useful Lemmas

We need the following lemmas in the proof of our main theorem.

**Lemma 1.** *For any planar graph $G$ of order $n$, $e(G) \leq 3n-6$ and $\delta(G) \leq 5$.*

(This well-known result can be found in almost every textbook on graph theory.)

**Lemma 2.** *Let $m \geq 1$ be a fixed integer. Suppose that any planar graph of order $mt$ is equitably $m$-colorable for any integer $t \geq 1$. Then any planar graph is also equitably $m$-colorable.*

**Proof:** We prove this lemma by induction on the order $n$ of $G$. By the assumption, we may assume that $mt < n < m(t+1)$. If $m \leq 5$, then $n = m(t+1)-j$ for some $0 < j < 5$. Now we consider the case that $m \geq 6$. By Lemma 1, $G$ has a vertex $u$ of degree at most 5. By the induction hypothesis, $G-u$ has an equitable $m$-coloring $\phi$. Let the color classes of $\phi$ be $V_1, V_2, \ldots, V_m$, where $|V_i| = t$ or $t+1$ for all $i \geq 1$. Since $d(u) \leq 5$, we may assume that $u$ is adjacent to only vertices in $V_1 \cup V_2 \cup \cdots \cup V_5$. If $|V_i| = t$ for some $i \geq 6$, then by adding $u$ to $V_i$, we get an equitable $m$-coloring of

$G$ (having color classes $V_1, \ldots, V_{i-1}, V_i \cup \{u\}, V_{i+1}, \ldots, V_m$). Hence we can assume that $|V_i| = t+1$ for all $i \geq 6$ and we also have $n = m(t+1) - j$, $0 < j < 5$. Finally, let $G' = G \cup K_j$. Then $G'$ is planar of order $m(t+1)$ and thus by the assumption, $G'$ is equitably $m$-colorable (and so is $G$). □

**Lemma 3.** *Let $m \geq 4$ and $t \geq 1$ be integers. Let $H$ be a graph of order $mt$ with vertex chromatic number $\chi \leq m$. If $e(H) \leq (m-1)t$, then $H$ is equitably $m$-colorable.*

**Proof:** Let $A$ be the set of all isolated vertices in $H$ and let $\delta = \delta(H - A)$. We first consider the case that $m \leq \delta$. Since $\chi \leq m$, $H - A$ has an $m$-coloring $\phi$ having color classes $U_1, \ldots, U_m$. Now $e(H) \leq (m-1)t < \delta t$ implies that $|U_i| < t$ for all $i = 1, 2, \ldots, m$. Hence we can get an equitable $m$-coloring of $H$ by adding in some isolated vertices to the color classes $U_i$, $i = 1, \ldots, m$.

Next we consider the case that $m \geq \delta + 1$. We prove this case by induction on $e(H)$. Since $H$ has an edge $xy$ where $d(x) = \delta$, by the induction hypothesis, $H - xy$ has an equitable $m$-coloring $\phi$ having color classes $V_1, V_2, \ldots, V_m$, where $|V_i| = t$ for all $i = 1, 2, \ldots, m$. Clearly we need only to consider the case that $x, y \in V_1$. Let $N(x) \subset V_1 \cup V_2 \cup \cdots \cup V_\delta$ and let $V_1' = V_1 \setminus \{x\}$. Suppose that there exists $z \in V_i$ for some $i \geq \delta + 1$, such that $e(z, V_1') = 0$. Then by transferring $z$ from $V_i$ to $V_1'$ and adding $x$ to $V_i \setminus \{z\}$, we get an equitable $m$-coloring of $H$. Thus for each $z \in V_{\delta+1} \cup \cdots \cup V_m$, $e(z, V_1') \geq 1$. Hence $e(\bigcup_{i=\delta+1}^m V_i, V_1') \geq (m - \delta)t$. If every $w \in V_j$, $j \in \{2, \ldots, \delta\}$, is adjacent to some vertex in $V_1'$, then $e(V_j, V_1') \geq t$. Otherwise there exists $w \in V_j$ such that $e(w, V_1') = 0$. Then we transfer $w$ from $V_j$ to $V_1'$ and by the above argument, $e(\bigcup_{i=\delta+1}^m V_i, V_j \setminus \{w\}) \geq (m - \delta)t \geq t$. Consequently $e(H) \geq (m - \delta)t + (\delta - 1)t + \delta = (m-1)t + \delta$, a contradiction to the assumption. □

**Lemma 4.** *Let $m$ and $s$ be positive integers. Suppose $G$ is a planar graph with $\Delta(G) \leq m$. If $G$ has an independent $s$-set $V'$ and there exists $A \subseteq V(G) \setminus V'$ such that $|A| > \frac{s(m+3)}{2}$ and $e(v, V') \geq 1$ for all $v \in A$, then $A$ contains two nonadjacent vertices $\alpha$ and $\beta$ which are adjacent to exactly one and the same vertex $\gamma \in V'$.*

**Proof:** Let $A_1 \subseteq A$ be such that each $v \in A_1$ is adjacent to exactly one vertex of $V'$. Let $r = |A_1|$. Then $r + 2(|A| - r) \leq ms$, from which it follows that $r \geq 2|A| - ms > 3s$. Hence $V'$ contains at least one vertex $\gamma$ which is adjacent to at least four vertices of $A_1$. Since $G$ is planar, $G$ does not induce $K_5$. Hence $A_1$ contains two independent vertices $\alpha$ and $\beta$ which are adjacent to $\gamma$. □

**Lemma 5.** *Let $m \geq 7$ and $t \geq 1$ be integers. Let $H$ be a planar graph of order $mt$ with maximum degree $\Delta$. If $e(H) \leq (2m-3)t - \max\{\Delta - 3, t\}$, then $H$ is equitably $m$-colorable.*

**Proof:** We prove this lemma by induction on $e(H)$. As $H$ is planar, $H$ has an edge $xy$ where $d(x) = \delta \le 5$. By the induction hypothesis, $H - xy$ has an equitable $m$-coloring $\phi$ having color classes $V_1, V_2, \ldots, V_m$, where $|V_i| = t$ for all $i = 1, 2, \ldots, m$. Clearly we need only to consider the case that $x, y \in V_1$. Let $N(x) \subset V_1 \cup V_2 \cup \cdots \cup V_\delta$ and $V_1' = V_1 \setminus \{x\}$. By an argument in the proof of Lemma 3, $e(\bigcup_{i=\delta+1}^m V_i, V_1') \ge (m-\delta)t$.

Suppose for each $v \in V_2 \cup \cdots \cup V_\delta$, $e(v, V_1') \ge 1$. Then $e(V_2 \cup \cdots \cup V_\delta, V_1') \ge (\delta - 1)t$. Let $X = \bigcup_{i=2}^m V_i \cup \{x\}$. Then $e(X, V_1') \ge (m-1)t + \delta$. Thus $e(H[X]) \le e(H) - e(X, V_1') \le (m-2)t - \max\{\Delta - 3, t\} - \delta$. Since $|X| = (m-1)t + 1 > \frac{(m+3)(t-1)}{2}$, by Lemma 4, $X$ contains two nonadjacent vertices $\alpha$ and $\beta$ which are adjacent to exactly one and the same vertex $\gamma \in V_1'$. Let $H_1 = H[(V_1' \setminus \{\gamma\}) \cup \{\alpha, \beta\}]$ and $H_2 = H[(X \setminus \{\alpha, \beta\}) \cup \{\gamma\}]$. Clearly $e(H_2) \le e(H[X]) + \Delta - 2 \le (m-2)t$. As $|H_2| = (m-1)t$, by Lemma 3 and the Four Color Theorem, $H_2$ is equitably $(m-1)$-colorable. Hence $H$ is equitably $m$-colorable.

Now we suppose that there exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$ for some $j \in \{2, \ldots, \delta\}$, say $j = 2$. By an argument in the proof of Lemma 3, $e(\bigcup_{i=\delta+1}^m V_i, V_2') \ge (m-\delta)t$ where $V_2' = V_2 \setminus \{v_2\}$. If every $w \in V_j$, $j \in \{3, \ldots, \delta\}$, is adjacent to some vertex in $V_k'$, $k = 1, 2$, then $e(V_j, V_k') \ge t$, $k = 1, 2$. Otherwise there exists $w \in V_j$ such that $e(w, V_1' \cup V_2') = 0$. Then by an argument in the proof of Lemma 3, $e(\bigcup_{i=\delta+1}^m V_i, V_j \setminus \{w\}) \ge (m-\delta)t \ge 2t$. Thus $e(H) \ge 2(m-\delta)t + 2(\delta-2)t + \delta > (2m-3)t - \max\{\Delta-3, t\} \ge e(H)$, which is a contradiction. □

## 3 Proof of Main Theorem

In this section we shall prove that any planar graph $G$ with maximium degree $\Delta \ge 13$ is equitably $m$-colorable for any $m \ge \Delta$. By Hajanal and Szemerédi's theorem, we only need to prove that $G$ is equitably $\Delta$-colorable. However, here we do not apply Hajanal and Szemerédi's theorem and instead we prove the general case because it does not increase the length of the proof.

**Theorem.** *Let $G$ be a planar graph with maximum degree $\Delta \ge 13$. Then $G$ is equitably $m$-colorable for any $m \ge \Delta$.*

**Proof:** By Lemma 2, we need only to consider the case that $n = |G| = mt$. We prove this theorem by induction on $e(G)$. As $G$ is planar, $G$ has an edge $xy$ where $d(x) \le 5$. By the induction hypothesis, $G - xy$ has an equitable $m$-coloring $\phi$ having color classes $V_1, V_2, \ldots, V_m$, where $|V_i| = t$ for all $i \ge 1$. Clearly we need only to consider the case that $x, y \in V_1$. Let $V_1' = V_1 \setminus \{x\}$ and assume that $N(x) \subset V_1 \cup V_2 \cup \cdots \cup V_5$.

It is clear now that we can assume that $e(z, V_1') \ge 1$ for any $z \in \bigcup_{i=6}^m V_i$.

Consequently

$$e(\bigcup_{i=6}^{m} V_i, V_1') \geq (m-5)t. \tag{1}$$

Suppose there exists $w \in V_j$, for some $j = 2,3,4,5$, such that $e(w, V_1') = 0$. Then following the proof of Lemma 3, we can assume that for any $z \in V_i$, $i \geq 6$, $e(z, V_j') \geq 1$, where $V_j' = V_j \setminus \{w\}$. Consequently

$$e(\bigcup_{i=6}^{m} V_i, V_j') \geq (m-5)t \tag{2}$$

Suppose there exists $v \in V_k$, where $k \in \{2,3,4,5\}$ and $k \neq j$, such that $e(v, V_j') = 0$. Then by transferring $v$ from $V_k$ to $V_j'$ and $w$ from $V_j$ to $V_1'$, we can make $V_k' = V_k \setminus \{v\}$ play the role of $V_1'$. Thus we can assume that $e(z, V_k') \geq 1$ for all $z \in V_i$, $i \geq 6$. Hence

$$e(\bigcup_{i=6}^{m} V_i, V_k') \geq (m-5)t. \tag{3}$$

It is clear that $e(\bigcup_{i=6}^{m} V_i \cup \{x\}, V_1') \leq \Delta(t-1) \leq m(t-1)$. Hence $(m-5)t + 1 \leq m(t-1)$, from which it follows that $t \geq \frac{m+1}{5}$. Thus

$$t \geq 3.$$

**Case 1.** There exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$, for each $j = 2,3,4,5$. In this case, from (1) and (2), we obtain

$$e(\bigcup_{i=6}^{m} V_i, \bigcup_{j=1}^{5} V_j') \geq 5(m-5)t, \text{ where } V_j' = V_j \setminus \{v_j\}. \tag{4}$$

Then $5(m-5)t + 1 \leq e(G) = 3n - 6 = 3mt - 6$, which is false because $m \geq 13$. Hence $G$ is equitably $m$-colorable.

**Case 2.** There exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$ for each $j = 2,3,4$ but $e(v, V_1') \geq 1$ for any $v \in V_5$.

Since $e(v_j, V_1') = 0$ for $j = 2,3,4$, by (1) and (2), $e(\bigcup_{i=6}^{m} V_i, \bigcup_{j=1}^{4} V_j') \geq 4(m-5)t$ where $V_j' = V_j \setminus \{v_j\}$. Next, if there exists $v_5 \in V_5$ such that $e(v_5, V_j') = 0$ for some $j = 2,3,4$, then by (3), $e(\bigcup_{i=6}^{m} V_i, V_5') \geq (m-5)t$. Thus (4) holds (which has been proved false). Otherwise $e(V_5, V_j') \geq t$ for all $j = 2,3,4$ and thus

$$e(\bigcup_{i=5}^{m} V_i, \bigcup_{j=1}^{4} V_j') \geq 4(m-5)t + 4t. \tag{5}$$

Let $A = \bigcup_{i=5}^{m} V_i \cup \{x\}$. Then $e(G[A]) \leq e(G) - e(A, V_1' \cup (\bigcup_{j=2}^{4} V_j)) \leq 3mt - 6 - (4(m-5)t + 4t + 1) = (16-m)t - 7$. Since $|A| = (m-4)t + 1 > \frac{(m+3)(t-1)}{2}$, by Lemma 4, $A$ contains two nonadjacent vertices $\alpha$ and $\beta$ which are adjacent to exactly one and the same vertex $\gamma \in V_1'$. Let

$$G_1 = G[((V_1' \setminus \{\gamma\}) \cup \{\alpha, \beta\}) \cup (\bigcup_{j=2}^{4} V_j)]$$

and

$$G_2 = G[(A \setminus \{\alpha, \beta\}) \cup \{\gamma\}].$$

Clearly $e(G_2) \leq e(G[A]) + m - 2 \leq (16-m)t + m - 9 \leq (m-5)t$. As $|G_2| = (m-4)t$, by Lemma 3, $G_2$ is equitably $(m-4)$-colorable. Hence $G$ is equitably $m$-colorable.

**Case 3.** There exists $v_j \in V_j$ such that $e(v_j, V_1') = 0$ for each $j = 2, 3$ but $e(v, V_1') \geq 1$ for any $v \in V_4 \cup V_5$.

By (2), $e(\bigcup_{i=6}^{m} V_i, V_j') \geq (m-5)t$ for $j = 2, 3$.

Suppose there exists $v_k \in V_k$, for some $k = 4, 5$, such that $e(v_k, V_j') = 0$ for some $j = 2, 3$. Then by (3), $e(\bigcup_{i=6}^{m} V_i, V_k') \geq (m-5)t$. Hence either (4) or (5) holds (each of which has been proved false).

It remains to settle the case that $e(v, V_2') \geq 1$ and $e(v, V_3') \geq 1$ for any $v \in V_4 \cup V_5$. We now have $e(V_4 \cup V_5, V_2' \cup V_3') \geq 4t$. Let $B = \bigcup_{i=4}^{m} V_i \cup \{x\}$. Since $e(V_4 \cup V_5, V_1') \geq 2t$, by (1) and (2), we have

$$e(B, V_1' \cup V_2' \cup V_3') \geq 3(m-5)t + 6t + 1. \tag{6}$$

Then $e(G[B]) \leq 3mt - 6 - e(B, V_1' \cup V_2' \cup V_3') \leq 9t - 7$. Since $|B| = (m-3)t + 1 > \frac{(m+3)(t-1)}{2}$, by Lemma 4, there exist two nonadjacent vertices $\alpha$ and $\beta$ in $B$ such that $\alpha$ and $\beta$ are adjacent to exactly one and the same vertex $\gamma \in V_1'$. Let

$$G_1 = G[((V_1' \setminus \{\gamma\}) \cup \{\alpha, \beta\}) \cup V_2 \cup V_3]$$

and

$$G_2 = G[(B \setminus \{\alpha, \beta\}) \cup \{\gamma\}].$$

Clearly $e(G_2) \leq e(G[B]) + m - 2 \leq 9t + m - 9 \leq (2(m-3)-3)t - \max\{\Delta - 3, t\}$, by Lemma 5, $G_3$ is equitably $(m-3)$-colorable. Consequently $G$ is equitably $m$-colorable.

**Case 4.** There exists $v_2 \in V_2$ such that $e(v_2, V_1') = 0$ and $e(v, V_1') \geq 1$ for any $v \in V_3 \cup V_4 \cup V_5$.

In this case $e(V_3 \cup V_4 \cup V_5, V_1') \geq 3t$. Clearly $e(\bigcup_{i=3}^{m} V_i \cup \{x\}, V_1') \leq \Delta(t-1) \leq m(t-1)$. Hence $(m-2)t + 1 \leq m(t-1)$, from which it follows that

$$t \geq 7.$$

By (2), $e(\bigcup_{i=6}^{m} V_i, V_2') \geq (m-5)t$.

Suppose there exists $v_j \in V_j$, for some $j = 3,4,5$, such that $e(v_j, V_2') = 0$. By (3), $e(\bigcup_{i=6}^{m} V_i, V_j') \geq (m-5)t$. If There exists $v_k \in V_k$, for some $2 < k \neq j \leq 5$, such that $e(v_k, V_j') = 0$, then by (3), $e(\bigcup_{i=6}^{m} V_i, V_k') \geq (m-5)t$. Thus at least one of (4), (5) and (6) holds (each of which has been proved false).

Now we suppose that for each $v \in V_3 \cup V_4 \cup V_5$, $e(v, V_2') \geq 1$. In this subcase, $e(V_3 \cup V_4 \cup V_5, V_2') \geq 3t$. Hence, $e(\bigcup_{i=3}^{m} V_i \cup \{x\}, V_1' \cup V_2') \geq 2(m-5)t + 6t + 1$. Let $D = \bigcup_{i=3}^{m} V_i \cup \{x\}$. Then $e(G[D]) \leq 3mt - 6 - e(D, V_1' \cup V_2') \leq (m+4)t - 7$. Since $|D| = (m-2)t + 1 > \frac{(m+3)(t-1)}{2}$, by Lemma 4, there exists two nonadjacent vertices $\alpha, \beta \in D$ such that both $\alpha$ and $\beta$ are adjacent to exactly one and the same vertex $\gamma \in V_1'$. Let $G_1 = G[((V_1' \setminus \{\gamma\}) \cup \{\alpha,\beta\}) \cup V_2]$ and $G_2 = G[(D \setminus \{\gamma\}) \cup \{\alpha,\beta\}]$. Clearly $e(G_2) \leq e(G[D]) + m - 2 \leq (m+4)t + m - 9 \leq (2(m-2)-3)t - \max\{\Delta - 3, t\}$, by Lemma 5, $G_2$ is eauitably $(m-2)$-colorable. Consequently $G$ is equitably $m$-colorable.

**Case 5.** For any $z \in V_2 \cup V_3 \cup V_4 \cup V_5$, $e(z, V_1') \geq 1$.

In this case $e(\bigcup_{i=2}^{5} V_i, V_1') \geq 4t$. Let $I = \bigcup_{i=2}^{m} V_i \cup \{x\}$. By (1), $(t-1)m \geq (t-1)\Delta \geq e(I, V_1') \geq (m-5)t + 4t + 1$, from which it follows that

$$t \geq m + 1 \geq 14.$$

Clearly $e(G[I]) \leq 3mt - 6 - e(I, V_1') \leq 2mt + t - 7$. Since $|I| = (m-1)t + 1 > \frac{(m+3)(t-1)}{2}$, by Lemma 4, there exists two nonadjacent vertices $\alpha, \beta \in I$ such that both $\alpha$ and $\beta$ are adjacent to exactly one and the same vertex $\gamma \in V_1'$. Let $G_1 = G[V_1' \setminus \{\gamma\} \cup \{\alpha,\beta\}]$ and $G_2 = G[I \setminus \{\alpha,\beta\} \cup \{\gamma\}]$. Then $e(G_2) \leq e(G[I]) + m - 2 \leq 2mt + m + t - 9$.

We shall next use induction on $e(G_2)$ to show that $G_2$ is equitably $(m-1)$-colorable. As $G_2$ is planar, $G_2$ has one edge $uv$ where $d(u) \leq 5$. By the induction hypothesis, $G_2 - uv$ has an equitable $(m-1)$-coloring having color classes $Y_1, Y_2, \ldots, Y_{m-1}$, where $|Y_i| = t$ for all $i \geq 1$. We assume that $u, v \in Y_1$ and $N(u) \in Y_1 \cup Y_2 \cup \cdots \cup Y_5$. Let $Y_1' = Y_1 \setminus \{u\}$. By (1), $e(\bigcup_{i=6}^{m-1} Y_i, Y_1') \geq (m-6)t$.

**Subcase 5.1.** There exist $y_2 \in Y_2$, $y_3 \in Y_3$ and $y_4 \in Y_4$ such that $e(y_2, Y_1') = e(y_3, Y_1') = e(y_4, Y_1') = 0$.

By (2), $e(\bigcup_{i=6}^{m-1} Y_i, Y_p') \geq (m-6)t$, where $Y_p' = Y_p \setminus \{y_p\}$ for any $p = 2,3,4$. If there exists $y_5 \in Y_5$ such that $e(y_5, Y_j') = 0$, for some $j = 1,2,3,4$, then by (2) and (3), $e(\bigcup_{i=6}^{m-1} Y_i, Y_5 \setminus \{y_5\}) \geq (m-6)t > 4t$. Otherwise $e(Y_5, Y_1' \cup Y_2' \cup Y_3' \cup Y_4') \geq 4t$. Hence $e(G_2) \geq 4(m-6)t + 4t + 1 > 2mt + m + t - 9 \geq e(G_2)$, which is a contradiction.

**Subcase 5.2.** There exist $y_2 \in Y_2, y_3 \in Y_3$ such that $e(y_2, Y_1') = e(y_3, Y_1') = 0$ and for any $w \in Y_4 \cup Y_5$, $e(w, Y_1') \geq 1$.

In this subcase $e(Y_4 \cup Y_5, Y_1') \geq 2t$. By (2), $e(\bigcup_{i=6}^{m-1} Y_i, Y_k') \geq (m-6)t$ for any $k = 2, 3$, where $Y_k' = Y_k \setminus \{y_k\}$.

Suppose there exists $y_j \in Y_j$, for some $j = 4, 5$, such that $e(y_j, Y_2') = 0$ or $e(y_j, Y_3') = 0$. By (3), $e(\bigcup_{i=6}^{m-1} Y_i, Y_j') \geq (m-6)t$, where $Y_j' = Y_j \setminus \{y_j\}$. Then $e(G_2) \geq 4(m-6)t + 4t + 1$, which is a contradiction (see Subcase 5.1).

Now we suppose for each $w \in Y_4 \cup Y_5$, $e(w, Y_2') \geq 1$ and $e(w, Y_3') \geq 1$. Then $e(Y_4 \cup Y_5, Y_2' \cup Y_3') \geq 4t$. Let $J = \bigcup_{i=4}^{m-1} Y_i \cup \{u\}$. Then $e(J, Y_1' \cup Y_2' \cup Y_3') \geq 3(m-6)t + 6t + 1$, and thus $e(G_2[J]) \leq e(G_2) - e(J, Y_1' \cup Y_2' \cup Y_3') \leq (13-m)t + m - 10$. Since $|J| = (m-4)t + 1 > \frac{(m+3)(t-1)}{2}$, by Lemma 4, there exists two nonadjacent vertices $\alpha, \beta \in J$ such that $\alpha$ and $\beta$ are adjacent to exactly one and the same vertex $\gamma \in Y_1'$. Let $G_3 = G[Y_1' \setminus \{\gamma\} \cup \{\alpha, \beta\} \cup Y_2 \cup Y_3]$ and $G_4 = G_2[J \setminus \{\alpha, \beta\} \cup \{\gamma\}]$. Clearly $e(G_4) \leq e(G_2[J]) + m - 2 \leq (13-m)t + 2m - 12 \leq (m-5)t$, by Lemma 3, $G_4$ is equitably $(m-4)$-colorable. Consequently $G$ is equitably $m$-colorable.

**Subcase 5.3.** There exists $y_2 \in Y_2$ such that $e(y_2, Y_1') = 0$ and for any $w \in Y_3 \cup Y_4 \cup Y_5$, $e(w, Y_1') \geq 1$.

In this subcase $e(Y_3 \cup Y_4 \cup Y_5, Y_1') \geq 3t$. By (2), $e(\bigcup_{i=6}^{m-1} Y_i, Y_2') \geq (m-6)t$, where $Y_2' = Y_2 \setminus \{y_2\}$.

Suppose there exists $y_j \in Y_j$, for some $j = 3, 4, 5$, such that $e(y_j, Y_2') = 0$. By (3), $e(\bigcup_{i=6}^{m-1} Y_i, Y_j') \geq (m-6)t$, where $Y_j' = Y_j \setminus \{y_j\}$. Thus $e(G_2) \geq 3(m-6)t + 6t + 1$, which is a contradiction (see Subcase 5.2).

Now we suppose for each $w \in Y_3 \cup Y_4 \cup Y_5$, $e(w, Y_2') \geq 1$. Then $e(Y_3 \cup Y_4 \cup Y_5, Y_2') \geq 3t$. Let $K = \bigcup_{i=3}^{m-1} Y_i \cup \{x\}$. Then $e(K, Y_1' \cup Y_2') \geq 2(m-6)t + 6t + 1$, and thus $e(G_2[K]) \leq e(G_2) - e(K, Y_1' \cup Y_2') \leq m + 7t - 10$. By Lemma 4, there exists two nonadjacent vertices $\alpha, \beta \in J$ such that $\alpha$ and $\beta$ are adjacent to exactly one and the same vertex $\gamma \in Y_1'$. Let $G_3 = G[Y_1' \setminus \{\gamma\} \cup \{\alpha, \beta\} \cup Y_2]$ and $G_4 = G_2[K \setminus \{\alpha, \beta\} \cup \{\gamma\}]$. Clearly $e(G_4) \leq e(G_2[K]) + m - 2 \leq 2m + 7t - 12 \leq (m-4)t$. By Lemma 3, $G_4$ is equitably $(m-3)$-colorable. Consequently $G$ is equitably $m$-colorable.

**Subcase 5.4.** For any $w \in \bigcup_{i=2}^{5} Y_i$, $e(w, Y_1') \geq 1$.

In this subcase $e(\bigcup_{i=2}^{5} Y_i, Y_1') \geq 4t$. Then $e(\bigcup_{i=2}^{m-1} Y_i \cup \{x\}, Y_1') \geq (m-6)t + 4t + 1$. Let $L = \bigcup_{i=2}^{m-1} Y_i \cup \{x\}$. Then $e(G_2[L]) \leq e(G_2) - e(L, Y_1') \leq (m+3)t + m - 10$.

By Lemma 4, there exists two nonadjacent vertices $\alpha, \beta \in K$ such that both $\alpha$ and $\beta$ are adjacent to exactly one and the same vertex $\gamma \in Y_1'$. Let $G_3 = G_2[Y_1' \setminus \{\gamma\} \cup \{\alpha, \beta\}]$ and $G_4 = G_2[L \setminus \{\alpha, \beta\} \cup \{\gamma\}]$. Then $e(G_4) \leq e(G_2[L]) + m - 2 \leq (m+3)t + 2m - 12 \leq (2(m-2)-3)t - \max\{\Delta - 3, t\}$. By Lemma 5, $G_4$ is equitably $(m-2)$-colorable. Consequently $G$ is equitably $m$-colorable. □

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