On the Nordhaus-Gaddum Problem for the n-Path-Chromatic Number of a Graph

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ABSTRACT. Let $\mathcal{G}(p)$ denote the class of simple graphs of order p. For a graph G, the complement of G is denoted by $\overline{G}$. For a positive integer n, the n-path-chromatic number $\chi_n(G)$ is the least number of colours that can be associated to the vertices of G such that not all the vertices on any path of length n receive the same colour. The Nordhaus-Gaddum Problem for the n-path-chromatic number of G is to find bounds for $\chi_n(G)+\chi_n(\overline{G})$ and $\chi_n(G)\cdot\chi_n(\overline{G})$ over the class $\mathcal{G}(p)$. In this paper we determine sharp lower bounds for the sum and the product of $\chi_n(G)$ and $\chi_n(\overline{G})$. Furthermore, we provide weak upper bounds for $\chi_2(G)+\chi_2(\overline{G})$ and $\chi_2(G)\cdot\chi_2(\overline{G})$.

1 Introduction

All graphs considered in this paper are undirected, finite, loopless and have no multiple edges. For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$ respectively. Let $\mathcal{G}(p)$ denote the class of graphs of order p. The complement of G is denoted by $\overline{G}$. A set $U \subseteq V(G)$ is said to be n-independent if $G[U]$, the subgraph induced on U, has no paths of length n. Note that a 1-independent set is an independent set in the usual sense. For the most part, our notation and terminology follow that of Bondy and Murty [1].

Chartrand, Geller and Hedetniemi [2] defined the n-path-chromatic number $\chi_n(G)$ of G to be the least number of colours needed to colour the vertices of G so that not all the vertices on any path of length n are coloured the same. We refer to such a colouring as an n-path-colouring of G. Note that $\chi_1(G)=\chi(G)$, the usual chromatic number of G. Thus $\chi_n(G)$ is a generalization of $\chi(G)$. In this paper we consider the Nordhaus-Gaddum problem

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[4] of determining sharp bounds for $\chi_n(G)+\chi_n(\overline{G})$ and $\chi_n(G)\cdot\chi_n(\overline{G})$ over the class $\mathcal{G}(p)$.

Let P_n denote a path of length n. The Ramsey number $R(P_m, P_n)$ is the smallest integer p such that for every graph $G \in \mathcal{G}(p)$ either G contains a path P_m or $\overline{G}$ contains a path P_n. For a real number x, the largest integer less than or equal to x is denoted by $\lfloor x \rfloor$. Similarly $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. The following theorem by Gerencsér and Gyárfás [3] determines $R(P_m, P_n)$.

Theorem 1. *For positive integers m and n with $m \geq n \geq 1$,*

$$R(P_m, P_n) = m + \left\lfloor \frac{n+1}{2} \right\rfloor .$$

□

The following theorem by Chartrand, Geller and Hedetniemi [2] determines the n-path-chromatic number of the complete graph K_p.

Theorem 2. *For positive integers p and n, $\chi_n(K_p) = \lceil \frac{p}{n} \rceil$.* □

In the next section we will show that

$$\chi_n(G)\cdot\chi_n(\overline{G}) \geq \left\lceil \frac{p}{R-1} \right\rceil$$

and

$$\chi_n(G)+\chi_n(\overline{G}) \geq \left\lceil 2\sqrt{\left\lceil \frac{p}{R-1} \right\rceil} \right\rceil$$

where $R = R(P_n, P_n)$. We will also establish the sharpness of the above bounds. Furthermore, we shall derive a weak upper bound for the sum and the product of $\chi_2(G)$ and $\chi_2(\overline{G})$.

2 Main Results

The lower bounds for $\chi_n(G)+\chi_n(\overline{G})$ and $\chi_n(G)\cdot\chi_n(\overline{G})$ are dependent on the Ramsey number $R(P_n, P_n)$. From Theorem 1, it follows that $R(P_n, P_n) = \lfloor \frac{3n+1}{2} \rfloor$. For notational convenience we denote $R(P_n, P_n)$ by R, understanding the value of n from the context.

Lemma 1. *Let $H \in \mathcal{G}(p)$ with $\chi_n(H) = 1$. Then $\chi_n(\overline{H}) \geq \frac{p}{R-1}$.*

Proof: Suppose that $\chi_n(\overline{H}) = t$. Let $V_1, V_2, \ldots, V_t$ be a partition of $V(\overline{H})$ induced by a valid n-path-chromatic colouring of the vertices of $\overline{H}$. From the properties of this partition and the fact that $\chi_n(H) = 1$, it follows that neither $\overline{H}[V_i]$ nor $H[V_i]$ has a path of length n, for $1 \leq i \leq t$. Thus from the definition of the Ramsey number $R = R(P_n, P_n)$, we have

$$|V_i| \leq R-1, \quad 1 \leq i \leq t.$$

Now, summing over all i, we get

$$p = \sum_{i=1}^{t} |V_i| \leq t(R-1).$$

Thus $\chi_n(\overline{H}) = t \geq \frac{p}{R-1}$. □

From the definition of the Ramsey number $R = R(P_n, P_n)$, it follows that for a positive integer $t \leq R-1$, there exists a graph H of order t such that neither H nor $\overline{H}$ contains a path of length n. We refer to such a graph as a Ramsey graph and denote it by $H[t]$. An example of such a graph is given below:

$$H[t] = \begin{cases} \overline{K}_t, & \text{if } 1 \leq t \leq n, \\ \overline{K}_n \vee K_{\lfloor (n-1)/2 \rfloor}, & \text{if } t = R-1, \\ \overline{K}_n \vee K_{t-n}, & \text{otherwise.} \end{cases}$$

Using these Ramsey graphs we develop a method to construct a graph G such that $\chi_n(G)$ and $\chi_n(\overline{G})$ are equal to specified positive integers. The method of constructing a graph G such that $\chi_n(G) = k$ and $\chi_n(\overline{G}) = k'$ is briefly described below.

Start with a $0-1$ matrix A of order kk' with α ones. Furthermore we assume that all the entries in the first row and the first column are equal to 1. From this matrix A, we construct a graph G' of order α such that $\chi(G') = k$ and $\chi(\overline{G'}) = k'$. Next we construct a graph G by replacing every vertex of G' by a suitable Ramsey graph. More precisely, the following procedure describes the method of constructing G in three phases.

Procedure 1.

Phase 1: Let α, k and k' be given positive integers such that $\alpha \geq k+k'-1$. Suppose that $A = ((a_{ij}))$ is a $0-1$ matrix with k rows and k' columns such that:

(i) $a_{ij} = 1$, if $i = 1$ or $j = 1$.

(ii) The number of 1's in A is α.

Phase 2: Using the matrix A, we construct a graph G' as follows:

(i) $V(G') = \{(ij) : a_{ij} = 1\}$.

(ii) The adjacency in G' is defined as follows:

- Any two vertices of G' coming from the same column of A are adjacent.
- No two vertices of G' coming from the same row of A are adjacent.

- Any two vertices coming from distinct rows and distinct columns of A may be adjacent.

Phase 3: For $1 \leq i \leq k$ and $1 \leq j \leq k'$, let x_{ij} be integers satisfying the following conditions:

$$a_{ij} \leq x_{ij} \leq (R-1)a_{ij}, \quad 1 \leq i \leq k, \ 1 \leq j \leq k' \tag{1}$$

$$\sum_{j=1}^{k'} x_{1j} \geq (R-1)(k'-1)+1 \tag{2}$$

$$\sum_{i=1}^{k} x_{i1} \geq (R-1)(k-1)+1 \tag{3}$$

Using these x_{ij}'s and the graph G' of Phase 2, we construct a graph G of order $\sum_{i=1}^{k}\sum_{j=1}^{k'} x_{ij}$ as follows:

(i) Every vertex (ij) of G' is replaced by a Ramsey graph $H_{ij} = H[x_{ij}]$.

(ii) If the vertex (ij) is adjacent to the vertex (st) in G', then every vertex of H_{ij} is joined to every vertex of H_{st} in G. Otherwise, no vertex of H_{ij} is joined to any vertex of H_{st} in G.

For given positive integers k, k' and α with $k+k'-1 \leq \alpha \leq kk'$, we define the set $T_n(k,k',\alpha)$ of graphs as the set of all graphs G produced by the above procedure. From Phase 2 of the procedure, it is easy to see that $G' \in \mathcal{G}(\alpha)$, $\chi(G') = k$ and $\chi(\overline{G'}) = k'$.

Lemma 2. *If $G \in T_n(k,k',\alpha)$, then $\chi_n(G) = k$ and $\chi_n(\overline{G}) = k'$.*

Proof: Let $G \in T_n(k,k',\alpha)$ and x_{ij}, $1 \leq i \leq k$, $1 \leq j \leq k'$ be the corresponding integers satisfying (1), (2) and (3) of Phase 3. We define a subgraph F_i of G for $1 \leq i \leq k$ as follows:

$$F_i \cong H[x_{i1}] \cup H[x_{i2}] \cup \cdots \cup H[x_{ik'}].$$

Since $H[x_{ij}]$ is a Ramsey graph, F_i is free of paths of length n. Thus we can assign the same colour to all the vertices of F_i. Now note that $V(G) = \bigcup_{i=1}^{k} V(F_i)$ and hence $\chi_n(G) \leq k$. Next consider the subgraph $G_1 \cong H[x_{11}] \vee H[x_{21}] \vee \cdots \vee H[x_{k1}]$ of G. Clearly $\chi_n(\overline{G}_1) = 1$. From (3), it follows that the order of G_1 is at least $(k-1)(R-1)+1$. Now from Lemma 1, we have $\chi_n(G_1) \geq k$. Thus $\chi_n(G) \geq \chi_n(G_1) \geq k$. Combining this with the inequality $\chi_n(G) \leq k$, we have $\chi_n(G) = k$.

Proceeding along similar lines, one can easily show that $\chi_n(\overline{G}) = k'$. □

In the next theorem, we establish a sharp lower bound for $\chi_n(G) \cdot \chi_n(\overline{G})$ over the class $\mathcal{G}(p)$.

Theorem 3. *Let $G \in \mathcal{G}(p)$. Then $\chi_n(G) \cdot \chi_n(\overline{G}) \geq \left\lceil \frac{p}{R-1} \right\rceil$ where $R = R(P_n, P_n)$. Furthermore, this lower bound is sharp.*

Proof: Let $\chi_n(G) = t$. Consider an n-path-colouring of G which uses t colours. Let $V_1, V_2, \ldots, V_t$ be the partition of $V(G)$ induced by the above n-path-colouring. Without any loss of generality, let $|V_1| = \max_i |V_i|$. Then

$$|V_1| \geq \frac{p}{t}. \tag{4}$$

Note that $\chi_n(G[V_1]) = 1$ and hence by Lemma 1,

$$\chi_n(\overline{G}[V_1]) \geq \frac{|V_1|}{R-1}.$$

Therefore $\chi_n(\overline{G}[V_1]) \geq \frac{p}{t(R-1)}$ by (4).
Since $\chi_n(\overline{G}) \geq |X_n(\overline{G}[V_1])$, it follows that

$$\chi_n(\overline{G}) \geq \frac{p}{t(R-1)}.$$

Therefore $\chi_n(G) \cdot \chi_n(\overline{G}) \geq \left\lceil \frac{p}{R-1} \right\rceil$.

To establish the sharpness, let us assume that $\alpha = \left\lceil \frac{p}{R-1} \right\rceil$ and let k and k' be integers such that $k\ k' = \alpha$. Now consider a graph G^* in $T_n(k, k', \alpha)$ of order p given by Procedure 1 where x_{ij} are chosen such that

$$\sum_i \sum_j x_{ij} = p.$$

In the following, we show the existence of such x_{ij}'s.

Let m and s be non-negative integers such that $p = m(R-1) + s$ and $0 \leq s < R-1$. Note that $\alpha = m$ or $m+1$ according as $s = 0$ or not. Now define

$$x_{ij} = \begin{cases} R-1, & \text{if } (ij) \neq (kk'), \\ R-1, & \text{if } (ij) = (kk') \text{ and } s = 0, \\ s, & \text{otherwise.} \end{cases}$$

It is easy to see that $\sum_i \sum_j x_{ij} = p$ and the inequalities (1), (2) and (3) are satisfied. Now from Lemma 2, $\chi_n(G^*) = k$ and $\chi_n(\overline{G^*}) = k'$. This completes the proof. □

In the next theorem, we determine a lower bound for $\chi_n(G) + \chi_n(\overline{G})$ over the class $\mathcal{G}(p)$.

Theorem 4. *Let $G \in \mathcal{G}(p)$. Then*

$$\chi_n(G) + \chi_n(\overline{G}) \geq \left\lceil 2\sqrt{\left\lceil \frac{p}{R-1} \right\rceil} \right\rceil .$$

Furthermore, this bound is sharp.

Proof: The above inequality follows from the inequality in Theorem 3 and the arithmetic mean-geometric mean inequality.

The sharpness of the above inequality is easily established in the case $p \leq R-1$ by choosing G to be the Ramsey graph $H[p]$. To establish the sharpness when $p > R-1$, let k and k' be integers such that

$$k + k' = \left\lceil 2\sqrt{\left\lceil \frac{p}{R-1} \right\rceil} \right\rceil , \tag{5}$$

$$k\, k' \geq \left\lceil \frac{p}{R-1} \right\rceil \tag{6}$$

and

$$p \geq (R-1)(k + k' - 3) + 2. \tag{7}$$

The existence of integers k and k' satisfying (5), (6) and (7) is guaranteed by the numbers

$$\left\lceil \frac{\left\lceil 2\sqrt{\left\lceil \frac{p}{R-1} \right\rceil} \right\rceil}{2} \right\rceil \quad \text{and} \quad \left\lfloor \frac{\left\lceil 2\sqrt{\left\lceil \frac{p}{R-1} \right\rceil} \right\rceil}{2} \right\rfloor .$$

Now let $\alpha = \left\lceil \frac{p}{R-1} \right\rceil$ and consider a graph G^* of order p from the class $T_n(k, k', \alpha)$ defined along the same lines as in the proof of Theorem 3. From Lemma 2, it follows that $\chi_n(G^*) + \chi_n(\overline{G^*}) = k + k' = \left\lceil 2\sqrt{\left\lceil \frac{p}{R-1} \right\rceil} \right\rceil$. □

The problem of determining a sharp upper bound for $\chi_n(G) + \chi_n(\overline{G})$ seems to be difficult. Hence we will restrict our attention to $n = 2$ and present an upper bound for $\chi_2(G) + \chi_2(\overline{G})$.

Theorem 5. *Let $G \in \mathcal{G}(p)$. Then*

$$\chi_2(G) + \chi_2(\overline{G}) \leq \frac{2p+4}{3}. \tag{8}$$

Proof: Let $G \in \mathcal{G}(p)$. We first provide a 2-path-colouring of G with k colours by partitioning $V(G)$ as follows:

Set V_i as the largest 2-independent set in the graph induced on $V(G) - (\bigcup_{\ell=1}^{i-1} V_\ell)$, $1 \le i \le k$. Note that $|V_1| \ge |V_2| \ge \cdots \ge |V_k|$ and $|V_{k-1}| \ge 2$. From this valid 2-path-colouring of G, it follows that

$$\chi_2(G) \le k. \tag{9}$$

Note that if $k \le 2$, then the inequality (8) follows easily. Thus we assume $k \ge 3$. Next we prove that

$$\chi_2(\overline{G}) \le \frac{p-k+2}{2}. \tag{10}$$

From the construction of V_i, observe that for $i \ge 2$, if $x_i \in V_i$, then there exist two vertices y_{i-1} and z_{i-1} in V_{i-1} such that G has a path M_i of length 2 on the set $V(M_i) = \{x_i, y_{i-1}, z_{i-1}\}$. Otherwise, $V_{i-1} \cup \{x_i\}$ is a 2-independent set in the graph induced on $V(G) - (\bigcup_{\ell=1}^{i-2} V_\ell)$, contradicting the maximality of V_{i-1}.

Case (1) $|V_{k-1}| \ge 3$.

Note that, in G we can collect $(k-1)$ vertex disjoint paths M_i of length 2, for $i = 2, \ldots, k$ by choosing x_i to be different from y_i and z_i since $|V_i| \ge 3$, $1 \le i \le k-1$. Now we provide a 2-path-colouring of $\overline{G}$ as follows:

- Colour the vertices of $V(M_i)$ with colour i, $2 \le i \le k$.
- Colour the remaining $p - 3(k-1)$ vertices with $\left\lceil \frac{p-3k+3}{2} \right\rceil$ colours. Thus

$$\chi_2(\overline{G}) \le k - 1 + \left\lceil \frac{p-3k+3}{2} \right\rceil \le \frac{p-k+2}{2}$$

and this proves (10) under Case (1).

Case (2) $|V_{k-1}| = 2$.

Let r be the smallest integer such that $|V_r| = 2$. Clearly $r \le k-1$, $|V_i| \ge 3$, $1 \le i \le r-1$ and $|V_i| = 2$, $r \le i \le k-1$. As before, in G we can start with a vertex x_r in V_r and collect $(r-1)$ vertex disjoint paths M_i of length 2, $2 \le i \le r$ by choosing x_i to be different from y_i and z_i. Note that this choice is possible since $|V_i| \ge 3$, $1 \le i \le r-1$.

Subcase (2a) $r \le k-2$.

From the construction of V_i and the definition of r, note that $\bigcup_{\ell=r}^{k} V_\ell$ is a 2-independent set in $\overline{G}$. Now we provide a 2-path-colouring of $\overline{G}$as follows:

- Colour the vertices of $V(M_i)$ with colour i, $2 \le i \le r$.

- Colour the vertices of $\bigcup_{\ell=r}^{k} V_\ell - \{x_r\}$ with colour $(r+1)$.

- Colour the remaining $\alpha = p - 3(r-1) - |\bigcup_{\ell=r}^{k} V_\ell| + 1$ vertices with $\lceil \frac{\alpha}{2} \rceil$ colours.

Using the facts that $|\bigcup_{\ell=r}^{k-1} V_\ell| = (k-r)2$ and $|V_k| = 1$ or 2, note that $\alpha = p - r - 2k + 2$ or $p - r - 2k + 3$. Thus

$$\chi_2(\overline{G}) \le r + \left\lceil \frac{\alpha}{2} \right\rceil \le r + \left\lceil \frac{p - r - 2k + 3}{2} \right\rceil \le \frac{p + r - 2k + 4}{2}.$$

Now since $r \le k - 2$, we have

$$\chi_2(\overline{G}) \le \frac{p - k + 2}{2}$$

and this proves (10) in Subcase (2a).

Subcase (2b) $r = k - 1$ and $|V_{k-2}| \ge 4$.

Consider the set $X = V(G) - \bigcup_{i=2}^{k-1}(M_i)$. Clearly X contains $|V_1| - 2$ vertices of V_1, at least one vertex of V_{k-2}, exactly one vertex of V_{k-1} and all the vertices of V_k. Thus $|X| \ge |V_1| + 1$. Since $|V_1|$ is the largest 2-independent set in G, it follows that there is a path M_k of length 2 in $G[X]$. Using the vertex disjoint paths $M_2, M_3, \ldots, M_k$ of G, once again it is easy to check that

$$\chi_2(\overline{G}) \le \frac{p - k + 2}{2}.$$

This proves (10) in Subcase (2b).

Subcase (2c) $r = k - 1$ and $|V_{k-2}| = 3$.

Let $V_{k-2} = \{u, v, w\}$, $V_{k-1} = \{a, b\}$ and $c \in V_k$. Note that $G[V_{k-1} \cup \{c\}]$ has a path P of length 2. Without loss of generality, assume that $(a, b) \in E(G)$ and $(b, c) \in E(G)$. We claim that there is a cycle C of length 4 in G on the set $V_{k-2} \cup V_{k-1} \cup V_k$ which does not involve at least one vertex of V_{k-2}.

If V_{k-2} is independent in G, then it follows that each of a, b, c is joined to at least 2 vertices of V_{k-2}. In this case, trivially we have a cycle C of length 4 without involving at least one vertex of V_{k-2}. Now suppose that V_{k-2} is not independent in G. Without any loss of generality, let us assume that $(u, v) \in E(G)$. From the definition of V_{k-2}, it follows that every vertex of $V_{k-1} \cup V_k$ must be joined to at least one of the two vertices u and v. If a and c have a common neighbour in V_{k-2}, then our claim is easily proved. Thus without loss of generality, we assume that (a, u) and (c, v) are edges of G. Now if $(b, u) \in E(G)$, then $\{u, v, c, b\}$ forms a cycle of length 4 without involving w of V_{k-2}. Similarly, if $(b, v) \in E(G)$, then $\{u, v, b, a\}$ is the required cycle.

Let C be a cycle of length 4 on the set $V_{k-2} \cup V_{k-1} \cup V_k$ which does not involve a vertex x_{k-2} of V_{k-2}. Start with x_{k-2} of V_{k-2} and collect vertex disjoint paths M_i of length 2 in G, for $2 \le i \le k-2$ by choosing x_i to be different from y_i and z_i. Now provide a 2-path-colouring of $\overline{G}$ as follows:

- Colour the vertices of $V(M_i)$ with colour i, $2 \le i \le k-2$.
- Colour the vertices of C with colour $k-1$.
- Colour the remaining $(p-3k+5)$ vertices with $\left\lceil \frac{p-3k+5}{2} \right\rceil$ colours.

Thus $\chi_2(\overline{G}) \le k-2+\left\lceil \frac{p-3k+5}{2} \right\rceil \le \frac{p-k+2}{2}$ establishing (10) in this subcase. Combining (9) and (10), we have

$$\chi_2(G) + 2\chi_2(\overline{G}) \le p+2.$$

Similarly, reversing the roles of G and $\overline{G}$, we have

$$\chi_2(\overline{G}) + 2\chi_2(G) \le p+2.$$

Combining the above two inequalities, we have

$$\chi_2(G) + \chi_2(\overline{G}) \le \frac{2p+4}{3}.$$

This completes the proof Theorem 5. □

It is easy to verify that the inequality (8) is sharp for $p \le 9$. For $p \ge 10$, it seems that $\frac{2p+4}{3}$ is a weak upper bound for $\chi_2(G) + \chi_2(\overline{G})$.

Using (8) we can easily arrive at the following inequality: $\chi_2(G) \cdot \chi_2(\overline{G}) \le \left(\left\lfloor \frac{2p+4}{3} \right\rfloor\right)^2$.

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