Almost parity graphs and claw-free parity graphs

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ABSTRACT. A graph G is well-covered if every maximum independent set of vertices of G has the same cardinality. A graph G_1 is an almost well-covered graph if it is not well-covered, but $G_1 \setminus \{v\}$ is well-covered, $\forall v \in V(G_1)$. Similarly, a graph H is a parity graph if every maximal independent set of vertices of H has the same parity and a graph H_1 is an almost parity graph if H_1 is not a parity graph but $H_1 \setminus \{h\}$ is a parity graph, $\forall h \in V(H_1)$. Here we will give a complete characterization of almost parity graphs. We also prove that claw-free parity graphs must be well-covered.

1 Introduction

A graph G is *well-covered* if every maximal independent set of vertices of G has the same cardinality. These graphs were introduced by Plummer [8] in 1970. Although the recognition problem of well-covered graphs in general is Co-NP-complete [1,4,10], it is polynomial for some classes of graphs, for instance, claw-free [12]. The reader is referred to Plummer [9], and more recently, Hartnell [6] for survey articles and further references to work on

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well-covered graphs. Caro, Ellingham and Ramey [2] defined a *minimal non-well-covered graph* as a graph that is not well-covered but once any vertex and its neighbourhood is removed it becomes well-covered. On the other hand, a graph T belongs to the class W_2 when it is well-covered and $T \setminus \{t\}$ is well-covered, $\forall t \in V(T)$. W_2 graphs were introduced by Staples [11] and studied further by Pinter [7]. In this paper, we introduce a concept similar to minimal non-well-covered and W_2 graphs. In particular we call a graph G_1 an *almost well-covered graph* if it is not well-covered, but $G_1 \setminus \{v\}$ is well-covered, $\forall v \in V(G_1)$. A graph H is a *parity graph* if every maximal independent set of vertices of H has the same parity and a graph H_1 is an *almost parity graph* if H_1 is not a parity graph but $H_1 \setminus \{h\}$ is a parity graph, $\forall h \in V(H_1)$. Caro [3] proved that the recognition problem of well-covered graphs is Co-NP-complete even for parity graphs. Finbow and Hartnell [5] gave a characterization of parity graphs with girth > 5. In section 2, we will give a complete characterization of almost parity graphs and, as a consequence, we also prove that $K_{1,2}$ is the unique almost well-covered graph. In section 3 we will prove that a claw-free parity graph must, in fact, be well-covered.

2 Characterization of almost parity graphs

First of all, we will prove (Lemma 1) that an almost parity graph must be bipartite. Theorem 1 will give the complete characterization of graphs in this family.

Lemma 1. *Let G be an almost parity graph and I_1 and I_2 be two maximal independent sets of G with different parity. Then:*

1) $I_1 \cap I_2 = \emptyset$, *and*

2) $G = I_1 \cup I_2$.

Proof:

1) Suppose $I_1 \cap I_2 \neq \emptyset$, and let $|I_1| < |I_2|$.

   $I_1 \cap I_2 \neq \emptyset \Rightarrow \exists g \in I_1$ and $g \in I_2$. Let $h \in N(g)$, so $h \notin I_1$, and $h \notin I_2$. Then I_1 and I_2 are maximal independent sets in $G \setminus \{h\}$ of different parity, contradiction.

2) $G \subseteq I_1 \cup I_2$.

   Suppose that there exists a vertex g such that $g \in V(G)$, but $g \notin I_1$ and $g \notin I_2$. There are vertices in the neighbourhood of g that are in I_1 and I_2. Then the sets I_1 and I_2 are maximal independent sets in $G \setminus \{g\}$ of different parity, contradiction.

In the following Theorem, recall that for $v \in V(G)$, $N(v)$ represents the set of vertices adjacent to v and $\overline{N(v)}$ those not adjacent to v.

Theorem 1. *G is an almost parity graph $\iff G = K_{r,s}$ with $r \equiv s+1 \pmod 2$.*

Proof:

1) It is easy to see that the graphs $K_{r,s}$ with $r \equiv s+1 \pmod 2$ are almost parity graphs.

2) Let G be an almost parity graph. By Lemma 1, G must be bipartite.

   Suppose that G is not complete. Let I_1 and I_2 be maximal independent sets of different parity in G. So by Lemma 1, $G = I_1 \cup I_2$. Since G is not complete, there must be a vertex $v \in I_1$, say, such that v is not adjacent to all vertices of I_2. Let $I = \{v\} \cup (\overline{N(v)} \cap I_2)$, $I \subseteq I_3$, with I_3 maximal and independent. Now,

   a) $I_1 \cap I_3 \supset \{v\}$, so I_1 and I_3 must have the same parity (by Lemma 1).

   b) $I_2 \cap I_3 \supset \overline{N(v)} \cap I_2 \neq \emptyset$, so I_2 and I_3 must also have the same parity.

   Then I_1 and I_2 have the same parity, which is a contradiction.

Corollary 1. *The graph $G = K_{1,2}$ is the unique almost well-covered graph.*

Proof: First we will prove that the class of almost well-covered graphs is in the class of almost parity graphs. Suppose that there is an almost well-covered graph G that it is not almost parity. So, one of the following two possibilities occur:

1) G is a parity graph and not well-covered. Then there exist two maximal independent sets I_1 and I_2 of the same parity with $|I_1| < |I_2|$. But removing a vertex $i \in I_2 \setminus I_1$, we obtain the set $I_2 \setminus \{i\} \subseteq I$, with I maximal and independent which implies $|I| > |I_1|$. Hence $G \setminus \{i\}$ is not well-covered, which is a contradiction.

2) G is not a parity graph, but $\exists g \in G$ such that $G \setminus \{g\}$ is not a parity graph. So $G \setminus \{g\}$ is not well-covered, contradiction.

Therefore, we need only to look for almost well-covered graphs in the family $K_{r,s}$, with $r \equiv s+1 \pmod 2$.

a) The family $K_{1,s}$ is not almost well-covered for $s > 3$.

b) Suppose that there is a $G = K_{r,s}$, $r \geq 2$, $r \leq s$, $r \equiv s+1 \pmod 2$, almost well-covered, so when we delete one vertex of the set I_r, we will have a maximal independent set of cardinality $(r-1)$ and another one with cardinality s, with $(r-1) < s$, and the resulting graph is not well-covered, contradiction.

3 Parity claw-free graphs

Tankus and Tarsi [12] proved, recently, that the recognition problem of well-covered graphs is polynomial for the class of claw-free graphs. Now, we will prove that well-covered graphs are the unique parity, claw-free graphs.

Theorem 2. *If G is a claw-free parity graph, then G is well-covered.*

Proof: Suppose that there is a claw-free graph G,which is a parity graph, but is not well-covered. Then there are maximal independent sets I_1, I_2 of different sizes in G, where, say $|I_1| < |I_2|$.

Let J_1 and J_2 be the following sets:

$$J_1 = I_1 \setminus (I_1 \cap I_2),$$

and

$$J_2 = I_2 \setminus (I_1 \cap I_2).$$

1) Now consider the subgraph H induced by $J_1 \cup J_2$. $H = \langle J_1 \cup J2 \rangle$ is a bipartite graph and claw-free, so H can have only even (length) paths, odd paths and even cycles.

2) Since $|J_1| < |J_2|$, H must have at least one even path P with one more vertex in J_2 than J_1.

Let I_3 be the set: $I_3 = (I_1 \setminus (P \cap I_1)) \cup (P \cap I_2)$.

First observe that $|I_3| = |I_1| + 1$, and I_3 is also independent.

If I_3 were not maximal, there would be a vertex w adjacent to a vertex of $P \cap I_1$, but not adjacent to any vertex of $I_1 \setminus (P \cap I_1)$ nor to any vertex of $P \cap I_2$. But since G is claw-free, this is impossible. Hence I_3 is a maximal independent set, contradicting the fact that G is a parity graph.

4 Further remarks and conclusions

A graph G is a *Z_m-well-covered graph*, see [2,3], if $|I_1| \equiv |I_2| \pmod m$, for all maximal independent sets I_1 and I_2 in $V(G)$. Similarly, a graph H is an *almost Z_m-well-covered graph* if it is not Z_m-well-covered, but $H \setminus \{x\}$ is Z_m-well-covered for all $x \in V(G)$. Using the same arguments as in the proof of Lemma 1, we prove that an almost Z_m-well-covered graph must be bipartite and using the idea of 2 of Theorem 1, it also must be complete. Now, if there exists an almost Z_m-well-covered graph H, let I_1 and I_2 be two maximal independent sets of $V(H)$, with $|I_1| \not\equiv |I_2| \pmod m$. Without loss of generality, let $|I_1| < |I_2|$, so $|I_2| - |I_1| = mr + k$, with $r \in N$, $1 \leq k \leq (m-1)$. Then if $1 \leq k \leq (m-2)$, when we remove a vertex of I_1, we will have a maximal independent set J_1, with $|J_1| = |I_1| - 1$, and $|I_2| - |J_1| = mr + (k+1)$. If $k = m-1$, when we remove a vertex of I_2, we will have a maximal independent set J_2, with $|J_2| = |I_2| - 1$, and $|J_2| - |I_1| = mr + (k-1)$. Thus we have proved the following general result:

Theorem 3. *There are no almost Z_m-well-covered graphs, for $m \geq 3$.*

Also using the same idea as in the proof of Theorem 2, we can establish the following result:

Theorem 4. *If G is a claw-free Z_m-well-covered graph, then G is well-covered.*

We conclude by observing that the result above is not true for $K_{1,r}$-free graphs, for $r > 3$, since $K_{1,3}$ is a parity graph and it is not well-covered.

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