Hypergraph Designs

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Dedicated to Anne Penfold Street.

Abstract

Let H_i be the 3-uniform hypergraph or 4 vertices with i hyperedges. In this paper we settle the existence of H_3-hypergraph designs of index λ, obtaining simple H_3-hypergraph designs when $\lambda = 2$, and obtaining a new proof of their existence when $\lambda = 1$. The existence of simple H_2-hypergraph designs of index λ is completely settled, as is the spectrum of H_2-hypergraph designs of index λ.

1 Introduction

In this paper we consider a generalization of graph designs. A G-design of a graph M is a partition of the edge-set of M for which each element of the partition induces a copy of the graph G. Let λK_n denote the multigraph on n vertices in which each pair of vertices is joined by exactly λ edges. A G-design of *order* n and *index* λ is a G-design of λK_n. The λ*-spectrum* of G is the set of integers n for which there exists a G-design of order n and index λ. Finding the spectrum of various graphs is a well studied question that began with Kirkman's proof in 1849 [11] of the result that K_3-designs of order n (Steiner triple systems) exist if and only if $n \equiv 1$ or $3 \pmod 6$. More recently, the spectrum of G has been found in the cases where G is K_4 [9], K_5 [9], a star [18], a path [19], and most cases where G is a graph with at most 5 vertices [2, 4]. Substantial progress has also been made in the case where G is a cycle [3, 5, 10, 12], but solving this case may be very difficult. See also [6, 16].

A *hypergraph* of *order* n is an ordered pair (V, E) where V is a set of n vertices, and E is a collection of subsets of V. Each element e of E is

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said to be a *hyperedge* of size $|e|$. The hypergraph $H = (V, E)$ is said to be x*-uniform* if each hyperedge in E has size x; throughout this paper, we let H^x denote the property that H is x-uniform. Let λK_n^x denote the *complete* x*-uniform hypergraph* of *order* n and *index* λ (so the hyperedge collection of λK_n^x contains each x-element subset of V exactly λ times).

It is natural for one to ask the same questions of "hypergraph designs" that have been considered for graph designs. An H^x*-hypergraph design* of a hypergraph M^x is a partition of the hyperedges of M^x, in which each element of the partition induces a copy of H^x. An H^x-hypergraph design of *order* n and *index* λ is an H^x-hypergraph design of λK_n^x. The λ*-spectrum* of H^x is the set of integers n for which there exists an H^x-hypergraph design of order n and index λ. An H^x-hypergraph design of index λ is said to be *simple* if no copy of H_x appears more than once. Throughout what follows, let H_e denote the unique (up to isomorphism) 3-uniform hypergraph on 4 vertices containing e hyperedges.

The spectrum of H_4-hypergraph designs was solved in 1960 by Hanani [7] in the guise of Steiner Quadruple Systems (notice that $H_4 = K_4^3$). Hanani showed that Steiner Quadruple Systems of order n exist if and only if $n \equiv 2$ or 4 (mod 6) or $n = 1$. A year later he settled [8] the λ-spectrum for H_4-hypergraph designs for all λ. More recently, Bermond Germa and Sotteau [1] solved the spectrum problem for H_2-hypergraph designs and for H_3-hypergraph designs.

Theorem 1.1 ([1]) *There exist H_3-hypergraph designs of order n and index 1 if and only if $n \equiv 0, 1$ or 2 (mod 9). There exist H_2-hypergraph designs of order n and index 1 if and only if $n \equiv 0, 1$ or 2 (mod 4).*

In their proof, they use the fact that Kirkman triple systems exist of all orders $n \equiv 3$ (mod 6), and remark that it would be interesting to find a proof that avoids using this heavy machinery.

In this paper we obtain such a proof, and then use it to obtain some new results. In Section 3 we provide this new proof, using it to obtain two H_3-hypergraph designs that have no copies of H_3 in common, for all possible orders (see Theorem 3.1). Of course, combining two such H_3-hypergraph designs of order n produces an H_3-hypergraph design of order n and index 2 that is simple. In Section 4 we obtain the λ-spectrum for H_3-hypergraph designs for all λ (see Theorem 4.1). Finally, in Section 5 we obtain a new proof of the existence of H_2-hypergraph designs of index 1 (this was first proved by Mouyart [14]), which we then use to solve the λ-spectrum problem for H_2-hypergraph designs (see Corollary 5.2), and to obtain necessary and sufficient conditions for the existence of a simple H_2-hypergraph design of index λ.

For any design theoretical terms not defined in this paper see [13].

2 Some Preliminary Constructions

Since the following constructions will need to refer to specific copies of H_3, we make the following definition. Let $H_3(a,b,c,d)$ be the hypergraph with vertex set $\{a,b,c,d\}$ and hyperdedge set $\{\{a,b,c\},\{a,b,d\},\{a,c,d\}\}$.

Lemma 2.1 *There exist two H_3-hypergraph designs of order 9 and index 1 that have no copies of H_3 in common.*

Proof: Let V = {1,2,3,4,5,6,7,8,9}. Let E_1 = {{1,5,6,9},{2,6,7,9},{3,7,1,9}, {4,1,2,9},{5,2,3,9}, {6,3,4,9}, {7,4,5,9},{1,2,8,9},{2,3,8,9},{3,4,8,9}, {4,5,8,9},{5,6,8,9},{6,7,8,9}, {7,1,8,9},{1,3,4,8}, {2,4,5,8},{3,5,6,8}, {4,6,7,8}, {5,7,1,8},{6,1,2,8},{7,2,3,8},{1,4,6,7}, {2,5,7,1},{3,6,1,2}, {4,7,2,3}, {5,1,3,4},{6,2,4,5},{7,3,5,6}}, and
E_2 = {{5,9,1,4}, {6,1,2,4}, {7,2,5,4}, {8,5,6,4}, {9,6,7,4}, {1,7,8,4}, {2,8,9,4}, {5,6,3,4}, {6,7,3,4}, {7,8,3,4}, {8,9,3,4}, {9,1,3,4}, {1,2,3,4}, {2,5,3,4}, {5,7,8,3}, {6,8,9,3}, {7,9,1,3}, {8,1,2,3}, {9,2,5,3}, {1,5,6,3}, {2,6,7,3}, {5,8,1,2}, {6,9,2,5}, {7,1,5,6}, {8,2,6,7}, {9,5,7,8}, {1,6,8,9}, {2,7,9,1}}. □

Lemma 2.2 *There exist two partial triple systems of order 11 with no common triples, each of which has leave consisting of two vertex disjoint 5-cycles.*

Proof: Let V = {0,1,2,3,4,5,6,7,8,9,A}. Let B_1 = {{A,0,1},{A,2,3}, {A,4,5}, {A,6,7},{A,8,9},{0,3,6}, {0,4,7},{0,5,9},{1,2,7},{1,4,8},{1,5,6}, {2,5,8},{2,6,9}, {3,4,9},{3,7,8}}, and B_2 = {{A,2,1} {A,4,3} {A,6,5} {A,8,7} {A,0,9} {2,3,8}, {2,6,7}, {2,5,9}, {1,4,7}, {1,6,0}, {1,5,8}, {4,5,0}, {4,8,9}, {3,6,9}, {3,7,0}}.
Then the leave of each is {(2,4,6,8,0), (1,3,5,7,9)}. □

Lemma 2.3 *There exists an H_3-hypergraph design of order 4 and index 3.*

Proof: Let V = {0,1,2,3}. Let E = {{0,1,2,3},{1,2,3,0},{2,3,0,1},{3,0,1,2}}. □

Lemma 2.4 *There exists an H_3-hypergraph design of order 5 and index 3.*

Proof: Let V = {0,1,2,3,4}. Let E = {{0,1,2,3},{1,2,3,4},{2,3,4,0},{3,4,0,1}, {4,0,1,2}, {0,2,4,1}, {1,3,0,2},{2,4,1,3},{3,0,2,4},{4,1,3,0}}}. □

Lemma 2.5 *There exists an H_3-hypergraph design of order 7 and index 3.*

Proof: Let V = {0,1,2,3,4,5,6}.
Let E = {{0,1,2,4},{1,2,3,5},{2,3,4,6}, {3,4,5,0},{4,5,6,1},

{5,6,0,2},{6,0,1,3},{0,1,2,4},{1,2,3,5}, {2,3,4,6},{3,4,5,0},
{4,5,6,1},{5,6,0,2},{6,0,1,3},{0,1,4,5}, {1,2,5,6},{2,3,6,0},
{3,4,0,1},{4,5,1,2},{5,6,2,3},{6,0,3,4}, {0,1,3,5},{1,2,4,6},
{2,3,5,0},{3,4,6,1},{4,5,0,2},{5,6,1,3}, {6,0,2,4},{0,1,2,3},
{1,2,3,4},{2,3,4,5}, {3,4,5,6},{4,5,6,0},{5,6,0,1},{6,0,1,2}}. □

We will need the following small design.

Lemma 2.6 *There exist a partial triple system of order 5 whose leave consists of a 4-cycle.*

Proof: Let $V = \{0,1,2,3,4\}$ and $T = \{\{0,1,3\},\{0,2,4\}\}$. Then the leave of (V,T) is the 4-cycle $(1,2,3,4)$. □

We will also need the following well-known result, a proof of which we include for completeness to verify that the result is not "heavy machinery."

Lemma 2.7 *For all $n \equiv 1$ or 3 (mod 6) there exist two Steiner triple systems (V,T_1) and (V,T_2) of order n that have no triples in common. Furthermore, if $n \equiv 3$ (mod 6) then T_i contains a parallel class $\pi_i (1 \leq i \leq 2)$ such that no edge (pair) is in a triple in both π_1 and π_2.*

Proof: If $n \equiv 3$ (mod 6), we use the Bose Construction (see [13]) as follows. Let $V = \mathbb{Z}_{2x+1} \times \mathbb{Z}_3$ and let $(V,\cdot)$ be a symmetric idempotent quasigroup of order $2x+1$. Let

$$\begin{aligned}
\pi_1 &= \{\{(a,1),(a,2),(a,3)\} \mid a \in \mathbb{Z}_{2x+1}\},\\
T_1 &= \pi_1 \cup \{\{(a,i),(b,i),(a\cdot b, i+1)\} \mid 0 \leq a < b \leq 2x, i \in \mathbb{Z}_3\},\\
\pi_2 &= \{\{(a,1),(a+1,2),(a+2,3)\} \mid a \in \mathbb{Z}_{2x+1}\}, \text{ and}\\
T_2 &= \pi_2 \cup \{\{(a,i),(b,i),(\alpha_i(a\cdot b), i-1)\} \mid 0 \leq a < b \leq 2x, i \in \mathbb{Z}_3\},
\end{aligned}$$

reducing the second component modulo 3, where $\alpha_1(c) = c+2$ and $\alpha_2(c) = \alpha_3(c) = c-1$ for each $c \in \mathbb{Z}_{2x+1}$. The Skolem Construction (see [13]) can be used similarly to handle the case where $n \equiv 1$ (mod 6). □

3 The 2-Spectrum of simple H_3-hypergraph designs

Let $H_3[V]$ denote an H_3-hypergraph design on the vertex set V.

Theorem 3.1 *H_3-hypergraph designs, of order n and index 2 with no repeated copies of H_3 exist if and only if $n \equiv 0, 1$ or 2 (mod 9).*

Remark: The following proof actually obtains two H_3-hypergraph designs of order n and index 1 that have no copies of H_3 in common.

Proof: To construct these H_3-hypergraph designs, we consider several cases in turn.

Case 1: $n \equiv 0 \pmod 9$

We will show the existence of H_3-hypergraph designs of order $n \equiv 0 \pmod 9$ and index 2 with no repeated copies of H_3 by induction. Let $n = 18z + 9$ or $18z$ for $z \geq 0$ or $z \geq 1$ respectively. The proof is by induction on z.

For $z = 0$, the existence of an H_3-hypergraph design of order 9 and index 2 with no repeated copies of H_3 is given by Lemma 2.1. So now, let $k \geq 1$ and assume that there exists an H_3-hypergraph design of order $18z + 9$ and index 2 with no repeated copies of H_3 for all z, $0 \leq z < k$.

The proof of this case is completed by showing there exist H_3-hypergraph designs of order $n = 18k$ and $n = 18k+9$; these two values of n are handled separately.

Construction I: to construct an H_3-hypergraph design of order $n = 18k$.

Let $X = \{1, 2, \ldots, 9\}$, $Y = \{10, 11, \ldots, 18k\}$, and $N = X \cup Y$.

Let $H_3[X]$ be an H_3-hypergraph design of order $|X| = 9$ and index 2 with no repeated copies of H_3 (this exists by Lemma 2.1), and let $H_3[Y]$ be an H_3-hypergraph design of order $|Y| = n\text{-}9 = 18k\text{-}9 = 18(k\text{-}1)\text{+}9$ and index 2 with no repeated copies of H_3 (this exists by the induction hypothesis). Note that $|X| \equiv |Y| \equiv 3 \pmod 6$ so for $1 \leq i \leq 2$ let STS(X,B_{X_i}) denote a Steiner triple system of order 9 with vertex set X and block set B_{X_i} such that $B_{X_1} \cap B_{X_2} = \emptyset$ (see Lemma 2.7) and similarly let STS(Y,B_{Y_i}) denote a Steiner triple system of order $n - 9$ such that $B_{Y_1} \cap B_{Y_2} = \emptyset$.

Define $H_3[N]$ as follows:

- **Type 1:** if $H_3(a, b, c, d) \in H_3[X]$ then $H_3(a, b, c, d) \in H_3[N]$,
- **Type 2:** if $H_3(a, b, c, d) \in H_3[Y]$ then $H_3(a, b, c, d) \in H_3[N]$,
- **Type 3:** for $1 \leq i \leq 2$, if $a \in X$, and $\{b, c, d\} \in B_{Y_i}$ then $H_3(a, b, c, d) \in H_3[N]$, and
- **Type 4:** for $1 \leq i \leq 2$, if $a \in Y$, and $\{b, c, d\} \in B_{X_i}$ then $H_3(a, b, c, d) \in H_3[N]$.

Claim: $H_3[N]$ is an H_3-hypergraph design with vertex set N of order $n = 18k$ and index 2 with no repeated copies of H_3. To prove this we need to show every hyperedge, t, of K_n^3 is in exactly two different copies of $H_3 \in H_3[N]$. If $t = \{a, b, c\} \subset X$ then since $H_3[X]$ is an H_3-hypergraph design of index 2 with no repeated copies of H_3 there exist two different copies of H_3 in $H_3[X]$ containing t; so t is in two different copies of H_3 that are of Type 1 in $H_3[N]$.

If $t = \{a, b, c\} \subset Y$ then since $H_3[Y]$ is an H_3-hypergraph design of index 2 with no repeated copies of H_3 there exist two different copies of H_3 in

$H_3[Y]$ containing t; so t is in two different copies of H_3 that are of Type 2 in $H_3[N]$.
If $t = \{a,b,c\}$ where $a \in X$ and $\{b,c\} \subset Y$, then there exist $d_1, d_2 \in Y$ such that $\{b,c,d_i\} \in B_{Y_i}$, where $d_1 \neq d_2$ (since $B_{Y_1} \cap B_{Y_2} = \emptyset$); so t is in $H_3(a,b,c,d_i)$ which is of Type 3 in $H_3[N]$.
If $t = \{a,b,c\}$ where $a \in Y$ and $\{b,c\} \subset X$, then there exist $d_1, d_2 \in X$ such that $\{b,c,d_i\} \in B_{X_i}$, where $d_1 \neq d_2$ (since $(B_{X_1} \cap B_{X_2} = \emptyset)$; so t is in $H_3(a,b,c,d_i)$ which is of Type 4 in $H_3[N]$.
So every hyperedge, t, is in at least two different copies of $H_3 \in H_3[N]$. The total number of copies of H_3 in $H_3[N]$ is: $2\binom{9}{3}/3$ of Type 1; $2\binom{n-9}{3}/3$ of Type 2; $2\cdot 9\binom{n-9}{2}/3$ of Type 3; and $2(n-9)\binom{9}{2}/3$ of Type 4. So since every hyperedge is in at least two copies of $H_3 \in H_3[N]$, and since $H_3[N]$ contains $2\binom{9}{3}/3 + 2\binom{n-9}{3}/3 + 2\cdot 9\binom{n-9}{2}/3 + 2(n-9)\binom{9}{2}/3 = 2(n^3 - 3n^2 + 2n)/18 = 2\binom{n}{3}/3$ copies of H_3, every hyperedge is in exactly two copies of $H_3 \in H_3[N]$. So $H_3[N]$ is an H_3-hypergraph design with vertex set N of order n and index 2 with no repeated copies of H_3 as claimed.

Construction II: To construct an H_3-hypergraph design of order $n = 18k+9$.

Let $X = \{1,2,\ldots,9\}$, $Y = \{10,11,\ldots,18k+9\}$, and $N = X \cup Y$. Let $H_3[X]$ be an H_3-hypergraph design of order $|X| = 9$ and index 2 with no repeated copies of H_3 (this exists by Lemma 2.1), and let $H_3[Y]$ be an H_3-hypergraph design of order $|Y| = 18k$ and index 2 with no repeated copies of H_3 (this exists by Construction I). For $1 \leq i \leq 2$ let $STS(X, B_{X_i})$ be a Steiner triple system of order 9 such that $B_{X_1} \cap B_{X_2} = \emptyset$, and let π_i be a parallel class of the $STS(X, B_{X_i})$ such that π_1 and π_2 have no edge in common (see Lemma 2.7). Since $18k \equiv 0 \pmod 6$, there exist two maximal packings, (Y, B_{Y_i}) for $1 \leq i \leq 2$, of K_{18k} with triples on the vertex set Y such that: the leave in (Y, B_{Y_i}) is a 1-factor M_{Y_i} with $M_{Y_1} \cap M_{Y_2} = \emptyset$; and $B_{Y_1} \cap B_{Y_2} = \emptyset$ (using Lemma 2.7, delete a point $y \in Y$ from two STSs of order $18k+1$). Define $H_3[N]$ as follows:
Type 1: if $H_3(a,b,c,d) \in H_3[X]$ then $H_3(a,b,c,d) \in H_3[N]$,
Type 2: if $H_3(a,b,c,d) \in H_3[Y]$ then $H_3(a,b,c,d) \in H_3[N]$,
Type 3: for $1 \leq i \leq 2$, if $a \in X$ and $\{b,c,d\} \in B_{Y_i}$ then $H_3(a,b,c,d) \in H_3[N]$,
Type 4: for $1 \leq i \leq 2$, if $a \in Y$ and $\{b,c,d\} \in B_{X_i} \backslash \pi_i$ then $H_3(a,b,c,d) \in H_3[N]$, and
Type 5: for $1 \leq i \leq 2$ if $\{a,b,c\} \in \pi_i$ with $a < b < c$, and $\{d,e\} \in M_{Y_i}$ then each of $H_3(a,b,d,e)$,
$H_3(b,c,d,e)$, and $H_3(c,a,d,e)$ are in $H_3[N]$.

Claim: $H_3[N]$ is an H_3-hypergraph design with vertex set N of order $n = 18k+9$ and index 2 with no repeated copies of H_3.

As before, we consider each hyperedge, t, in turn. If $t = \{a,b,c\} \subset X$ then since $H_3[X]$ is an H_3-hypergraph design of index 2 with no repeated copies of H_3 there exist two different copies of H_3 in $H_3[X]$ containing t, so t is in two different copies of H_3 that are of Type 1 in $H_3[N]$. Similarly, if $t = \{a, b, c\} \subset Y$ then t is in two different copies of H_3 that are of Type 2 in $H_3[N]$.

If $t = \{a, b, c\}$ where $a \in X$ and $\{b, c\} \subset Y$ then for $1 \leq i \leq 2$: either there exists some $d_1, d_2 \in Y$ such that $\{b, c, d_i\} \in B_{Y_i}$, where $d_1 \neq d_2$ (since $B_{Y_1} \cap B_{Y_2} = \emptyset$), in which case $H_3(a, b, c, d_i)$ is of Type 3 in $H_3[N]$; or $\{b, c\} \in M_{Y_i}$ (this happens for at most one value of i, since $M_{Y_1} \cap M_{Y_2} = \emptyset$), in which case there exists some $\{d, e\} \subset X$, such that $\{a, d, e\} \in \pi_i$, so t is in a copy of Type 5 in $H_3[N]$. So $\{a, b, c\}$ occurs in two different copies in this case.

If $t = \{a, b, c\}$ where $a \in Y$ and $\{b, c\} \subset X$ then there exist $d_1, d_2 \in X$ with $d_1 \neq d_2$ such that $\{b, c, d_i\} \in B_{X_i}$. If $\{b, c, d_i\} \notin \pi_i$ then $H_3(a, b, c, d_i)$ is of Type 4 in $H_3[N]$; and otherwise, since M_{Y_i} is a 1-factor, there exists some $e \in Y$ such that $\{a, e\} \in M_{Y_i}$ (this happens for at most one value of i since π_1 and π_2 are edge-disjoint) so t is in a copy of H_3 that is of Type 5 in $H_3[N]$. Since π_1 and π_2 are edge-disjoint, $\{a, b, c\}$ occurs in two different copies in this case.

So every hyperedge in in at least two different copies of $H_3 \in H_3[N]$. The total number of copies of H_3 in $H_3[N]$ is $2\binom{9}{3}/3$ of Type 1; $2\binom{n-9}{3}/3$ of Type 2; $2(9\binom{n-9}{2} - (n-9)/2)/3$ of Type 3; $2(n-9)\left(\binom{9}{2}/3 - 3\right)$ of Type 4; and $2 \cdot 3 \cdot 3 \cdot (n-9)/2$ of Type 5. Since every hyperedge is in at least two copies of $H_3 \in H_3[N]$, and since $H_3[N]$ contains $2\binom{9}{3}/3 + 2\binom{n-9}{3}/3 + 2(9\binom{n-9}{2} - (n-9/2)/3 + 2(n-9)\left(\binom{9}{2}/3 - 3\right) + 2 \cdot 3 \cdot 3 \cdot (n-9)/2 = 2(n^3 - 3n^2 + 2n)/18 = 2\binom{n}{3}/3$ copies of H_3, every hyperedge is in exactly two copies of $H_3 \in H_3[N]$. So $H_3[N]$ is an H_3-hypergraph design with vertex set N of order n and index 2 with no repeated copies of H_3 as claimed. Therefore, there exists an H_3-design of order n and index 2 with no repeated copies of H_3 for all $n \equiv 0 \pmod 9$. This completes Case 1.

Case 2: $n \equiv 10 \pmod{18}$

Construction III: to construct an H_3-hypergraph design of order $n = 18k + 10$.

Let $X = \{1, 2, \ldots, 18k + 9\}$ and let $N = X \cup \{\infty\}$. Let $H_3[X]$ be an H_3-hypergraph design of order $|X| = 18k + 9$ and index 2 with no repeated copies of H_3; this was shown to exist in Case 1. For $1 \leq i \leq 2$ let $STS(X, B_{X_i})$ be a Steiner triple system of order $|X| = 18k + 9$ such that $B_{X_1} \cap B_{X_2} = \emptyset$ (see Lemma 2.7). Define $H_3[N]$ as follows:

Type 1: if $H_3(a, b, c, d) \in H_3[X]$ then $H_3(a, b, c, d) \in H_3[N]$, and

Type 2: for $1 \leq i \leq 2$ if $\{a, b, c\} \in B_{X_i}$ then $H_3(\infty, a, b, c) \in H_3[N]$.

Claim: $H_3[N]$ is an H_3-hypergraph design with vertex set N and order $n = 18k + 10$ and index 2 with no repeated copies of H_3.

As before, we consider each hyperedge, t, in turn. If $t = \{a, b, c\} \subset X$ then since $H_3[X]$ is an H_3-hypergraph design, of index 2 with no repeated copies of H_3, there exist two different copies of H_3 in $H_3[X]$ containing t; so t is in two different copies of H_3 that are of Type 1 in $H_3[N]$.

If $t = \{\infty, a, b\}$ then there exist $c_1, c_2 \in X$ such that $\{a, b, c_i\} \in B_{X_i}$ for $1 \leq i \leq 2$, with $c_1 \neq c_2$ since $B_{X_1} \cap B_{X_2} = \emptyset$, so t is in two different copies of H_3 of Type 2 in $H_3[N]$. So every hyperedge, t, is in at least two different copies of $H_3 \in H_3[N]$. The total number of copies of H_3 in $H_3[N]$ is $2\binom{n-1}{3}/3$ of Type 1 and $2\binom{n-1}{2}/3$ of Type 2. So since every hyperedge is in at least two different copies of $H_3 \in H_3[N]$, and $H_3[N]$ contains $2\binom{n}{3}/3$ copies of H_3, every hyperedge in in exactly two different copies of $H_3 \in H_3[N]$. So $H_3[N]$ is an H_3-hypergraph design with vertex set N of order n and index 2 with no repeated copies of H_3 as claimed.

Case 3: $n \equiv 11 \pmod{18}$

Construction IV: to construct an H_3-hypergraph design of order $n = 18k + 11$.

Let $X = \{1, 2, \ldots, 18k + 9\}$ and let $N = X \cup \{\infty_1, \infty_2\}$. Let $H_3[X]$ be an H_3-hypergraph design of order $|X| = 18k + 9$; this was shown to exist in Case 1. For $1 \leq i \leq 2$ let $STS(X, B_{X_i})$ be a Steiner triple system of order $|X| = 18k + 9$, such that $B_{X_1} \cap B_{X_2} = \emptyset$ and in which π_i is a parallel class of triples in B_i such that π_1 and π_2 have no edges in common (see Lemma 2.7). Define $H_3[N]$ as follows:

Type 1: if $H_3(a, b, c, d) \in H_3[X]$ then $H_3(a, b, c, d) \in H_3[N]$,

Type 2: for $1 \leq i \leq 2$ if $\{a, b, c\} \in B_{X_i} \backslash \pi_i$ then $H_3(\infty_1, a, b, c)$ and $H_3(\infty_2, a, b, c) \in H_3[N]$, and

Type 3: for $1 \leq i \leq 2$ if $\{a, b, c\} \in \pi_i$ with $a < b < c$ then $H_3(a, b, \infty_1, \infty_2)$, $H_3(b, c, \infty_1, \infty_2)$, and
$H_3(c, a, \infty_1, \infty_2) \in H_3[N]$.

Claim: $H_3[N]$ is an H_3-hypergraph design with vertex set N of order $18k + 11$ and index 2 with no repeated copies of H_3.

As before, we consider each hyperedge, t, in turn. If $t = \{a, b, c\} \subset X$, then since $H_3[X]$ is an H_3-hypergraph design of index 2 with no repeated copies of H_3, there exist two different copies of H_3 in $H_3[X]$ containing t; so t in in two different copies of H_3 that are of Type 1 in $H_3[N]$.

If $t = \{\infty_j, a, b\}$ ($j \in \{1, 2\}$), then there exist $c_1, c_2 \in X$ such that $\{a, b, c_i\} \in B_{X_i}$ for $1 \leq i \leq 2$ with $c_1 \neq c_2$ since $B_{X_1} \cap B_{X_2} = \emptyset$. If $\{a, b, c_i\} \notin \pi_i$, then t is in an H_3 of Type 2 in $H_3[N]$. If $\{a, b, c_i\} \in \pi_i$, then t is in an H_3 of Type 3 in $H_3[N]$. Since π_1 and π_2 are edge-disjoint $\{\infty_j, a, b\}$ occurs in different copies in this case.

If $t = \{\infty_1, \infty_2, a\}$ then for $1 \leq i \leq 2$ there exists some $\{b_i, c_i\} \subset X_i$ such that $\{a, b_i, c_i\} \in \pi_i$, with $\{b_1, c_1\} \cap \{b_2, c_2\} = \emptyset$ since π_1 and π_2 are edge-disjoint; so t is two different copies of H_3 of Type 3 in $H_3[N]$.

So every hyperedge, t, is in at least two different copies of $H_3 \in H_3[N]$. The total number of copies of H_3 in $H_3[N]$ is $2\binom{n-2}{3}/3$ of Type I, $2 \cdot 2((\binom{n-2}{2}/3) - (n-2)/3)$ of Type 2, and $2 \cdot 3((n-2)/3)$ of Type 3. So since every hyperedge is in at least one copy of $H_3 \in H_3[N]$, and since the total number of copies of H_3 is $2\binom{n}{3}/3$, every hyperedge is in exactly two copies of $H_3 \in H_3[N]$. So $H_3[N]$ is an H_3-hypergraph design with vertex set N of order n and index 2 with no repeated copies of H_3 as claimed.

Case 4: $n \equiv 1 \pmod{18}$

Construction V: to construct an H_3-hypergraph design of order $n = 18k+1$.

Let $X = \{1,2,\ldots,18k-9\}$, $Y = \{18k-9, 18k-8, \ldots, 18k+1\}$ (so $X \cap Y = \{18k-9\}$), and $N = X \cup Y$. Let $H_3[X]$ be an H_3-hypergraph design of order $|X| = 18k-9$ and index 2 with no repeated copies of H_3; this was show to exist in Case 1. Let $H_3[Y]$ be an H_3-hypergraph design of order $|Y| = 11$ and index 2 with no repeated copies of H_3; this was shown to exist in Case 3. For $1 \le i \le 2$: let $STS(X, B_{X_i})$ be a Steiner triple system of order $|X| = 18k-9$ such that $B_{X_1} \cap B_{X_2} = \emptyset$ (see Lemma 2.7); let α_i be the set of all triples in B_{X_i} that contain the vertex $18k-9$; and let $L_i = \{\{j,\ell\} \mid \{18k-9, j, \ell\} \in B_{X_i}\}$ (note $L_1 \cap L_2 = \emptyset$). For $1 \le i \le 2$ let $TS(Y, B_{Y_i})$ be a partial triple system of order 11 such that $B_{Y_1} \cap B_{Y_2} = \emptyset$ with leave $M_1 = \{(18k-8+s, 18k-7+s, 18k-6+s, 18k-5+s, 18k-4+s) \mid s \in \{0,5\}\}$ (this exists by Lemma 2.2). Define $H_3[N]$ as follows:

Type 1: if $H_3(a,b,c,d) \in H_3[X]$ then $H_3(a,b,c,d) \in H_3[N]$,
Type 2: if $H_3(a,b,c,d) \in H_3[Y]$ then $H_3(a,b,c,d) \in H_3[N]$,
Type 3: for $1 \le i \le 2$, if $\{a,b,c\} \in B_{X_i} \backslash \alpha_i$ then $H_3(y,a,b,c) \in H_3[N]$, for each $y \in Y \backslash \{18k-9\}$,
Type 4: for $1 \le i \le 2$, if $\{a,b,c\} \in B_{Y_i}$ then $H_3(x,a,b,c) \in H_3[N]$, for each $x \in X \backslash \{18k-9\}$, and
Type 5: for $1 \le i \le 2$, if $(a,b,c,d,e) \in M_i$ with $a > b > c > d > e$ then $\{H_3(a,b,j,\ell), H_3(b,c,j,\ell), H_3(c,d,j,\ell), H_3(d,e,j,\ell), H_3(e,a,j,\ell)\} \subseteq H_3[N]$, for each $\{j,k\} \in L_i$.

Claim: $H_3[N]$ is an H_3-hypergraph design with vertex set N of order $n = 18k+1$ and index 2 with no repeated copies.

As before, we consider each hyperedge, t, in turn. If $t = \{a,b,c\} \subset X$ or $t = \{a,b,c\} \subset Y$, then t is in two different copies of $H_3 \in H_3[N]$ that is of Type 1 or Type 2 respectively.

If $t = \{a,b,c\}$, where $\{b,c\} \subset X$ and $a \in Y \backslash \{18k-9\}$, then there exist $d_1, d_2 \in X$ such that $\{b,c,d_i\} \in B_{X_i}$ for $1 \le i \le 2$. If $d_i = 18k-9$, then there exists some 5-cycle in M_i containing a, so t is in a copy of H_3 of Type 5 in $H_3[N]$. Otherwise t is in an H_3 of Type 3 in $H_3[N]$. These copies are different since $B_{X_1} \cap B_{X_2} = \emptyset$.

If $t = \{a,b,c\}$, where $a \in X \backslash \{18k-9\}$ and $\{b,c\} \subset Y$ (possibly $18k-9 \in \{b,c\}$), then: either there exists $d_1, d_2 \in Y$ such that $\{b,c,d_i\} \in B_{Y_i}$ for $1 \le i \le 2$, with $d_1 \ne d_2$ since $B_{Y_1} \cap B_{Y_2} = \emptyset$, in which case t is in two different copies of H_3 of Type 4 in $H_3[N]$; or $\{b,c\}$ is an edge in a 5-cycle in M_1 and in M_2, in which case there exist $g_1, g_2 \in X$ such that $\{a, g_i\} \in L_i$, for $1 \le i \le 2$ with $g_1 \ne g_2$ since $L_1 \cap L_2 = \emptyset$, so t is in two different copies of H_3 of Type 5 in $H_3[N]$.

So every hyperedge t is in at least two different copies of $H_3 \in H_3[N]$. The total number of copies of H_3 in $H_3[N]$ is $2\binom{n-10}{3}/3$ of Type 1, $2\binom{11}{3}/3$ of Type 2, $2(n-11)((\binom{11}{2}-10)/3)$ of Type 3, $2\cdot 10((\binom{n-10}{2}/3)-(n-11)/2)$ of Type 4, and $2\cdot 2\cdot 5(n-11)/2$ of Type 5. So $H_3[N]$ contains $2\binom{n}{3}/3$ copies of H_3. Therefore $H_3[N]$ is an H_3-hypergraph design with vertex set N of order n of index 2 with no repeated copies of H_3 as claimed.

Case 5: $n \equiv 2 \pmod{18}$

Construction VI: to construct an H_3-hypergraph design of order $n = 18k+2$.

Let $X = \{1, 2, \ldots, 18k+1\}$ and let $N = X \cup \{\infty\}$. Let $H_3[X]$ be an H_3-hypergraph design of order $|X| = 18k+1$ and index 2 with no repeated copies of H_3; this was shown to exist in Case 4. For $1 \le i \le 2$ let $STS(X, B_{X_i})$ be a Steiner triple system of order $|X| = 18k+1$ such that $B_{X_1} \cap B_{X_2} = \emptyset$ (see Lemma 2.7). Define $H_3[N]$ as follows:

Type 1: if $H_3(a,b,c,d) \in H_3[X]$ then $H_3(a,b,c,d) \in H_3[N]$,

Type 2: for $1 \le i \le 2$, if $\{a,b,c\} \in B_{X_i}$ then $H_3(\infty,a,b,c) \in H_3[N]$.

Claim: $H_3[N]$ is an H_3-hypergraph design with vertex set N of order $n = 18k+2$ and index 2 with no repeated copies of H_3.

As before, we consider each hyperedge, t, in turn. If $t = \{a,b,c\} \subset X$ then t is in two different copies of H_3 that are of Type 1 in $H_3[N]$, and if $t = \{\infty, a, b\}$ then there exist $c_1, c_2 \in X$ such that $\{a,b,c_i\} \in B_{X_i}$, $1 \le i \le 2$ so t is in an H_3 of Type 2 in $H_3[N]$. The copies containing t are different since $B_{X_1} \cap B_{X_2} = \emptyset$. So every hyperedge, t, is in at least two different copies of $H_3 \in H_3[N]$. Since the total number of copies of H_3 in $H_3[N]$ is $2\binom{n-1}{3}/3$ of Type 1 and $2\binom{n-1}{2}/3$ of Type 2, which is $2\binom{n}{3}/3$ altogether. So $H_3[N]$ is an H_3-hypergraph design with vertex set N of order n and index 2 with no repeated copies of H_3 as claimed. □

4 The λ-spectrum of H_3-hypergraph designs for all values of λ

Theorem 4.1 *H_3-hypergraph designs of order n and index λ exist if and only if*

$$\begin{cases} n \equiv 0,1,2, \pmod{9}, \text{ or} \\ \lambda \equiv 0 \pmod{3} \text{ and } n \ne 3. \end{cases}$$

Proof: The existence of H_3-hypergraph designs of order n, $n = 0,1,2 \pmod{9}$, and index 1 was solved in Theorem 1.1. To obtain an H_3-hypergraph design of order n, $n \equiv 0,1,2 \pmod{9}$, for any λ, just take λ copies of each H_3. For $\lambda \equiv 0 \pmod{3}$ we need only consider when $\lambda = 3$ and

$n \equiv 3,4,5,6,7,8 \pmod 9$, since taking $\lambda/3$ copies of H_3 will produce an H_3-hypergraph design of index λ. To construct these H_3-hypergraph designs of index 3, we will consider two cases.

Case 1: $n \equiv 1 \pmod 2$, $n \neq 3$.

We will show the existence of of H_3-hypergraph designs of order $n \equiv 1 \pmod 2$ and index $\lambda = 3$ by induction. Let $n = 2z+5$ for $z \geq 0$. The proof is by induction on z.

For $z = 0$ the existence of an H_3-hypergraph design of order 5 is given by Lemma 2.4. For $z = 1$ the existence of an H_3-hypergraph design of order 7 is given by Lemma 2.5. So now, let $k \geq 2$ and assume there exists an H_3-hypergraph design of order $2z + 5$ for all z, $0 \leq z < k$.

Construction VII to construct an H_3-hypergraph design of order $n = 2k + 5$.

Let $X = \{1,2,3,4,5\}$, $Y = \{5,6,\ldots,2k+5\}$, and $N = X \cup Y$ (so $X \cap Y = \{5\}$).

Let $H_3[X]$ be an H_3-hypergraph design of order $|X| = 5$ and index 3 (this exists by Lemma 2.4). Let $H_3[Y]$ be an H_3-hypergraph design of order $|Y| = n - 4 = 2k + 1 = 2(k - 2) + 5$ and index 3 (this exists by the induction hypothesis). Let $\mathrm{TS}(Y, B_Y)$ be a three-fold triple system of order $|Y| = 2k + 1$; this exists since $|Y| = 1 \pmod 2$ (see [12], for example). Let π be the set of all triples in B_Y that contain the vertex 5. Let L be the multiset given by $L = \{\{j,\ell\} | \{5,j,\ell\} \in B_Y\}$. Let $\mathrm{TS}(X, B_X)$ be a partial triple system of order $|X| = 5$ with leave the 4-cycle $M = (1,2,3,4)$; this exists by Lemma 2.6.

Define $H_3[N]$ as follows:

Type 1: if $H_3(a,b,c,d) \in H_3[X]$ then $H_3(a,b,c,d) \in H_3[N]$,

Type 2: if $H_3(a,b,c,d) \in H_3[Y]$ then $H_3(a,b,c,d) \in H_3[N]$,

Type 3: if $\{a,b,c\} \in B_X$ then three copies of $H_3(y,a,b,c) \in H_3[N]$, for each $y \in Y \setminus \{5\}$,

Type 4: if $\{a,b,c\} \in B_Y \setminus \pi$ then $H_3(\ell,a,b,c) \in H_3[N]$, for each $\ell \in X \setminus \{5\}$, and

Type 5: $\{H_3(1,2,j,\ell), H_3(2,3,j,\ell), H_3(3,4,j,\ell), H_3(4,1,j,\ell)\} \subset H_3[N]$, for each $\{j,\ell\} \in L$.

Claim: $H_3[N]$ is an H_3-hypergraph design with vertex set N of order $n = 2k + 5$ and index 3.

As before, we consider each hyperedge, t, in turn. If $t = \{a,b,c\} \subset X$ or $t = \{a,b,c\} \subset Y$, then t is in three copies of $H_3 \in H_3[N]$ that is of Type 1 or Type 2 respectively.

If $t = \{a,b,c\}$, where $\{b,c\} \subset X$ (possibly $5 \in \{b,c\}$) and $a \in Y \setminus \{5\}$, then: either there exists some $d \in X$ such that $\{b,c,d\} \in B_X$, in which case t is in three copies of $H_3 \in H_3[N]$ of Type 3; or $\{b,c\}$ is an edge in M, in which case there exist three edges in L that contain a (since there are three triples in B_Y containing $\{5,a\}$), so t is in three copies of $H_3 \in H_3[N]$

of Type 5.

If $t = \{a, b, c\}$, where $a \in X \setminus \{5\}$ and $\{b, c\} \subset Y \setminus \{5\}$, then there exist $d_1, d_2, d_3 \in Y$ such that $\{b, c, d_i\} \in B_Y$ for $1 \le i \le 3$. If $d_i = 5$, t is in a copy of $H_3 \in H_3[N]$ of Type 5; otherwise t is in a copy of $H_3 \in H_3[N]$ of Type 4. So every hyperedge t is in at least three copies of $H_3 \in H_3[N]$. The total number of copies of $H_3 \in H_3[N]$ is 10 of Type I, $\binom{n-4}{3}$ of Type 2, $3 \cdot 2(n-5)$ of Type 3, $4(\binom{n-4}{2} - 3(n-5)/2)$ of Type 4, and $4 \cdot 3(n-5)/2$ of Type 5, which is $\binom{n}{3}$ altogether. So $H_3[N]$ is an H_3-hypergraph design with vertex set N of order n and index 3 as claimed.

Case 2: $n \equiv 0 \pmod 2$

Construction VIII: to construct an H_3-hypergraph design of order $n \equiv 0 \pmod 2$ and index 3.

Let $X = \{1, 2, 3, \ldots n-1\}$ and let $N = X \cup \{\infty\}$. Let $H_3[X]$ be an H_3-hypergraph design of order $|X| = n - 1$; this was shown to exist in Construction VII. Let $TS(X, B_X)$ be a three-fold triple system of order $|X| \equiv 1 \pmod 2$. Define $H_3[N]$ as follows:
Type 1: if $H_3(a, b, c, d) \in H_3[X]$ then $H_3(a, b, c, d) \in H_3[N]$, and
Type 2: if $\{a, b, c\} \in B_X$ then $H_3(\infty, a, b, c) \in H_3[N]$.

Claim: $H_3[N]$ is an H_3-hypergraph design with vertex set N of order n and index 3. As before, we consider each hyperedge, t, in turn. If $t = \{a, b, c\} \subset X$ then t is in three copies of H_3 that are of Type 1 in $H_3[N]$. If $t = \{\infty, a, b\}$ then there exists some $\{c_1, c_2, c_3\} \subset X$ such that $\{\{a, b, c_1\}, \{a, b, c_2\}, \{a, b, c_3\}\} \subset B_x$, so t is in three copies of H_3 of Type 2 in $H_3[N]$. So every hyperedge, t, is in at least three copies of $H_3 \in H_3[N]$. The total number of copies of H_3 in $H_3[N]$ is $\binom{n-1}{3}$ of Type 1 and $\binom{n-1}{2}$ of Type 2, which is $\binom{n}{3}$ altogether. So $H_3[N]$ is an H_3-hypergraph design of order n and index 3 as claimed. □

5 The Spectrum of H_2

Since the following construction will need to refer to specific copies of H_2, we make the following definitions. Let $H_2(t_1, t_2)$ be the hypergraph with vertex set $\{t_1 \cup t_2\}$ and hyperedge set $\{\{t_1\}, \{t_2\}\}$.

Theorem 5.1 *H_2-hypergraph designs, of order n and index λ that are simple exist if and only if $\lambda \le 3n - 9$ and*

$$\begin{cases} n \equiv 0, 1, 2 \pmod 4 & \text{or} \\ \lambda \equiv 0 \pmod 2 & \text{and } n \neq 3, \end{cases}$$

Proof: The necessity of the conditions follows because the number of triples $\lambda\binom{n}{3}$ must be divisible by 2, and because each copy of H_2 that contains the triple $\{a,b,c\}$ must contain one of the $3(n-3)$ triples $\{a,b,x\}$, $\{a,c,y\}$ or $\{b,c,z\}$ for some x,y and z.

We begin the proof of the sufficiency by considering $\lambda \le 2$, and in the process produce a new proof for the spectrum of H_2-designs of index 1. There are various ways that one might establish the existence for H_2-hypergraph designs. We decided the most interesting and direct way is to produce such a hypergraph design explicitly by finding a hamilton cycle C in the graph $G(n)$ with vertex set being the set of all triples in $\{1,2,\dots,n\}$ and with two triples t_1 and t_2 being joined if and only if $|t_1 \cap t_2| = 2$. For then let $C = (t_1, t_2, \dots, t_{\binom{n}{3}})$: if $n \equiv 0, 1$, or 2 (mod 4) then C has even length, so taking the copies of each H_2 in $E = \{H_2(t_{2i-1}, t_{2i}) | 1 \le i \le \binom{n}{3}/2\}$ produces the required H_2-hypergraph design of index 1; and if $n \equiv 3$ (mod 4) then $\lambda = 2$, so taking the copies of each H_2 in $\{H_2(t_{2i-1}, t_{2i}) | 1 \le i \le \binom{n}{3}\}$ (reducing subscripts mod $\binom{n}{3}$) produces the required H_2-hypergraph design.

Let $G(k,n)$ be the graph with vertex set equal to the two element subsets of $\{k+1, k+2, \dots, n\}$ in which two vertices v_1 and v_2 are joined if and only if $|v_1 \cap v_2| = 1$. There exists a directed Hamilton path $P^+(k,n)$ in $G(k,n)$ from $\{k+1, k+2\}$ to $\{n-1, n\}$ defined by Figure 1.

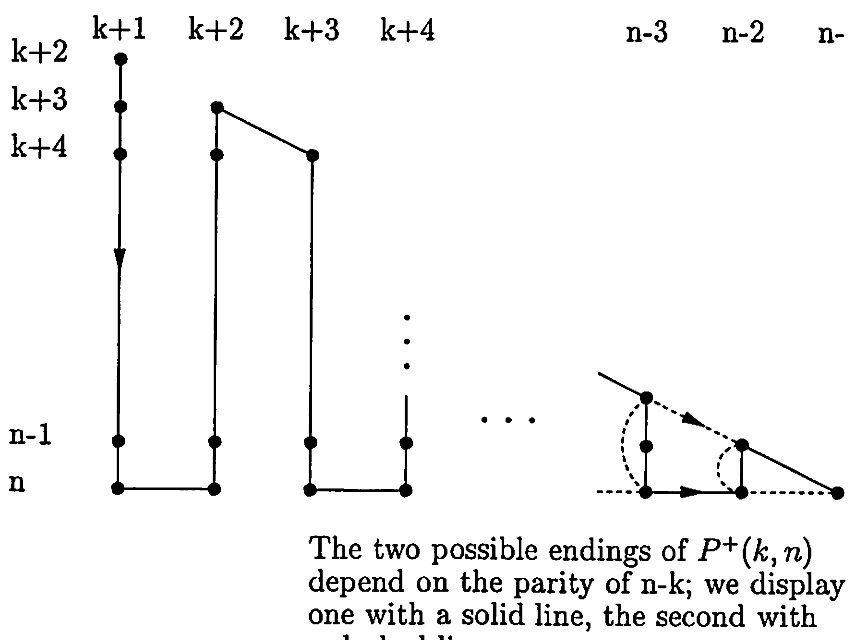

Figure 1: $P^+(k,n)$

Let $P^-(k,n)$ be the directed path in $G(k,n)$ from $\{n-1,n\}$ to $\{k+1,k+2\}$ formed by reversing the orientation of $P^+(k,n)$. If $P^* = P^+(k,n)$ or $P^-(k,n)$ then let $k+P^*$ be formed by adding a third element k to each vertex in P^* (so each vertex in $k+P^*$ is a triple containing k). Finally let P be the concatenation of the paths $1+P^-(1,n), 2+P^+(2,n), 3+P^-(3,n), \dots, n-2+P^{\pm}(n-1,n)$ where $P^{\pm}(n-1,n) = P^+(n-1,n) = P^-(n-1,n)$. Then P is a Hamilton path in $G(n)$ from $\{1,n-1,n\}$ to $\{n-2,n-1,n\}$, so adding the edge $\{\{1,n-1,n\},\{n-2,n-1,n\}\}$ to P forms C.

To prove the theorem itself, suppose $\lambda \leq 3n-9$. Note that $G(n)$ is $(3n-9)$-regular. If there exists a λ-factor F of $G(n)$, then it immediately follows that the set of edges of F form an H_2-hypergraph design of order n and index λ which is clearly simple, so the result would follow. So it remains to form F.

If $G(n)$ has an even number of vertices, then let F_1 and F_2 be two 1-factors of $G(n)$ that partition the edges of C. In any case, by Peterson's Theorem, there exists a 2-factorization $\{T_1, \dots, T_{\lfloor (3n-9)/2 \rfloor}\}$ of $G(n)$ or of $G(n) - F_1$ if $G(n)$ has even or odd degree respectively; furthermore, if $G(n)$ has even degree and an even number of vertices, then we can choose $T_{(3n-9)/2} = F_1 \cup F_2$. Then, since whenever the number of vertices is odd (so $n \equiv 3 \pmod 4$) the necessary conditions require λ to be even, we can define F by:

$$F = \begin{cases} \bigcup_{i=1}^{\lambda/2} E(T_i) & \text{if } \lambda \text{ is even, and} \\ (\bigcup_{i=1}^{\lfloor \lambda/2 \rfloor} E(T_i)) \cup E(F_1) & \text{if } \lambda \text{ is odd.} \end{cases}$$

□

Corollary 5.2 *There exists an H_2-hypergraph design of order n and index λ if and only if*

$$\begin{cases} n \equiv 0, 1 \text{ or } 2 \pmod 4, \text{ or} \\ \lambda \equiv 0 \pmod 2 \text{ and } n \neq 3. \end{cases}$$

Proof: The necessity follows from the fact that the number of triples $\lambda\binom{n}{3}$ must be even, and since clearly $n \neq 3$. The sufficiency follows from λ (or $\lambda/2$) copies of an H_2-hypergraph design of index 1 (or 2) if $n \equiv 0, 1$ or 2 (mod 4) (or $\lambda \equiv 0$ (mod 2)) found in Theorem 5.1. □

References

[1] J. C. Bermond, A. Germa, and D. Sotteau, *Hypergraph-designs*, Ars Combin., 3 (1977), 47-66.

[2] J.C. Bermond, C. Huang, A. Rosa and D. Sotteau, *Decomposition of complete graphs into isomorphic subgraphs with five vertices*, Ars Combin., 10 (1980), 211-254.

[3] J.C. Bermond, C. Huang and D. Sotteau, *Balanced cycle and circuit designs: even cases*, Ars Combin., 5 (1978), 293-318.

[4] J.C. Bermond and J. Schönheim, *G-decomposition of K_n, where G has four vertices or less*, Discrete Math., 19 (1977), 113-120.

[5] J.C. Bermond and D. Sotteau, *Balanced cycle and circuit designs: odd cases*, Proc. Colloq. Oberhof Illmenau (1978), 11-32.

[6] J. Bosák, *Decompositions of Graphs*, Kluwer Academic Publishers, Boston, 1990.

[7] H. Hanani, *On quadruple systems*, Canad, J. Math., 12 (1960), 145-157.

[8] H. Hanani, *On some tactical configurations*, Canad, J. Math., 15 (1961), 702-722.

[9] H. Hanai, *Balanced incomplete block designs and related designs*, Discret Math., 11 (1975), 255-369.

[10] D.G. Hoffman, C.C. Lindner, and C.A. Rodger, *On the construction of odd cycle systems*, J. Graph Theory, 13 (1989), 417-426.

[11] Rev. T.P. Kirkman, *On a problem in combinatorics*, Cambridge and Dublin Math. Journal, 2 (1847), 191-204.

[12] C.C. Lindner and C. A. Rodger, *Decomposition into Cycles II: Cycle Systems*, Contemporary Design Theory: A collection of Surveys, 1992, 325-368.

[13] C.C. Lindner and C. A. Rodger, *Design Theory*, CRC Press, 1997.

[14] A. F. Mouyart, Decomposition of the complete hypergraph into hyperclaws, Discrete Math, 76 (1989), 137-150.

[15] D.K. Ray-Chaudhuri and R.M. Wilson, *Solution of Kirkman's schoolgirl problem*, Proc. Symp. Pure Math. Amer. Math. Soc., 19 (1971), 187-204.

[16] C.A. Rodger, *Graph Decompositions*, Graphs, Designs and Combinatorial Geometries, Second Catania Combinatorial Conference Invited Lectures, 1989.

[17] J. Schönheim, *On coverings*, Pacific J. Math., 14 (1964), 1405-1411.

[18] M. Tarsi, *Decomposition of complete multigraphs into stars*, Discret Math., 26 (1979), 273-278.

[19] M. Tarsi, *Decomposition of the complete multigraph into simple paths: non-balanced handcuffed designs*, J. Combinatorial Theory (A), 34 (1983), 60-70