Projective Properties of Small Hadamard Matrices and Fractional Factorial Designs

Derek Goh and Deborah J Street
School of Mathematical Sciences
University of Technology, Sydney
New South Wales 2007 AUSTRALIA

Dedicated to Anne Penfold Street.

Abstract

We consider the projective properties of small Hadamard matrices when viewed as two level *OAs* of strength two. We show that in some cases sets of rows with the same type of projection form balanced incomplete block designs.

1 Introduction

An *orthogonal array* $OA(N, k, s, t)$ is a $k \times N$ array with entries from a set of s distinct symbols arranged so that for any t rows of the array each of the s^t column vectors appear equally often. Thus we see that $s^t|N$. We call N the *number of runs* in the OA, k the *number of factors*, s the *number of levels* for each factor and t the *strength* of the array. Sometimes $N/s^t = \lambda$ is called the *index* of the array. Table 1 gives an example of an $OA(16, 15, 2, 2)$.

An array on 2 symbols with k rows and N columns is an (N, k, p) *screening design* if for each choice of p rows, each of the 2^p column vectors appears at least once. Thus an $OA(\lambda 2^k, k, 2, t)$ is a $(\lambda 2^k, k, t)$ screening design. An example of a $(16, 14, 3)$ screening design can be obtained from Table 1 by removing the first row. However, this $(16, 14, 3)$ screening design is not an orthogonal array of strength 3 since the combination of rows 1, 3 and 5 from this screening design gives three copies of each of (1,1,2), (1,2,1), (2,1,1) and (2,2,2) but only one copy of each of the combinations (1,1,1), (1,2,2), (2,1,2) and (2,2,1). This is an example of a (1,3) projection.

2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1
2	2	2	2	1	1	1	1	2	2	2	2	1	1	1	1
2	2	2	2	1	1	1	1	1	1	1	1	2	2	2	2
2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1
2	2	1	1	2	2	1	1	1	1	2	2	1	1	2	2
2	2	1	1	1	1	2	2	2	1	2	1	2	1	2	1
2	2	1	1	1	1	2	2	1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1	2	2	1	1	1	1	2	2
2	1	2	1	2	1	2	1	1	1	2	2	2	2	1	1
2	1	2	1	1	2	1	2	2	1	1	2	1	2	2	1
2	1	2	1	1	2	1	2	1	2	2	1	2	1	1	2
2	1	1	2	2	1	1	2	2	1	2	1	1	2	1	2
2	1	1	2	2	1	1	2	1	2	1	2	2	1	2	1
2	1	1	2	1	2	2	1	2	1	1	2	2	1	1	2
2	1	1	2	1	2	2	1	1	2	2	1	1	2	2	1

Table 1: An $OA(16, 15, 2, 2)$

If the possible p-dimensional columns vectors each appear either i times or j times then we talk of a projection of type (i, j). If i or j is 0 then the design is not of projectivity p.

To determine if the projectivity of a design is p, each subset of p rows must be checked to see if each of the distinct column vectors appears at least once. For example, the $(16, 14, 3)$ screening design obtained from Table 1 has projectivity three, since if any three rows are selected then each of the 2^3 column vectors appears at least once. It does not have projectivity four since rows 1, 2, 3, and 4 of the screening design do not contain any of the quadruples with three 1's and one 2.

A screening design of projectivity 2 is sometimes called a *covering array*; see [10].

An N by N Hadamard matrix H is an orthogonal matrix with entries $+1$ or -1. Thus $H^T \cdot H = N \cdot I_N$ where I_N is the $N \times N$ identity matrix. Without loss of generality we can insist that the first row of any Hadamard matrix contain only 1's. Then by removing this row we obtain an $OA(N, N-1, 2, 2)$. Some OAs of this form were introduced by Plackett and Burman (1946) and are termed *Plackett-Burman* designs.

A *balanced incomplete block design* (BIBD) with parameters (v, b, r, k, λ) consists of b blocks each of size k. The number of treatments in the design is v, all treatments are replicated r times and every pair of treatments appears in λ blocks. Thus some restrictions are placed on the values of

v, b, r, k, and λ. In particular, counting the occurrences of treatments, $vr = bk$ and by counting pairs of treatments, $\lambda(v-1) = r(k-1)$. Because of these relationships we usually talk about a (v, k, λ) BIBD.

Some work has been done on projective properties of a few small designs. Lin and Draper [8] consider the projection properties of the Plackett-Burman designs with 12 runs in detail and give results for Plackett-Burman designs of orders $N = 20$ and $N = 24$ projected into $p = 2, 3, 4$ or 5 dimensions, and for values of N to 36 for $p = 3$.

Box and Tyssedal [1] consider the projection properties of two-level OAs with $k = N - 1$. They show that any design $OA(2N, N-1, 2, 2)$ obtained by taking the Kronecker product of

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

with an $OA(N, N-1, 2, 2)$ is always of projectivity 2. If an $OA(N, N-1, 2, 2)$ is obtained by cyclically developing an initial row then that OA may have projectivity 2 or 3. Any $OA(4m, 4m-1, 2, 2)$ with m odd is of projectivity 3.

Cheng [2] gives some theoretical results about the projections of $OA(N, k, 2, t)$ onto $t+1$ and $t+2$ rows. In particular, he shows that if the projection onto some $t+1$ rows does not contain all the 2^{t+1} column vectors at least once then the projection onto any other set of $t+1$ rows with t rows in common with the first set will contain at least one copy of each of the 2^{t+1} column vectors (Lemma 2.2).

In this paper we look at the structure of sets of p rows of Hadamard matrices of small order which either do not contain at least one copy of each of the 2^p distinct column vectors, or which all have the same projection type. We find that these sets are often examples of BIBDs. In the final section we briefly consider the same problem for some of the small designs listed in Dey [5].

2 Results for Hadamard matrices of order 2^N

There is only one non-isomorphic Hadamard matrix of order 8 [3] and we give it in Table 2. (Two Hadamard matrices are said to be *isomorphic* if one of the matrices can be obtained from the other by permuting the rows, or the columns, or by multiplying any row or column by -1.)

The first row of all 1's is removed to obtain an $OA(8, 7, 2, 2)$. We can view this OA as an (8,7,2) screen. However, no (8,7,3) screen can be obtained, since if rows 1, 2 and 3 are selected from the OA, for instance, there are 2 copies of the triple $(1,-1,-1)$ and so some triples do not appear. But for rows 5, 6 and 7 there is one copy of each of the 8 triples. In Table 3, we list the triples of rows of projection type (0,2). We see that they form a (7,7,3,3,1) BIBD.

1	1	1	1	1	1	1	1
1	1	-1	-1	1	1	-1	-1
1	-1	1	-1	1	-1	1	-1
1	-1	-1	1	1	-1	-1	1
1	1	1	1	-1	-1	-1	-1
1	1	-1	-1	-1	-1	1	1
1	-1	1	-1	-1	1	-1	1
1	-1	-1	1	-1	1	1	-1

Table 2: The Hadamard matrix of order 8

1	2	3
1	4	5
1	6	7
2	4	6
2	5	7
3	4	7
3	5	6

Table 3: A (7,7,3,3,1) BIBD

Let $OA_{2,N}$ be the orthogonal array obtained by taking the Kronecker product of the Hadamard matrix of order two N times and discarding the first row. Then we have the following result.

Theorem 1 *The set of triples of rows of $OA_{2,N}$ which have projection type (0,2) form a $(2^N - 1, 3, 1)$ BIBD for $N > 2$.*

Proof: We prove the result by induction. We have established the result for $N = 3$. Suppose the result is true for N and consider $N + 1$. By induction there is a $(2^N - 1, 3, 1)$ BIBD from rows 1 to $2^N - 1$. For each triple (i, j, m) in this BIBD we obtain the triples of rows $(i, j+2^N, m+2^N)$, $(i+2^N, j, m+2^N)$, and $(i+2^N, j+2^N, m)$ each of which has projection type (0,2). The other triples of rows are those of the form $(i, i+2^N, 2^N)$ since the triples from here are of the form $(a, a, 1)$ or $(a, -a, -1)$. From Lemma 2.2 of Cheng [2] we know that no pair of rows can be in a triple of projection type (0,2) more than once. We have found $(2^N - 1) + 4(2^N - 1)(2^{N-1} - 1)/3 =$

$(2^{N+1}-1)(2^N-1)/3$ triples of rows of projection type (0,2) and so the result follows.

Clearly the designs obtained from this result have at least one subdesign embedded in them but for larger designs there may be more. A recursive construction for obtaining a $(2v+1,3,1)$ from a $(v,3,1)$ appears in Stanton and Goulden [11]. Applying Theorem 1 to $OA_{2,4}$ gives a (15, 35, 7, 3, 1) with 15 $(7,7,3,3,1)$ BIBDs embedded in it, and so is the first of the (15,35,7,3,1)'s listed in Colbourn and Dinitz [3], Table 1.1.22.

Now consider those sets of four rows of the Hadamard matrix of order 8 which have an element-wise product of 1 in each position, such as $\{1,2,4,7\}$. There are seven such sets, given in Table 4, which we see form a (7,7,4,4,2) BIBD.

1	2	4	7
1	2	5	6
1	3	4	6
1	3	5	7
2	3	4	5
2	3	6	7
4	5	6	7

Table 4: A (7,7,4,4,2) BIBD

Theorem 2 *The sets of quadruples of rows of $OA_{2,N}$ which have an element-wise product of 1 form a $(2^N-1,4,2(2^{N-2}-1))$ BIBD for $N>2$.*

Proof: We prove the result by induction. We have established the result for $N=3$. Suppose the result is true for N and consider $N+1$. By induction there is a $(2^N-1,4,2(2^{N-2}-1))$ BIBD from rows 1 to 2^N-1. For each quadruple (i,j,m,n) in this BIBD we obtain the quadruples of rows $(i,j,m+2^N,n+2^N)$, $(i,j+2^N,m+2^N,n)$, $(i,j+2^N,m,n+2^N)$, $(i+2^N,j,m,n+2^N)$, $(i+2^N,j,m+2^N,n)$, $(i+2^N,j+2^N,m,n)$ each of which has an element-wise product of 1. Other quadruples of rows are those obtained from the triples of the $(2^N-1,3,1)$ BIBD constructed in Theorem 1. Each triple gives rise to four quadruples. These are $(i,j,m+2^N,2^N)$, $(i,j+2^N,m,2^N)$, $(i+2^N,j,m,2^N)$ and $(i+2^N,j+2^N,m+2^N,2^N)$. The final quadruples are of the form $(i,i+2^N,j,j+2^N)$, for $1\le i,j\le 2^N-1$.

Applying this result when N=4 gives a (15, 105, 28, 4, 6) BIBD. Taking any of the sets of three rows from the (15,35,7,3,1) constructed in Theorem 1 and adjoining a fourth row gives a quadruple of rows in which some column vectors are repeated. There are 420 of these sets and they form a (15, 420, 112, 4, 24). Together these 525 quadruples account for all the quadruples of rows in which there are repeated column vectors.

The next result looks at sets of five rows.

Theorem 3 *The sets of quintuples of rows of $OA_{2,N}$ which have an element-wise product of 1 form a $(2^N - 1, 5, 2(2^{N-2} - 1)(2^N - 8)/3)$ BIBD for $N > 3$.*

Proof: The triples given in Theorem 1 all have element-wise product 1. If these rows are represented by a, b and c then we write $a \cdot b \cdot c = 1$. The quadruples from Theorem 2 satisfy $a \cdot b \cdot c \cdot d = 1$. The quintuples arise by adjoining the triple $\{a, b, c\}$ and the quadruple $\{c, x, y, z\}$ to get $\{a, b, x, y, z\}$ (where a, b, x, y and z must all be distinct). Straight-forward counting then gives the result.

When $N = 4$ the previous result gives a BIBD (15,168, 56, 5, 16) and when $N = 5$ it gives a BIBD $(31, 5208, 840, 5, 112)$.

3 Some other results for Hadamard matrices

Hall [6] found that there are exactly five nonisomorphic Hadamard matrices of order 16. The representative matrices of each of these five classes may be found in Seberry [9]. The first matrix, H16.1, corresponds to $OA_{2,4}$.

For these five matrices, we remove the first row of all 1's to obtain five OA $(16, 15, 2, 2)$s. The OA obtained from H16.4 in this manner is shown in Table 1. For the second, third and fifth OAs, if the first three rows are removed, $(16, 12, 3)$ screens can be produced. Only the first row needs to be removed from the fourth design matrix in order to produce a $(16, 14, 3)$ screen, as discussed in Section 1.

There are three nonisomorphic Hadamard matrices of order 20 which may be found in [9]. For all three OAs, (20,19,3) screens can be obtained. However, 57 combinations of three rows produce (1,4) projections. For each of the three OAs, these 57 combinations of three rows produce a BIBD $(19, 57, 9, 3, 1)$. Recall that there are over 1.1×10^9 $(19, 3, 1)$ BIBDs [3].

There are 60 Hadamard matrices of order 24. From the one given in Table 2 of Hedayat and Wallis [7] a (24,23,3) screen can be obtained. From these 1771 combinations of three rows, 759 are (2,4) projections and form a BIBD $(23, 759, 99, 3, 9)$ while the remaining 1012 have all triples of columns equally replicated and form a BIBD $(23, 1012, 132, 3, 12)$.

When attempting to find (24,23,4) screens, 5060 combinations of four rows have projection type (0,1,2,3) and form a BIBD $(23, 5060, 880, 4, 120)$.

The remaining 3795 sets of four rows have projection type (1,2) and form a BIBD (23, 3795, 660,4,90).

4 Results for several orthogonal fractional factorial designs

Several orthogonal fractional designs given by Dey [5] were checked for the existence of screening designs. In particular we considered the symmetrical orthogonal resolution III designs which correspond to an $OA(81, 40, 3, 2)$, an $OA(16, 5, 4, 2)$, an $OA(64, 21, 4, 2)$ and an $OA(25, 6, 5, 2)$.

In the $OA(81, 40, 3, 2)$ there is no (81, 40, 3) screen but there is a (40, 3, 2) BIBD from the sets of three rows which have at least one of the column vectors not represented. Similarly the sets of four rows which do not have all the column vectors appearing exactly once form a (40, 4, 217) BIBD.

These are the only non-trivial BIBDs obtained from the designs in Dey [5].

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