Some Constructions of Block Designs

Malcolm Greig
Greig Consulting,
5685 Daffodil Drive,
West Vancouver,
B.C., Canada, V7W 1P2

Dinesh G. Sarvate*
Department of Mathematics,
University of Charleston,
Charleston, SC 29424

Dedicated to Anne Penfold Street.

Abstract

Using a blend of Drake's and Saha's techniques, we construct a BTD$(n^2/4; (n^2+n)/2; 2n-4, 3, 2n+2; n; 8)$ whenever n is a power of 2, as well as some new symmetric BTDs. It is known that the necessary condition $v \equiv 1 \pmod 2$ is sufficient for the existence of simple BIBD(v,3,3). In the second part of this paper we give a very simple construction based on graph factorization to prove this result whenever v is not divisible by 3. We then expand upon this result to exhibit further constructions of BTDs.

Key words and phrases: Simple balanced incomplete block designs, Balanced ternary designs, Affine planes
AMS 1991 *subject classifications*: Primary 05B05

1 Introduction

A Balanced Ternary Design, BTD $(V; B; \rho_1, \rho_2, R; K; \Lambda)$, is an arrangement of V elements into B multisets, or blocks, each of cardinality K, satisfying:

*Corresponding Author.

a. Each element appears $R = \rho_1 + 2\rho_2$ times altogether, with multiplicity one in exactly ρ_1 blocks, with multiplicity two in exactly ρ_2 blocks and

b. Every pair of distinct elements appears Λ times.

Note that in a block $\{a, a, b, c, c, c\}$, the pair (a,b) is said to occur twice, the pair (b,c) is said to occur three times and the pair (a,c) is said to have occured 6 times. A BTD(6; 12; 4, 1, 6; 3; 2) on elements $\{1, 2, 3, 4, 5, 6\}$ is given below where a block, say, $\{1, 1, 2\}$ is written as 112.

$$\{112, 133, 144, 156, 156, 223, 255, 246, 246, 366, 345, 345\}$$

Two excellent survey papers on BTDs are by Billington [3], [4].

In the next section we will construct a new family of BTD$(n^2/4; (n^2 + n)/2; 2n - 4, 3, 2n + 2; n; 8)$ whenever n is a power of 2. Observe that the BTDs we constructed can not be a multiple of smaller designs as ρ_2 is 3 and $\Lambda = 8$ is even.

It is well known that Hanani [12] showed that the necessary condition $v \equiv 1 \pmod 2$ is sufficient for the existence of BIBD(v,3,3).

Van Buggenhaut [23] gave constructions of simple BIBD$(v, 3, 3)$ $\forall\, v \equiv 1 \pmod 2$. Street [21] gave recursive constructions of simple BIBD$(v, 3, 2)$ and simple BIBD$(v, 3, 3)$.

Of course, there is a vast literature on simple BIBDs including Stinson and Wallis [20] and many others as well as a flawed/unrepairable proof by Sarvate [16], [17] that all simple BIBDs with block size 3 exists, but it is not our intention to give here even a brief survey of the literature. We just wish to give a very simple construction motivated by a paper of Professor Street [21] to suit the occasion. We then exploit the ideas in our construction to construct more BTDs.

2 Construction of Balanced Ternary Designs

We begin this section by presenting a standard construction for $AG(2, q)$ where q is a prime power. Let x be a primitive element for $GF(q)$.

$$\mathcal{P} = GF(q) \times GF(q)$$

$$\begin{aligned} G &= \{(0,0)(0,x^0)(0,x^1)\ldots(0,x^{q-2})\} \bmod (q,-) \\ C &= \{(0,0)(x^0,0)\ldots(x^{q-2},0)\} \bmod (-,q) \\ B_\alpha &= \{(0,0)(x^0,x^\alpha)(x^1,x^{\alpha+1})\ldots(x^{q-2},x^{\alpha+q-2})\} \bmod (-,q) \end{aligned}$$

$$\mathcal{B} = \text{dev}(G) \cup \text{dev}(C) \cup \bigcup_{\alpha=0}^{\alpha=q-2} \text{dev}(B_\alpha)$$

We make an observation here, that an affine plane of order q is also an RBIBD$(q^2; q^2+q; q+1; q; 1)$.

We next make the general observation that if we have a $(v; b; r; k; \lambda)$ BIBD with $v = am$, and replace a spanning collection of disjoint sets of m points by a new points, then we get a $(a; b; rm; k; \lambda m^2)$ $(m+1)$-ary design. However, this design does not, in general, have an equi-replicate pattern over the new points, although if the design degenerates into a ternary design, we will have an equi-replicate pattern with $\rho_2 = \lambda\binom{m}{2}$.

If $q = n$ is a power of 2, we found a way to collapse sets of 4 old points onto 1 new point so that we got a BTD$(n^2/4; n^2+n; 4n-8, 6, 4n+4; n; 16)$. It turns out that this design can be partitioned into two identical halves. This construction can be considered a blend of well-known constructions [7], [15].

Firstly, each of the above base blocks generates a resolution set in $AG(2,q)$ and if we remove a resolution set from an affine plane we get a resolvable transversal design $RTD(q;q)$. Secondly, if $q = p^t$, we can represent points in $GF(q)$ by vectors of dimension t with elements from $GF(p)$, these vectors normally being thought of as coefficients of a degree t polynomial in x (we take the constant term last). A standard constuction for an $RTD_\lambda(p^t; p^t/\lambda)$ where $\lambda = p^s < p^t$, is to take our set of base blocks, and remove G to provide the groups, and collapse points that only differ in the last s places of the second (vector) element. Now, although a p^s to 1 collapse of points in a design normally multiplies the original value of λ by p^{2s}, here we only need a factor of p^s as we have p^s identical copies, and we only need one of them. To see this, we let $GF^0(p^{t-s})$ be the subset of elements of $GF(p^t)$ whose vector representation has a zero in the last s places. Now we need only develop the base blocks over $GF^0(p^{t-s})$ rather than $GF(p^t)$, since, if $y \in GF^0(p^{t-s})$, then adding y and say, $y+1$, produce the same block after collapsing.

Lastly, Saha [15] constructed n-ary designs by collapsing points within a group of a PBIBD with group association scheme; (he paid particular attention to groups of size 2 so as to ensure the n-ary design were, in fact, ternary designs).

Now we put these ideas together. Our aim is to collapse four points onto one; we can think of the four points as a quadrangle; we will get all the verticals from G, all the horizontals from C, and the diagonals from B_0. In one sense, we can select different resolution sets of $AG(2,q)$, treat them as group definers, and collapse pairs inside the groups, which helps maintain the ternary nature of our design. We now summarize construction.

Theorem 2.1 *A* $BTD(n^2/4; (n^2+n)/2; 2n-4, 3, 2n+2; n; 8)$ *whenever* n *is a power of 2.*

Proof: It suffices to exhibit a 4 to 1 mapping of the points of $AG(2,n)$ onto new points, then show the resulting design is actually a BTD, (not say, a 5-ary design), and that every block is duplicated, so we can just take half the blocks. Let $n = 2^m$.

Let a, b be $(m-1)$-tuples of binary elements. Then we map the points $(a0, b0)$, $(a0, b1)$, $(a1, b0)$, $(a1, b1)$ onto (a, b). The six pairs of old points ocurred as follows:

Pair		Block
$(a0, b0)$	$(a0, b0)$	$G + (a0, 0)$
$(a0, b0)$	$(a1, b0)$	$C + (0, b0)$
$(a0, b0)$	$(a1, b1)$	$B_0 + (0, b0 - a0)$
$(a0, b1)$	$(a1, b0)$	$B_0 + (0, b1 - a0)$
$(a0, b1)$	$(a1, b1)$	$C + (0, b1)$
$(a1, b0)$	$(a1, b1)$	$G + (a1, 0)$

Since these 6 pairs all occurred on distinct blocks, the new design is actually a BTD.

Now let $s \in GF(2^m)$, and note that $s - 1 = s + 1$. Then, after replacing the old points by the new ones, we have:

$$\begin{aligned} G + (s, 0) &= G + (s + 1, 0) \\ C + (0, s) &= C + (0, s + 1) \\ B_i + (0, s) &= B_i + (0, s + 1) \qquad \forall i \end{aligned}$$

Taking only those shifts whose s value ends in 0 gives the desired half of the design. ∎

3 More BTD Constructions

We begin this section by presenting a construction for an $RTD_3(9; 3)$ constructed by the collapsing method of Section 2. Firstly we present an array of symbols; replace the symbols by circulants as follows: replace 0, 1, 2 by 100, 010, 001 (and treat the symbol -0 as 0). (We will give circulants by their first row.) This gives the RTD. If we now add J_3 to a transversal we get a BTD; say replace -0 by 211, rather than 100 in the above construction, and we get a BTD(27; 27; 10, 1, 12; 12; 5) which is BR121 in [5]; no design was previous known for these parameters.

−0	0	0	0	0	0	0	0	0
0	−0	1	1	2	0	2	2	1
0	1	1	2	0	2	2	1	−0
0	1	2	0	2	2	1	−0	1
0	2	0	2	2	1	−0	1	1
0	0	2	2	1	−0	1	1	2
0	2	2	1	−0	1	1	2	0
0	2	1	−0	1	1	2	0	2
0	1	−0	1	1	2	0	2	2

The next construction uses two $RTD(4;1)$'s; since 4 is not a prime, we have a more complicated development. Firstly we treat each of the 8 rows of 8 pairs of symbols in the array below as eight 2 by 2 circulants, and develop to get 16 rows, each with 16 symbols. We then replace each symbol by a 2 by 2 circulant: replace 0, 1, J and − by 20, 02, 11, and 00, and we get the incidence matrix of a $\mathrm{BTD}(32;32;4,4,12;12;4)$ which is BR164 in [5]; this design was previously constructed, using different terminology, in [10].

00	00	00	00	JJ	−−	−−	−−
00	01	10	11	−−	JJ	−−	−−
00	10	11	01	−−	−−	JJ	−−
00	11	01	10	−−	−−	−−	JJ
JJ	−−	−−	−−	00	00	00	00
−−	JJ	−−	−−	00	01	10	11
−−	−−	JJ	−−	00	10	11	01
−−	−−	−−	JJ	00	11	01	10

The next construction uses two $RTD(5;1)$'s; but one is written with its circulants as the multiplicative inverse of the other. The diagonal elements are additive complements. We replace each symbol by a 5 by 5 circulant: replace 0, 1, 2, 3, 4, A, B and − by 20000, 02000, 00200, 00020, 00002, 12002, 10220 and 00000, giving the incidence matrix of a $\mathrm{BTD}(50;50;1,7,15;15;4)$ which is BR342 in [5]; no design was previously known for these parameters.

A	−	−	−	−	0	0	0	0	0
−	A	−	−	−	0	1	2	4	3
−	−	A	−	−	0	2	4	3	1
−	−	−	A	−	0	4	3	1	2
−	−	−	−	A	0	3	1	2	4
0	0	0	0	0	B	−	−	−	−
0	4	3	1	2	−	B	−	−	−
0	3	1	2	4	−	−	B	−	−
0	1	2	4	3	−	−	−	B	−
0	2	4	3	1	−	−	−	−	B

4 A Construction of simple BIBD$(v,3,3)$ for odd v not divisible by 3

As mentioned in the Introduction the necessary condition $v \equiv 1 \pmod 2$ is sufficient for the existence of a simple BIBD$(v,3,3)$. We present a new construction that establishes this whenever v is not divisible by 3.

Theorem 4.1 *If $v = 2n-1$ is odd, and not divisible by 3, then a simple BIBD$(v,3,3)$ exists.*

Proof: Let $v = 2n-1$ with the point set $0,1,\cdots,2n-2$. Let K_{2n} be the complete graph on the set of points $Z_{2n-1} \cup \{\infty\}$. Consider the well known one-factorization (see [19, Theorem 2.1]) where the one-factors are specified for $i = 0,1,\cdots,2n-2$ as follows:

$$F_i = (\infty, i), (i-j, i+j), \quad \text{where } j = 1,2,\cdots,n-1.$$

Note that arithmetic is done in Z_{2n-1}. The set of blocks of the required BIBD are:

$$\{\{a,b,c\}\text{: the edge } (a,b) \text{ is in the factor } F_c\}.$$

We only need to check that the blocks are distinct. Note that $0 \leq a,b,c < 2n-1$. If the edge (a,b) is in F_c and the edge (a,c) is in F_b, it will imply that $a+b \equiv 2c \pmod{2n-1}$ and $a+c \equiv 2b \pmod{2n-1}$, hence, after subtracting and rearranging, we have $3b \equiv 3c \pmod{2n-1}$. If 3 does not divide $2n-1$, this implies $b = c$. Note that if $2n-1 = 3T$ and $0 \leq y < T$, then the block $(y, T+y, 2T+y)$ is repeated 3 times (all with $j = T$). ∎

5 Designs from Frames

The one-factors given in the previous section can alternatively be considered as a BIBD$(2n-1,2,1)$, with the blocks defined by the vertices of each edge. The factorization defines a partial resolution set, with the set F_i having a hole consisting of the point $\{i\}$. This is a special instance of a frame, and we have filled in the hole (with the point $\{i\}$) to produce a BIBD$(2n-1,3,3)$. We will now exploit this idea to construct more BTDs.

A $\{k,\lambda\}$-frame is a $\{k,\lambda\}$-GDD with group divisible association scheme, and the additional property that its blocks can be partioned into holey resolution sets; if G_i is the i-th group, then there are $\lambda|G_i|/(k-1)$ resolution sets that each cover all the points except those in G_i. In the case where every group size is one, the resulting frame is a BIBD, and when

$\lambda = k-1$ is known as a *near resolvable design*, or NRB$(v,k,k-1)$. For more about frames, see [1, Section I.6] or [11]. Frames are useful objects for recursive constructions of RBIBDs, but the important thing, from our point of view, is that frames have been studied to some extent, and we can make use of this. Several standard frame constructions are known: given a RBIBD$(v,k,1)$, deleting any point and using its blocks for groups gives a frame (where the old resolution sets, less the deleted block, form the holey resolution sets); an NRB$(4t-1,2t-1,2t-2)$ is obtained from a skew-Hadamard of order $4t$ (with diagonal omitted) and its transpose; a BIBD$(v,v-1,v-2)$ is a NRB; if $v=kt+1$ is a prime power, the cyclotomic classes form the base blocks for a NRB$(v,k,k-1)$; the last construction we will state is rather less obvious, so we give a proof for it.

Theorem 5.1 *If $q = ef+1$ is a prime power, then a $\{q,f\}$-frame of type e^{q+1} exists.*

Proof: Let $q = ef+1$. Let $(GF(q)\cup\{\infty\})\times\{j : 0\le j<e\}$ be our point set. Note that $e=1$ or $f=1$ is permitted. We will use our earlier representation of $AG(2,q)$ as a starting point. After developing the base blocks, remove dev(G) which will be the indirect basis for the groups. Now replace every element (y,x^{ie+j}) by (y,j) and omit every element of the form $(y,0)$. (Effectively we are taking logs of the second element, and reducing modulo e, with the undefined log of zero avoided; alternatively we are replacing the second element by its multiplicative coset label). All the blocks generated by B_a have had one element removed; all the blocks generated by B_{ie+j} are augmented with the point (∞,j). All elements with the same (new) second element are in the same group. After carefully considering the infinite elements, it is clear we have effectively performed an f to 1 collapsing of points, and have produced a (q,f^2) GDD of type e^{q+1}. We will now show there are f copies of each block, so that by taking only one, we will get the required (q,f) GDD. Consider B_a and B_{a+te}. Suppose $d = x^{a+i}$ and $d' = x^{a+te+i}$. The blocks $B_a - d$ and $B_{a+te} - d'$ both are missing points with a first element of x^i. Consider the points with first elements of x^j with $j\neq i$; the second elements are $x^{a+j}-d$ and $x^{a+te+j}-d'$ respectively. Now $x^{a+te+j}-d' = x^{te}(x^{a+j}-d)$ so these elements are in the same coset, and thus B_a and B_{a+te} all produce the same blocks, although not generated in the same order. Finally, it is clear that $G+d$ and $G+d'$ are identical and $G+0$ has been removed entirely. Thus every block has f copies, and we have shown the existence of the desired GDD. This construction (of a GDD) is due to Wilson [13, Theorem 15.7.4]. Careful counting shows that the design is actually a frame, with the blocks that are disjoint from any group actually forming the holey resolution set for that group. ∎

Our intention is to form a frame of doubletons (singletons) and fill each hole with singletons (doubletons); clearly we will get a group associative PBTD; if there are b blocks in each of the h holey resolution sets for each hole, then the first associates' index will be $4bh$ (bh for singleton fill), and the second associates' index will be increased by $4h$; we now have to pick the right frames so that the two indices are the same. Note that the first associates' index is not relevant for NRBs. We arrive at several designs easily.

Theorem 5.2 *If p is an odd prime power, then a $BTD((p^2+1)(p+1)/2; B = V; p^2, (p+1)/2, p^2+p+1; K = R; 2(p+1))$, and a $BTD((p^2+1)(p+1); B = V; 2(p+1), p^2, 2(p^2+p+1); K = R; 2(p+1))$ exist.*

Proof: In Theorem 5.1, take $q = p^2$, and $e = (p+1)/2$ and fill with doubletons, or $f = (p-1)/2$ and fill with singletons. ∎

Remark 5.3 An instance of the first design, BR184, with $p = 3$, was given in [5]; the smallest second design has parameters $(80; 80; 8, 9, 26; 26; 8)$, when $p = 3$.

Theorem 5.4 *A $BTD(10; 30; 9, 3, 15; 5; 6)$ exists.*

Proof: A $NRB(10, 3, 2)$ exists (see [1, Example I.6.33]); fill in the holes with doubletons to get the BTD. Previously no example was known for this design (BR296). ∎

Actually, we do not need all the structure of frames in this sort of construction; since we are filling in the holes uniformly, all we really need is that all the holey resolution classes for a hole be jointly α-resolvable. We can exploit a construction of Wallis [24] or see [2, Theorem II.8.16]; we can view this design as a PBIBD consisting of holey α-resolution sets; here we also have $\lambda_1 = \lambda_2$ since it is a BIBD. The Wallis construction takes an affine resolvable BIBD, and forms the v by v adjacency matrix of the cliques defined by the blocks of a resolution set, and adds in the diagonal; a Latin square of order $r+1$ is taken, and the i-th symbol is replaced by the matrix formed from the i-th resolution set, with the last symbol (say 0) being replaced by a zero matrix. The zero transversal defines the groups (of size v) and the holey k-resolution sets. A BTD results from taking the initial design as $AG(2,4)$, using the Wallis design for the doubletons, and filling the holes with singletons, which we state as:

Theorem 5.5 *A $BTD(96; 96; 16, 20, 56; 56; 32)$ exists.*

6 Odds and Ends

In [5, p.252], it is remarked that there are still two open cases in Tocher's list [22], and that these cannot be equireplicate designs. Tocher attempted to construct ternary block designs, with constant block size, and constant pair-wise index. The two open cases are $(V = 12, K = 5, \Lambda = 2)$ and $(V = 13, K = 5, \Lambda = 2)$. We can resolve these cases; the former does not exist and the latter does.

For $(V = 12, K = 5, \Lambda = 2)$ we need 132 pairs, and every block contains at most one doubleton, since $\Lambda < 4$. A block with no doubletons contributes 10 pairs, and a block with one doubleton contributes 9 pairs, so there must be exactly 14 blocks, with 8 containing a doubleton. Each point needs to meet 22 other (non-distinct) points, but there are at least 4 points that have no doubletons, and if such a design exists, each of these points should meet 4 points per block; since 22 is not a multiple of 4, no design is possible.

For $(V = 13, K = 5, \Lambda = 2)$ we first construct a PBIBD$(12, 5, (1, 2))$ with a group association type of 3^4; Clatworthy[6, R145] gives a base block of $(1, 2, 4, 6, 7)$ in Z_{12}; we then fill in the groups with the aid of an infinite point by $(\infty, \infty, 0, 4, 8)$ (with a short orbit).

In [14], a tabulation was made of the possible values of b_2, the number of blocks containing doubletons, for the designs with parameters listed in [5]. We take this opportunity to note some typos in [14, Table IV.2.2];

- for KS28 $b_2 = 20$ is known and $b_2 = 10$ is open;
- for KS34, KS46, KS70, KS94, KS122, KS161, $b_2 = 10$ is known;
- for KS100, KS134, KS172, $b_2 = 13$ is known;
- for KS203, the b_2 entry should be "none", (see [9, p. 146]).

We have also resolved the status of a number of the open b_2's in [14]; these will appear in later papers. We have also resolved the status of several of the open parameter sets in [5]. To the best of our knowledge 17 of the 77 cases open there are now resolved.

Design	Exists?	Authority
BR94, BR164	Yes	[10]
BR106, BR197, BR262	No	
BR121, BR296, BR342	Yes	Constructed above
BR153, BR212, BR302	Yes	[8]
BR203	No	[9]
BR209, BR245	Yes	[18]
BR223, BR334	Yes	
BR266	Yes	[9]

References

[1] R.J.R. Abel and S.C. Furino, Resolvable and near resolvable designs, in *The CRC Handbook of Combinatorial Designs* (C.J. Colbourn and J.H. Dinitz, eds.) (CRC Press, Boca Raton, FL, 1996), 87–94.

[2] T. Beth, D. Jungnickel and H. Lenz, *Design Theory,* (Cambridge University Press, Cambridge, England, 1986).

[3] E.J. Billington, Balanced n-ary designs: a combinatorial survey and some new results, *Ars Combinatoria* **17**A (1984), 37–72.

[4] E.J. Billington, Designs with repeated elements: a survey and some recent results, *Congressus Numerantium* **68** (1989), 123–146.

[5] E.J. Billington and P.J. Robinson, A list of balanced ternary designs with $R \leq 15$, and some necessary existence conditions, *Ars Combinatoria* **16** (1983), 235–258.

[6] W.H. Clatworthy, Tables of two-associate-class partially balanced designs, *Nat. Bur. Standards Appl. Math Series* **63** (1973).

[7] D.A. Drake, Partial λ-geometries and generalized Hadamard matrices over groups, *Canadian J. Math.* **31** (1979), 617–627.

[8] J.F. Dillon and M.A. Wertheimer, Balanced ternary designs derived from other combinatorial designs, *Congressus Numerantium* **47** (1985), 285–298.

[9] J.D. Fanning, Symmetric balanced ternary designs with $\rho_1 = 1$ or 2, *Aequationes Math.* **47** (1994), 143–149.

[10] J.D. Fanning, The coexistence of some binary and N-ary designs, *Ars Combinatoria* **45** (1997), 217–227.

[11] S.C. Furino, Y. Miao and J. Yin, *Frames and Resolvable Designs,* (CRC Press, Boca Raton, FL, 1996).

[12] H. Hanani, Balanced incomplete block designs, *Discrete Math.* **11** (1975), 255–369.

[13] M. Hall, Jr., *Combinatorial Theory* (2nd ed.) John Wiley and Sons, New York, 1986).

[14] T. Kunkle and D.G. Sarvate, Balanced (part) ternary designs, in *The CRC Handbook of Combinatorial Designs* (C.J. Colbourn and J.H. Dinitz, eds.) (CRC Press, Boca Raton, FL, 1996), 233–238.

[15] G.M. Saha, On construction of balanced ternary designs, *Sankhyā (B)* **37** (1975), 220–227.

[16] D.G. Sarvate, Block designs without repeated blocks, *Ars Combinatoria* **21** (1986), 71–87.

[17] D.G. Sarvate, All simple BIBDs with block size 3 exist, *Ars Combinatoria* **21A** (1986), 257–270.

[18] K. Sinha, A construction of balanced ternary designs, *Ars Combinatoria* **33** (1992), 276–278.

[19] R.G. Stanton and I.P. Goulden, Graph factorization, general triple systems, and cyclic triple systems, *Aequationes Math.* **22** (1981), 1–28.

[20] D.R. Stinson and W.D. Wallis, Two-fold triple systems withoout repeated blocks, *Discrete Math.* **47** (1983), 125–128.

[21] A.P. Street, Some designs with block size three, *Lecture Notes in Mathematics* **829**, (Springer-Verlag, Berlin 1980), 224–237.

[22] K.D. Tocher, The design and analysis of block experiments, *J. Royal Statist. Soc. Ser B* **14** (1952), 45–100.

[23] J. Van Buggenhaut, Existence and constructions of 2-designs $S_3(2,3,v)$ without repeated blocks, *J. Geometry* **4** (1974), 1–10.

[24] W.D. Wallis, Constructions of strongly regular graphs using affine designs, *Bull. Austr. Math. Soc.* **4** (1971), 41–49.