4-Cycle Decompositions of the Cartesian Product of Two Complete Graphs

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Dedicated to Anne Penfold Street.

Abstract

In this paper we establish necessary and sufficient conditions on m and n in order for $K_m \times K_n$, the cartesian product of two complete graphs, to be decomposable into cycles of length 4. The main result is that $K_m \times K_n$ can be decomposed into cycles of length 4 if and only if either $m, n \equiv 0 \pmod 2$, $m, n \equiv 1 \pmod 8$, or $m, n \equiv 5 \pmod 8$.

1 Introduction

All graphs considered in this paper are finite and have no loops or multiple edges. By $V(G)$ we denote the vertex set of the graph G. By K_n we denote the complete graph on n vertices, and by $K_{m,n}$ we denote the complete bipartite graph with m vertices in one part and n vertices in the other part.

The *cartesian product* of two graphs, G_1 and G_2, is the graph $G_1 \times G_2$ having vertex set $V(G_1) \times V(G_2)$ and in which vertex (u_1, u_2) is adjacent to (v_1, v_2) if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2, or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

*Research supported by ONR grant N00014–95–1–0769.
†Research supported in part by NSERC (Canada).

A *cycle* is a 2-regular connected graph (or subgraph of a graph). A t-*cycle* is a cycle containing exactly t edges. A t-*cycle decomposition* of a graph G consists of a set of t-cycles of G which partition the edge set of G.

Cycle decompositions of graphs has been a topic of much research [1, 12], dating back to the now classic result that K_n is 3-cycle decomposable if and only if $n \equiv 1$ or 3 (mod 6) [11]. More recently, it has been shown that K_n is 4-cycle decomposable if and only if $n \equiv 1$ (mod 8) [9, 15].

To date, decomposition results for cartesian products of graphs appear to have been limited to Hamilton decompositions [2, 3, 10, 14, 17]. In this paper we consider the question:

Question 1 *For a given value of t, what values of m and n are necessary and sufficient for the graph $K_m \times K_n$ to be t-cycle decomposable?*

It is clear that for $t = 3$, the necessary and sufficient conditions for $K_m \times K_n$ to be 3-cycle decomposable are $m \equiv 1$ or 3 (mod 6) and $n \equiv 1$ or 3 (mod 6). For larger values of t, the question becomes more difficult. We focus on the case in which $t = 4$, proving as our main result the following theorem:

Theorem 1 *$K_m \times K_n$ is 4-cycle decomposable if and only if either*

1. *$m, n \equiv 0$ (mod 2),*
2. *$m, n \equiv 1$ (mod 8), or*
3. *$m, n \equiv 5$ (mod 8).*

It is interesting to observe that $K_m \times K_n$ is the line graph of $K_{m,n}$, and so our result also serves to establish necessary and sufficient conditions for $L(K_{m,n})$ to be 4-cycle decomposable. Cycles in line graphs have been another topic of study, albeit primarily concerning line graphs of complete graphs [6, 7, 8].

Before proceeding, we introduce some terminology. A *pure* 4-cycle in $K_m \times K_n$ is a 4-cycle whose edges are all contained within one copy of K_m or one copy of K_n. If we consider $K_m \times K_n$ as having its vertices arranged in a rectangular grid with m rows and n columns, a pure 4-cycle thus contains four vertical edges or four horizontal edges. A *mixed* 4-cycle is a 4-cycle which is not pure; it contains two vertical edges and two horizontal edges.

Also, we will use the following result, due to Sotteau [16]:

Theorem 2 *$K_{m,n}$ is t-cycle decomposable if and only if $t \geq 4$, $m \equiv n \equiv t \equiv 0$ (mod 2), $t \leq 2m$, $t \leq 2n$, and $t \mid mn$.*

In particular, we are interested in the following corollary of this theorem:

Corollary 1 *$K_{m,n}$ is 4-cycle decomposable if and only if $m \equiv n \equiv 0$ (mod 2), $m \geq 2$, and $n \geq 2$.*

2 Necessary Conditions

Lemma 1 *Given that $K_m \times K_n$ is 4-cycle decomposable, then either*

1. *$m, n \equiv 0$ (mod 2),*
2. *$m, n \equiv 1$ (mod 8), or*
3. *$m, n \equiv 5$ (mod 8).*

Proof. We first observe that $K_m \times K_n$ has mn vertices, each having degree $m + n - 2$. Hence $K_m \times K_n$ has $\frac{(mn)(m+n-2)}{2}$ edges.

Given that $K_m \times K_n$ is 4-cycle decomposable, not only must each vertex in the graph have even degree, but the number of edges in the graph must be divisible by 4. Hence $m \equiv n$ (mod 2) and $8 \mid ((mn)(m + n - 2))$; these conditions are both satisfied precisely when

1. $m, n \equiv 0$ (mod 2),
2. $m, n \equiv 1$ (mod 8),
3. $m \equiv 3$ (mod 8) and $n \equiv 7$ (mod 8),
4. $m \equiv 7$ (mod 8) and $n \equiv 3$ (mod 8), or
5. $m, n \equiv 5$ (mod 8).

Consider now the case in which $m \equiv 3$ (mod 8) and $n \equiv 7$ (mod 8). Observe that each pure 4-cycle in $K_m \times K_n$ uses an even number of horizontal edges (0 edges if the 4-cycle is vertical, 4 if it is horizontal) and that each mixed 4-cycle uses two horizontal edges. Thus the total number of horizontal edges used by all 4-cycles will be even. The number of horizontal edges present in $K_m \times K_n$ is $\frac{mn(n-1)}{2}$. Note that m is odd, n is odd, and that $\frac{n-1}{2} \equiv 3$ (mod 4). We find that the total number of horizontal edges in $K_m \times K_n$ is odd, and so we have a contradiction.

The case in which $m \equiv 7$ (mod 8) and $n \equiv 3$ (mod 8) is similar to that in which $m \equiv 3$ (mod 8) and $n \equiv 7$ (mod 8). □

3 Sufficient Conditions

To show that the stated necessary conditions are sufficient, we consider each in turn, and show a means of constructing a 4-cycle decomposition of $K_m \times K_n$.

Lemma 2 *If $m, n \equiv 0$ (mod 2) then $K_m \times K_n$ is 4-cycle decomposable.*

Proof. Each column of vertices in $K_m \times K_n$ will induce a subgraph isomorphic to K_m, while each row is isomorphic to K_n. In each column and each row we use the maximum number of pure 4-cycles that is possible; given that m and n are both even, we thus use all edges but a 1-factor in each of K_m and K_n [9, 15]. By using the same decomposition in each row (resp. column), each row (resp. column) will have the same 1-factor left over, say F (resp. F'). To complete the 4-cycle decomposition we use the mixed 4-cycles of $F \times F'$. □

Lemma 3 *If $m, n \equiv 1 \pmod 8$ then $K_m \times K_n$ is 4-cycle decomposable.*

Proof. Each column of vertices corresponds to K_m. Since $m \equiv 1 \pmod 8$, we can completely decompose K_m into 4-cycles. Hence we can use all of the vertical edges of $K_m \times K_n$ in the construction of pure 4-cycles.

Likewise, since $n \equiv 1 \pmod 8$, all of the horizontal edges of $K_m \times K_n$ can be used by pure 4-cycles. □

Lemma 4 *If $m, n \equiv 5 \pmod 8$ then $K_m \times K_n$ is 4-cycle decomposable.*

Proof. Since $K_m \times K_n$ and $K_n \times K_m$ are isomorphic, we assume, without loss of generality, that $m \leq n$. Thus, with m and n both equivalent to 5 (mod 8), we have precisely two cases:

1. $m = n$
2. $m < n$

We consider each case separately.

Case 1 ($m = n$). Consider first the case of $K_5 \times K_5$. We present a 4-cycle decomposition of $K_5 \times K_5$, obtained by the unique 4-cycle decompositions of each of the subgraphs of $K_5 \times K_5$ shown in Figure 1.

For $m = n$ with $m > 5$, we use an iterative construction. From $K_m \times K_m$, we first remove four embedded copies of $K_{\frac{m-3}{2}} \times K_{\frac{m-3}{2}}$ and one copy of $K_5 \times K_5$. Pictorially, we remove the four copies of $K_{\frac{m-3}{2}} \times K_{\frac{m-3}{2}}$ from the four corners of $K_m \times K_m$ and the $K_5 \times K_5$ from the centre, as illustrated in Figure 2.

$K_{\frac{m-3}{2}} \times K_{\frac{m-3}{2}}$ can be assumed to be 4-cycle decomposable since $m \equiv 5 \pmod 8$ implies that either $\frac{m-3}{2} \equiv 1 \pmod 8$ in which case a 4-cycle decomposition exists by Lemma 3, or else $\frac{m-3}{2} \equiv 5 \pmod 8$ in which case we note that $5 \leq \frac{m-3}{2} < m$ and so we can assume the existence of a 4-cycle decomposition by induction.

Now consider the three middle rows and three middle columns, each of which is now isomorphic to $K_m \setminus K_5$. Each of these rows and columns can

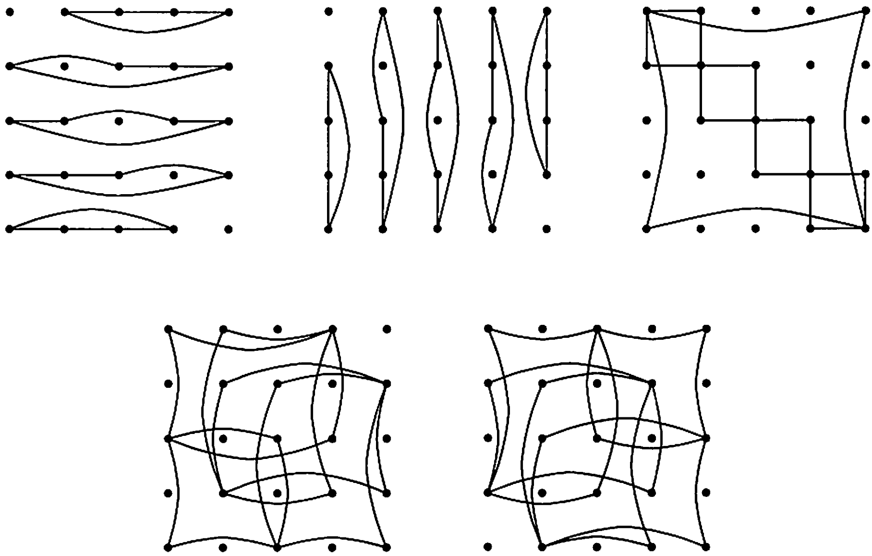

Figure 1: A 4-Cycle Decomposition of $K_5 \times K_5$

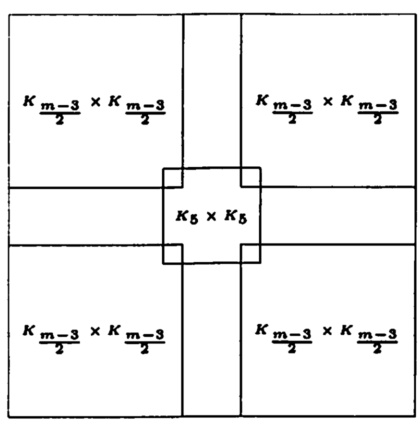

Figure 2: Subgraphs Embedded Within $K_m \times K_m$

be decomposed into pure 4-cycles. To see that this is so, let A be the set of five vertices from the deleted K_5 and let B be the other $m-5$ vertices. Fix one vertex v of A. The subgraph induced by $B \cup \{v\}$ is a complete graph on $m-4$ vertices, which is 4-cycle decomposable since $m-4 \equiv 1 \pmod 8$. The remaining edges now induce a complete bipartite graph with partition $(A \setminus \{v\}, B)$; this graph is isomorphic to $K_{4,m-5}$, which is 4-cycle decomposable by Corollary 1.

The two rows (resp. columns), one on each side of the three middle rows (resp. columns), are now isomorphic to K_m with one copy of K_5 and two copies of $K_{\frac{m-3}{2}}$ deleted, such that the two copies of $K_{\frac{m-3}{2}}$ are disjoint from each other but each shares a single vertex with the K_5. To see that each of these rows and columns is 4-cycle decomposable, let A and B be the sets of $\frac{m-3}{2}$ vertices from the deleted $K_{\frac{m-3}{2}}$'s and let X be the set of five vertices from the deleted K_5. Let $\{u\} = A \cap X$ and $\{v\} = B \cap X$. Consider now the complete bipartite graphs with partitions $(A \setminus \{u\}, B \setminus \{v\})$, $(A \setminus \{u\}, X \setminus \{u\})$, and $(B \setminus \{v\}, X \setminus \{v\})$; each is 4-cycle decomposable by Corollary 1.

All that now remains is to consider the $\frac{m-5}{2}$ top-most rows, the $\frac{m-5}{2}$ bottom-most rows, the $\frac{m-5}{2}$ left-most columns, and the $\frac{m-5}{2}$ right-most columns. Note that each of these rows and columns contains m vertices and $\frac{m^2+6m-15}{4}$ edges; in particular, note that $\frac{m^2+6m-15}{4} \equiv 2 \pmod 4$ and hence we cannot fully decompose the remaining edges into pure 4-cycles. However, the removal of a 6-cycle from each of these rows and columns does leave the proper number of edges for the rest of the edges to be decomposed into pure 4-cycles.

Let X be the set of the middle three vertices of some row (resp. column), A be a set of three of the $\frac{m-5}{2}$ left-most (resp. top-most) vertices of the row (resp. column), and B be a set of three of the $\frac{m-5}{2}$ right-most (resp. bottom-most) vertices of the row (resp. column). Then let P and Q be sets of $\frac{m-9}{2}$ vertices such that $A \cup P$ and $B \cup Q$ form the sets of $\frac{m-3}{2}$ vertices corresponding to the two copies of $K_{\frac{m-3}{2}}$ that have been deleted. The complete bipartite graphs having partitions $(P, B \cup X)$, $(Q, A \cup X)$, and (P, Q) are each decomposable into 4-cycles by Corollary 1. All of the remaining edges are incident only with vertices of A, B, and X; the subgraph induced by the vertex set $A \cup B \cup X$ is isomorphic to K_9 with two disjoint 3-cycles deleted.

Now fix a 1-factorisation of the complete bipartite graph having partition (A, B). We choose two of these 1-factors to form the 6-cycle we desire. We pair each of the three edges of the remaining 1-factor with one of the three edges having both end-vertices in X. Each of these pairs of edges can then be extended into a 4-cycle by the addition of one edge between A and X and one edge between B and X. The remaining twelve edges can be uniquely decomposed into three 4-cycles.

At this point we have only to handle the set of 6-cycles which remain in each of the outer $\frac{m-5}{2}$ rows and columns; their combination we wish to decompose into mixed 4-cycles. Given that the edges of the 6-cycles are incident only with vertices belonging both to one of the $\frac{m-5}{2}$ outer rows and one of the $\frac{m-5}{2}$ outer columns, we consider the $m-5$ by $m-5$ grid of vertices formed by the deletion of the middle 5 rows and 5 columns of our original m by m grid. In order to successfully obtain a decomposition into mixed 4-cycles, we need to have chosen each set A and B as well as each 6-cycle carefully in our previous step. Enumerate the first $\frac{m-5}{2}$ rows (resp. columns) in our present $m-5$ by $m-5$ grid, from top to bottom (resp. left to right) with the odd integers from 1 to $m-6$, and the following $\frac{m-5}{2}$ rows (resp. columns) with the even integers from 2 to $m-5$. Given the choice available in our selection of 6-cycles, we can assume to have selected our 6-cycles such that rows (resp. columns) 1 and 2 each have the 6-cycle $(1,2,3,4,5,6,1)$. For rows (resp. columns) $2i-1$ and $2i$, we have selected the 6-cycle $\sigma^{i-1}(1,2,3,4,5,6,1)$, where σ is the permutation $(m-5,m-7,\dots,6,4,2)(m-6,m-8,\dots,5,3,1)$, for $2 \le i \le \frac{m-5}{2}$. Having chosen our 6-cycles in such a fashion, we find that the edges from these 6-cycles can be uniquely decomposed into the mixed 4-cycles

$$((\sigma^{-i}(1),\sigma^{i}(1)),(\sigma^{-i}(1),\sigma^{i}(6)),(\sigma^{-i}(2),\sigma^{i}(6)),(\sigma^{-i}(2),\sigma^{i}(1)),(\sigma^{-i}(1),\sigma^{i}(1))),$$
$$((\sigma^{-i}(1),\sigma^{i}(1)),(\sigma^{-i}(1),\sigma^{i}(2)),(\sigma^{-i}(6),\sigma^{i}(2)),(\sigma^{-i}(6),\sigma^{i}(1)),(\sigma^{-i}(1),\sigma^{i}(1))),$$
$$((\sigma^{-i}(1),\sigma^{i}(3)),(\sigma^{-i}(1),\sigma^{i}(4)),(\sigma^{-i}(m-5),\sigma^{i}(4)),(\sigma^{-i}(m-5),\sigma^{i}(3)),(\sigma^{-i}(1),\sigma^{i}(3))),$$
$$((\sigma^{-i}(1),\sigma^{i}(3)),(\sigma^{-i}(1),\sigma^{i}(2)),(\sigma^{-i}(2),\sigma^{i}(2)),(\sigma^{-i}(2),\sigma^{i}(3)),(\sigma^{-i}(1),\sigma^{i}(3))),$$
$$((\sigma^{-i}(1),\sigma^{i}(5)),(\sigma^{-i}(1),\sigma^{i}(6)),(\sigma^{-i}(m-5),\sigma^{i}(6)),(\sigma^{-i}(m-5),\sigma^{i}(5)),(\sigma^{-i}(1),\sigma^{i}(5))),$$
$$((\sigma^{-i}(1),\sigma^{i}(5)),(\sigma^{-i}(1),\sigma^{i}(4)),(\sigma^{-i}(2),\sigma^{i}(4)),(\sigma^{-i}(2),\sigma^{i}(5)),(\sigma^{-i}(1),\sigma^{i}(5))),$$

for $0 \le i \le \frac{m-7}{2}$, where each ordered pair (a,b) refers to the vertex in row number a and column number b of the $m-5$ by $m-5$ grid.

Case 2 ($m < n$). We begin by removing from the graph $K_m \times K_n$ an embedded $K_m \times K_{n-8}$, induced by the vertices of the right-most $n-8$ columns of our m by n grid of vertices. Since $n-8 \equiv 5 \pmod 8$ and $5 \le n-8 < n$, we can inductively assume that the edges of this $K_m \times K_{n-8}$ are 4-cycle decomposable.

If $m > 5$ then we decompose the remaining edges in each of the bottom-most $m-5$ rows into pure 4-cycles. Let A be the set of the left-most eight vertices, let B be the set of the right-most $n-8$ vertices, and let $v \in B$. The complete bipartite graph with partition $(A, B\backslash\{v\})$ is 4-cycle decomposable by Corollary 1. The remaining edges now constitute a complete graph, K_9, on the vertex set $A \cup \{v\}$, which is 4-cycle decomposable.

Also if $m > 5$, then in each of the left-most eight columns we obtain several pure 4-cycles. Let A be the set of the top-most five vertices, let B be the set of the bottom-most $m-5$ vertices, and let $v \in A$. The complete bipartite graph with partition $(A \setminus \{v\}, B)$ is 4-cycle decomposable by Corollary 1. Furthermore, the vertices $B \cup \{v\}$ induce a complete graph on $m-4$ vertices, which is 4-cycle decomposable since $m - 4 \equiv 1 \pmod 8$. Each of these columns is thus left with the edges of a K_5, contained within the subgraph induced by the top-most five vertices A.

The remainder of the construction (including when $m = 5$) involves only the top-most five rows. In each of these five rows, we choose two disjoint sets of vertices, A and B, each consisting of four of the eight left-most vertices in the row. Let C be the set of the $n-8$ right-most vertices in the row and let u, v, w be distinct vertices in C. First we note that we can use Corollary 1 to obtain pure 4-cycles from the two complete bipartite graphs with partitions $(A, C \setminus \{u, v, w\})$ and $(B, C \setminus \{u, v, w\})$.

What now remains in each row is the subgraph induced by the set of vertices $A \cup B \cup \{u, v, w\}$; this subgraph is isomorphic to K_{11} with one 3-cycle deleted. We remove the copy of $K_{4,4}$ having partition (A, B) but we do not decompose the edges of this $K_{4,4}$ into 4-cycles; rather they will be used to form mixed 4-cycles. The 36 remaining edges can now be decomposed into pure 4-cycles as shown in Figure 3.

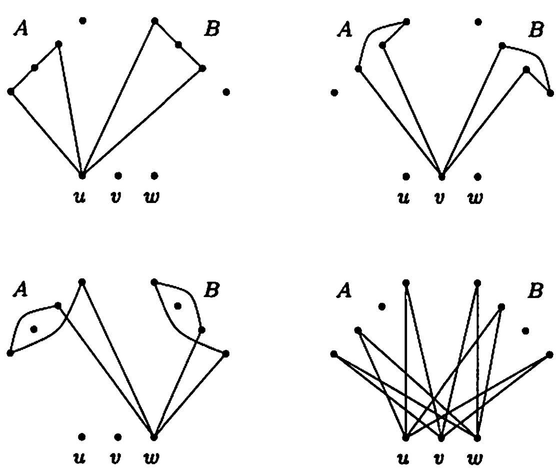

Figure 3: Nine Pure 4-Cycles in Each Row

The only edges in the $K_5 \times K_n$ which have yet to be decomposed into 4-cycles are the ten edges in each of the left-most eight columns and the sixteen edges associated with the $K_{4,4}$ remaining in each of the five rows. Together, these 160 edges will be decomposed into mixed 4-cycles. Focussing on the vertices of the $K_5 \times K_n$ which are incident with these edges, we need only consider the left-most 5 by 8 grid of vertices; let H be the subgraph induced by these forty vertices. Note that we had choice in our selection of the sets A and B, and hence in the vertices forming the partition of each of the copies of $K_{4,4}$; we may thus assume that the $K_{4,4}$ in each row was selected such that the eight vertices in each row of H fall into two colour classes as shown in Figure 4.

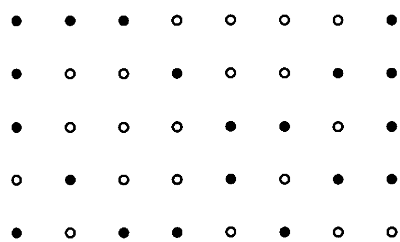

Figure 4: Vertices of the Graph H

A 4-cycle decomposition of H can now be obtained by the unique 4-cycle decompositions of each of the subgraphs of H shown in Figure 5. □

4 Main Result

As previously stated, the main result of this paper is the following theorem:

Theorem 3 *$K_m \times K_n$ is 4-cycle decomposable if and only if either*

1. $m, n \equiv 0 \pmod 2$,

2. $m, n \equiv 1 \pmod 8$, or

3. $m, n \equiv 5 \pmod 8$.

Proof. That the stated conditions are necessary is proved in Lemma 1; their sufficiency is shown in Lemmata 2, 3, and 4. □

5 A Second Construction for $m, n \equiv 5 \pmod 8$

We note that the proof of Lemma 4 is recursive. In this section we present a direct construction technique which uses only a handful of smaller decompositions; specifically those for the graphs $K_r \times K_s$ where $r, s \in \{5, 13, 21, 29, 37\}$, each of which we can obtain from Lemma 4.

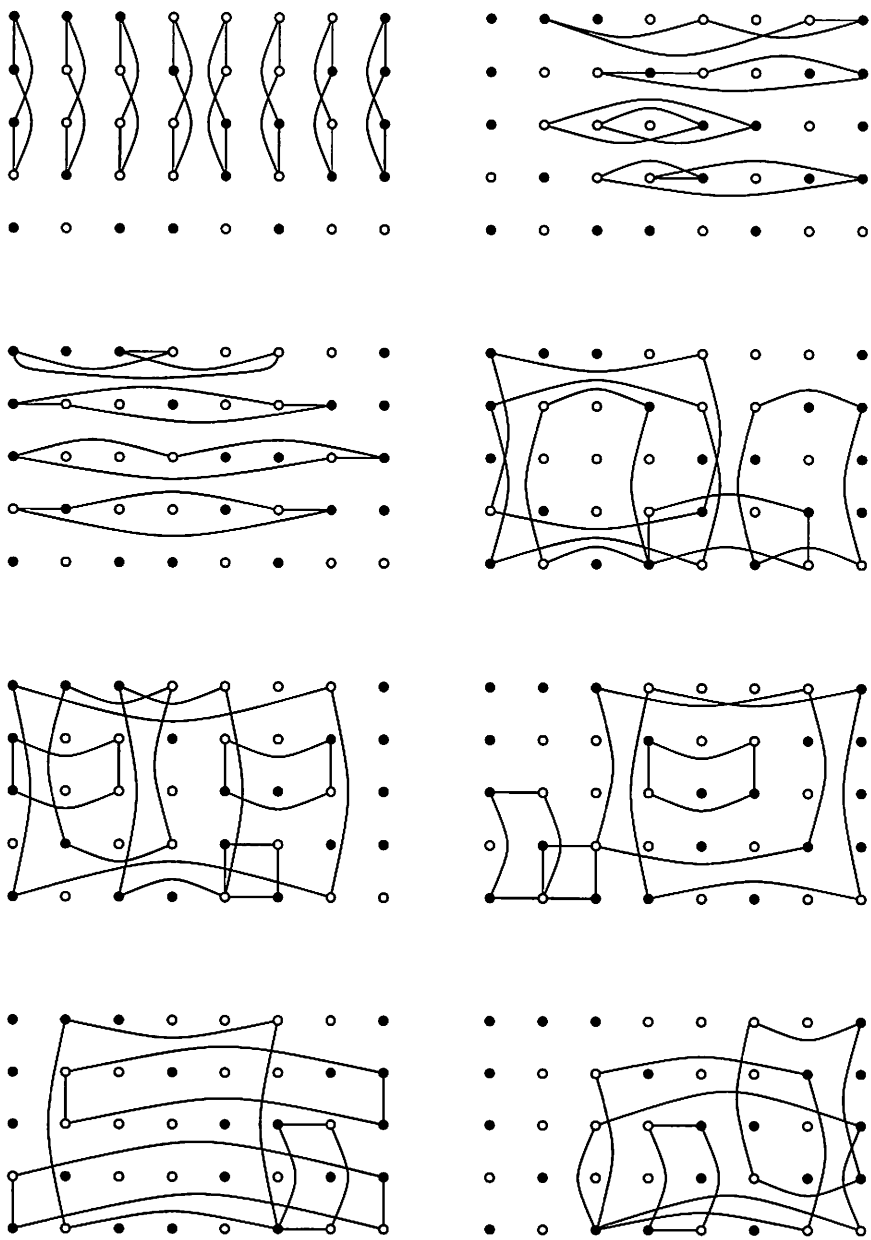

Figure 5: A 4-Cycle Decomposition of the Graph H

Again, we visualise an m by n grid of vertices. Let $r, s \in \{5, 13, 21, 29, 37\}$ such that $r \equiv m \pmod{40}$ and $s \equiv n \pmod{40}$. In each of the m rows, identify $\frac{n-s}{5}$ disjoint sets of five vertices, C_i for $i = 1, \ldots, \frac{n-s}{5}$, and one set of s vertices, C_0. Likewise, in each of the n columns, identify $\frac{m-r}{5}$ disjoint sets of five vertices, R_j for $j = 1, \ldots, \frac{m-r}{5}$, and one set of r vertices, R_0.

Using the sets C_i to index the columns of vertices, and the sets R_j to index the rows of vertices, consider now the vertices found in the intersection of C_i and R_j, for $i = 1, \ldots, \frac{n-s}{5}$ and $j = 1, \ldots, \frac{m-r}{5}$; these vertices form a 5 by 5 grid inducing the subgraph $K_5 \times K_5$, which we decompose into 4-cycles. The vertices found in the intersection of C_0 and R_j, for $j = 1, \ldots, \frac{m-r}{5}$, form a 5 by s grid inducing the subgraph $K_5 \times K_s$, which we decompose into 4-cycles. The vertices found in the intersection of C_i and R_0, for $i = 1, \ldots, \frac{n-s}{5}$, form an r by 5 grid inducing the subgraph $K_r \times K_5$, which we decompose into 4-cycles. And the intersection of C_0 and R_0 forms an r by s grid that induces the subgraph $K_r \times K_s$, which again we decompose into 4-cycles.

All remaining 4-cycles in our decomposition of $K_m \times K_n$ will be pure.

Notice that $\frac{n-s}{5}$ is divisible by 8, and so the number of sets of vertices, C_i, in each row is congruent to 1 (mod 8). If we identify each set into a single point, we thus obtain a complete graph, $K_{1+\frac{n-s}{5}}$, which is 4-cycle decomposable. We use a 4-cycle decomposition of this complete graph to dictate the manner in which we obtain pure 4-cycles for our decomposition of $K_m \times K_n$. Specifically, if $(C_w, C_x, C_y, C_z, C_w)$ is a 4-cycle in the 4-cycle decomposition of $K_{1+\frac{n-s}{5}}$, then we obtain pure 4-cycles in $K_m \times K_n$ by decomposing the complete bipartite graph having the partition $(C_w \cup C_y, C_x \cup C_z)$ into 4-cycles, using Corollary 1.

Similarly, $\frac{m-r}{5}$ is divisible by 8, and so we can obtain a 4-cycle decomposition of the complete graph, $K_{1+\frac{m-r}{5}}$, obtained by identifying each set R_j into a single vertex. Each 4-cycle $(R_w, R_x, R_y, R_z, R_w)$ in this $K_{1+\frac{m-r}{5}}$ is then used to obtain pure 4-cycles in $K_m \times K_n$ by decomposing the complete bipartite graph on partition $(R_w \cup R_y, R_x \cup R_z)$ into 4-cycles, again using Corollary 1.

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