# On Matching Equivalent Graphs

Jonathan Earl Holland and Earl Glen Whitehead, Jr.
Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260, USA

**Dedicated to Anne Penfold Street.**

**Abstract**

Two graphs are matching equivalent if they have the same matching polynomial. We prove that several infinite families of pairs of graphs are pairwise matching equivalent. We also prove some divisibility relations among matching polynomials. We also show that matching polynomials of certain graphs are a polynomial model for the Fibonacci numbers and for the Lucas numbers.

## 1 Introduction.

Most of this section is based on Farrell's introductory paper [1]. Let $G$ be a graph. A matching $\mathcal{M}$ is a spanning subgraph of $G$ with components consisting of nodes and edges only. A $k$-matching is a matching containing $k$ edges. If $G$ has $p$ nodes, then $0 \leq k \leq \lfloor p/2 \rfloor$ where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$. When $G$ has $p$ nodes, a $k$-matching of $G$ will contain $p - 2k$ nodes (as components of the matching).

Let $\mathcal{M}$ be a $k$-matching of $G$ where $G$ has $p$ nodes. To each node of $\mathcal{M}$, we assign the weight $w_1$. To each edge of $\mathcal{M}$, we assign the weight $w_2$. Thus we associate with $\mathcal{M}$ the weight $w_1^{p-2k} w_2^k$. (The weight associated with a matching is the product of the weights of its components.) The matching polynomial of $G$, denoted $M(G; \vec{w})$, is obtained by summing over all matchings of $G$. Let $a_k$ denote the number of $k$-matchings of $G$. Thus we have

$$M(G; \vec{w}) = \sum_{k=0}^{\lfloor p/2 \rfloor} a_k w_1^{p-2k} w_2^k.$$

The weight vector $\vec{w}$ is given by $\vec{w} = (w_1, w_2)$. If we let $w_1 = w_2 = x$, then the resulting polynomial is called the simple matching polynomial of $G$, denoted $M(G; x)$.

JCMCC **28** (1998), pp. 227-235

From the definition of a matching, we know that the edges of a matching are independent. Thus, the inclusion of an edge $\alpha$ in a matching $\mathcal{M}$ implies the exclusion of all edges adjacent to $\alpha$. The set of all matchings of a graph $G$ can be partitioned into two classes: $(i)$ those matchings which contain the edge $\alpha$ and $(ii)$ those matchings which do not contain the edge $\alpha$. Thus, we have the following theorem which is Theorem 1 in [1].

**Theorem 1** *Let $\alpha$ be an edge of $G$. Let $G'$ be the graph obtained from $G$ by deleting $\alpha$. Let $G''$ be the graph obtained from $G$ by removing the endnodes of $\alpha$. Then, we have*

$$M(G;\vec{w}) = M(G';\vec{w}) + w_2 M(G'';\vec{w})$$

The following theorem is Theorem 2 in [1].

**Theorem 2** *Let $G$ be a graph containing a node $y$ with valency $v_y$. Let $G-\{y\}$ be the graph obtained from $G$ by removing $y$. Let $G_i''$ be the graph obtained by removing node $y$ and an adjacent node $i$. Then*

$$M(G;\vec{w}) = w_1 M(G-\{y\}) + w_2 \sum_{i=1}^{v_y} M(G_i'';\vec{w}).$$

Let $P_p$ denote the path with $p$ nodes. The following equation is Equation 1 in [1].

$$M(P_p;\vec{w}) = w_1 M(P_{p-1};\vec{w}) + w_2 M(P_{p-2};\vec{w}) \quad for \quad p \geq 2 \tag{1.1}$$

The following theorem is Theorem 9 in [1]. This theorem shows that the binomial coefficients are the coefficients of $M(P_p;\vec{w})$.

**Theorem 3** *Let $P_p$ be a path with $p(\geq 0)$ nodes. Then*

$$M(P_p;\vec{w}) = \sum_{k=0} \binom{p-k}{k} w_1^{p-2k} w_2^k.$$

The following theorem is Theorem 10 in [1].

**Theorem 4** $M(P_{n+k};\vec{w}) = M(P_n;\vec{w})M(P_k;\vec{w}) + w_2 M(P_{n-1};\vec{w})M(P_{k-1};\vec{w})$

Let $C_p$ denote the cycle with $p$ nodes. By removing any edge of $C_p$ where $p \geq 3$, we obtain the path $P_p$. The following theorem is Lemma 1 in [1].

**Theorem 5** $M(C_p;\vec{w}) = M(P_p;\vec{w}) + w_2 M(P_{p-2};\vec{w})$ *for* $p \geq 2$.

The following theorem is part of Theorem 12 in [1].

**Theorem 6** *$M(C_p;\vec{w})$ satisfies the following recurrence*

$$M(C_p;\vec{w}) = w_1 M(C_{p-1};\vec{w}) + w_2 M(C_{p-2};\vec{w}) \quad for \quad p \geq 3,$$

*with the initial conditions $M(C_1;\vec{w}) = w_1$ and $M(C_2;\vec{w}) = w_1^2 + 2w_2$.*

A $\theta$-graph is a connected graph consisting of 3 edge-disjoint paths between two vertices of degree 3. All other vertices have degree 2. Let these paths have $p$, $q$ and $r$ nodes. We denote this graph as $\theta_{p,q,r}$. We observe that $\theta_{p,q,r}$ has $p+q+r-4$ nodes and $p+q+r-3$ edges. The following theorem is a correction of Theorem 14 in [1]. This corrected theorem is proven in the next section.

**Theorem 7**

$$M(\theta_{p,q,r};\vec{w}) = M(P_{p+q+r-4};\vec{w}) + w_2(M(P_{p-2};\vec{w})M(P_{q+r-4};\vec{w}) +$$
$$M(P_{q-2};\vec{w})M(P_{p+r-4};\vec{w}) + M(P_{r-2};\vec{w})M(P_{p+q-4};\vec{w}) -$$
$$M(P_{p-2};\vec{w})M(P_{q-2};\vec{w})M(P_{r-2};\vec{w})).$$

The following theorem is Theorem 8 in [2].

**Theorem 8** *For all positive integers $r$,*

$$M(P_r;\vec{w})M(C_{r+1};\vec{w}) = M(P_{2r+1};\vec{w})$$

# 2 Main Results.

Before we derive our main results, we need a few definitions and some notation. Let $G$ and $H$ be two graphs whose vertex sets are disjoint. Let $G \cup H$ be the graph whose vertex set is union of the vertex set of $G$ and the vertex set of $H$ and whose edge set is the union of the edge set of $G$ and the edge set of $H$. Let $Y_{i,j,k}$ denote the y-shaped graph obtained by taking a path on $i$ nodes, a path on $j$ nodes and a path on $k$ nodes and identifying one of the endnodes of each of these three paths. The resulting graph has $i+j+k-2$ nodes. Let $C^*_{i,j}$ denote the graph obtained by taking a cycle on $i$ nodes along with a path on $j$ nodes and identifying one of the endnodes the path with one of the nodes of the cycle. The resulting graph has $i+j-1$ nodes. In the proofs of Theorems 7, 13, 14 and 15, we will use the notation for a graph to mean the matching polynomial of that graph.

**Proof of Theorem 7.**

$$\theta_{p,q,r} = C^*_{q+r-2,p-1} + w_2 Y_{p-2,q-1,r-1}$$
$$= P_{p-2} \cup C_{q+r-2} + w_2 P_{p-3} \cup P_{q+r-3}$$

$$+w_2M(P_{r-2}\cup P_{p+q-4}+w_2^2P_{p-3}\cup P_{q-2}\cup P_{r-3})$$
$$=(P_{p-2}\cup P_{q+r-2}+w_2P_{p-2}\cup P_{q+r-4}+w_2P_{p-3}\cup P_{q+r-3}$$
$$+w_2P_{p-2}\cup P_{q-2}\cup P_{r-2}+w_2^2P_{p-3}\cup P_{q-3}\cup P_{r-2}+w_2^2P_{p-3}\cup P_{q-2}\cup P_{r-3}$$
$$=P_{p-2}\cup P_{q+r-2}+w_2P_{p-2}\cup P_{q-2}\cup P_{r-2}+w_2^2P_{p-2}\cup P_{q-3}\cup P_{r-3}+w_2P_{p-3}\cup P_{q+r-3}$$
$$+w_2M(P_{p-2}\cup P_{q-2}\cup P_{r-2};\vec{w})+w_2^2M(P_{p-3}\cup P_{q-3}\cup P_{r-2};\vec{w})$$
$$+w_2^2M(P_{p-3}\cup P_{q-2}\cup P_{r-3};\vec{w})$$
$$=P_{p+q+r-4}+2w_2P_{p-2}\cup P_{q-2}\cup P_{r-2}+w_2^2(P_{p-2}\cup P_{q-3}\cup P_{r-3}+$$
$$P_{p-3}\cup P_{q-2}\cup P_{r-3}+P_{p-3}\cup P_{q-3}\cup P_{r-2})$$

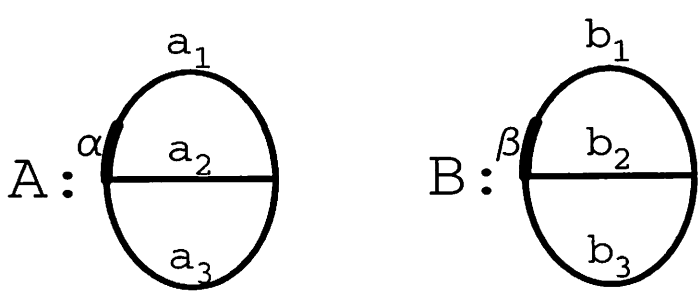
since $P_{p+q+r-4}=P_{p-2}\cup P_{q+r-2}+w_2P_{p-3}\cup P_{q+r-3}$. We add and subtract $w_2P_{p-2}\cup P_{q-2}\cup P_{r-2}$.

$$\theta_{p,q,r}=P_{p+q+r-4}+3w_2P_{p-2}\cup P_{q-2}\cup P_{r-2}+$$
$$w_2^2(P_{p-2}\cup P_{q-3}\cup P_{r-3}+P_{p-3}\cup P_{q-2}\cup P_{r-3}+P_{p-3}\cup P_{q-3}\cup P_{r-2})-$$
$$w_2P_{p-2}\cup P_{q-2}\cup P_{r-2}$$
$$=P_{p+q+r-4}+w_2(P_{p-2}\cup P_{q-2}\cup P_{r-2}+w_2P_{p-2}\cup P_{q-3}\cup P_{r-3}+$$
$$P_{p-2}\cup P_{q-2}\cup P_{r-2}+w_2P_{p-3}\cup P_{q-2}\cup P_{r-3}+$$
$$P_{p-2}\cup P_{q-2}\cup P_{r-2}+w_2P_{p-3}\cup P_{q-3}\cup P_{r-2})-w_2P_{p-2}\cup P_{q-2}\cup P_{r-2}$$
$$=P_{p+q+r-4}+w_2(P_{p-2}\cup P_{q+r-4}+P_{q-2}\cup P_{p+r-4}+$$
$$P_{r-2}\cup P_{p+q-4}-P_{p-2}\cup P_{q-2}\cup P_{r-2})\quad\Box$$

**Theorem 9** *$\theta_{x_1,x_2,x_3}$ and $\theta_{y_1,y_2,y_3}$ are matching equivalent if there exists multisets $\{a_1,a_2,a_3\}=\{x_1,x_2,x_3\}$ and $\{b_1,b_2,b_3\}=\{y_1,y_2,y_3\}$ such that the following conditions hold:*

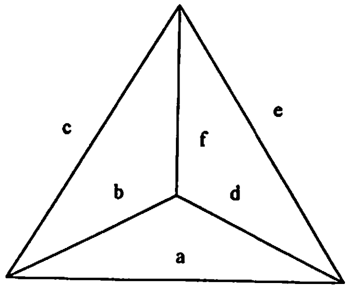
*(i)* $a_1+1=b_2+b_3$ *and* $b_1+1=a_2+a_3$;

*(ii)* $\{a_1-1,a_2,a_3\}=\{b_1-1,b_2,b_3\}$.




**Proof.** In the figure above, graph $A$ is $\theta_{a_1,a_2,a_3}$ with one edge denoted $\alpha$ and graph $B$ is $\theta_{b_1,b_2,b_3}$ with one edge denoted $\beta$. We apply Theorem 1 to graph $A$ obtaining $M(A;\vec{w}) = M(A';\vec{w}) + w_2 M(A'';\vec{w})$. Letting edge $\beta$ in graph $B$ play the role of edge $\alpha$ in Theorem 1, we apply Theorem 1 to graph $B$ obtaining $M(B;\vec{w}) = M(B';\vec{w}) + w_2 M(B'';\vec{w})$. By Theorem 7 in [2], we have $M(A';\vec{w}) = M(B';\vec{w})$. Since graph $A''$ is isomorphic to graph $B''$, we have $M(A'';\vec{w}) = M(B'';\vec{w})$. □

$K_4(a,b,c,d,e,f)$ :




**Theorem 10** *$K_4(a,1,1,2,1,f)$ and $K_4(a',1,1,2,1,f')$ are matching equivalent whenever $a+f=a'+f'$.*

**Proof.** It suffices to show that $K_4(a-1,1,1,2,1,f+1)$ and $K_4(a,1,1,2,1,f)$ are matching equivalent. Applying Theorem 1 to the edge "c" in both graphs, we obtain

$$M(K_4(a-1,1,1,2,1,f+1);\vec{w}) = M(\theta_{a,2,f+2};\vec{w}) + w_2 M(P_{a+f};\vec{w})$$

and

$$M(K_4(a,1,1,2,1,f);\vec{w}) = M(\theta_{a+1,2,f+1};\vec{w}) + w_2 M(P_{a+f};\vec{w})$$

By Theorem 9, $\theta_{a,2,f+2}$ and $\theta_{a+1,2,f+1}$ are matching equivalent. Thus, $K_4(a-1,1,1,2,1,f+1)$ and $K_4(a,1,1,2,1,f)$ are matching equivalent. □

**Theorem 11** *$K_4(1,1,1,d,e,f)$ and $K_4(1,1,1,d',e',f')$ are matching equivalent whenever $d+e+f=d'+e'+f'$.*

**Proof.** By symmetry, it suffices to show that $K_4(1,1,1,d,e,f)$ and $K_4(1,1,1,d-1,e,f+1)$ are matching equivalent. Applying Theorem 1 to the edge "a" in both graphs, we obtain

$$M(K_4(1,1,1,d,e,f);\vec{w}) = M(\theta_{d+e,2,f};\vec{w}) + w_2 M(P_{d+e+f-2};\vec{w})$$

and

$$M(K_4(1,1,1,d-1,e,f+1);\vec{w}) = M(\theta_{d+e-1,2,f+1};\vec{w}) + w_2 M(P_{d+e+f-2};\vec{w})$$

By Theorem 9, $\theta_{d+e,2,f}$ and $\theta_{d+e-1,2,f+1}$ are matching equivalent. Thus, $K_4(1,1,1,d,e,f)$ and $K_4(1,1,1,d-1,e,f+1)$ are matching equivalent. □

Let $K_1$ denote the complete graph on one node.

**Theorem 12** *$C_n \cup K_1$ and $Y_{n-1,2,2}$ are matching equivalent for every positive integer $n \geq 3$.*

**Proof.** This result can be easily verified by applying Theorem 1 to appropriately selected edges. □

**Theorem 13** *$M(P_n;\vec{w})|M(P_{k(n+1)+n};\vec{w})$ for $k \geq 0$.*

**Proof.** When $k = 0$, the statement of this theorem becomes the triviality $P_n|P_n$. Assume that $P_n|P_{j(n+1)+n}$ for some $j \geq 0$. Let us prove this statement for $k = j+1$. Changing the notation in Theorem 4, we obtain $P_{a+b} = P_aP_b + w_2P_{a-1}P_{b-1}$. Let $a = j(n+1)+n$ and $b = n+1$. Then,

$$P_{a+b} = P_{(j+1)(n+1)+n} = P_{j(n+1)+n}P_{n+1} + w_2P_{j(n+1)+n-1}P_n.$$

The assumption that $P_n|P_{j(n+1)+n}$ is equivalent to $P_{j(n+1)+n} \equiv 0 \pmod{P_n}$ whence

$$P_{(j+1)(n+1)+n} \equiv w_2P_{j(n+1)+n-1}P_n \equiv 0 \pmod{P_n}.$$ □

**Theorem 14** *$M(C_{a+b};\vec{w}) = M(P_a;\vec{w})M(C_b;\vec{w}) + w_2M(P_{a-1};\vec{w})M(C_{b-1};\vec{w})$ for $b \geq 2$.*

**Proof.** By letting $p = a+b$ in Theorem 5, we have

$$C_{a+b} = P_{a+b} + w_2P_{a+b-2}.$$

By applying Theorem 4 twice, we obtain

$$\begin{aligned} C_{a+b} &= (P_aP_b + w_2P_{a-1}P_{b-1}) + w_2(P_aP_{b-2} + w_2P_{a-1}P_{b-3}) \\ &= P_a(P_b + w_2P_{b-2}) + w_2P_{a-1}(P_{b-1} + w_2P_{b-3}) \\ &= P_aC_b + w_2P_{a-1}C_{b-1}. \end{aligned}$$ □

**Theorem 15** *$M(C_n;\vec{w})|M(C_{2kn+n};\vec{w})$ for $k \geq 0$.*

**Proof.** We proceed by induction on $k$. If $k = 0$, the statement of this theorem becomes the triviality $C_n|C_n$. Let us prove that the theorem holds for $k = 1$. By Theorem 14, taking $a = 2n$ and $b = n$, we obtain

$$C_{3n} = P_{2n}C_n + w_2P_{2n-1}C_{n-1} \equiv w_2P_{2n-1}C_{n-1} \pmod{C_n}.$$

By letting $r = n - 1$ in Theorem 8, we have $C_{3n} \equiv 0 \pmod{C_n}$. Thus, the theorem holds for $k = 1$. Assume that it holds for $0 \leq k \leq j$ where $j \geq 1$. Applying Theorem 14, we obtain

$$C_{2jn+n} = P_{2jn-n}C_{2n} + w_2P_{2jn-n-1}C_{2n-1}.$$

From our assumption, it follows that

$$P_{2jn-n}C_{2n} + w_2P_{2jn-n-1}C_{2n-1} \equiv 0 \pmod{C_n}. \tag{2.1}$$

Let $k = j + 1$ and apply Theorem 14, obtaining

$$C_{2kn+n} = C_{2jn+3n} = P_{2jn+n}C_{2n} + w_2P_{2jn+n-1}C_{2n-1}.$$

Now we apply Theorem 4 to $P_{2jn+n}$ and to $P_{2jn+n-1}$, obtaining

$$C_{2jn+3n} = (P_{2jn-n}P_{2n}+w_2P_{2jn-n-1}P_{2n-1})C_{2n}+w_2(P_{2jn-n-1}P_{2n}+w_2P_{2jn-n-2}P_{2n-1})C_{2n-1}$$
$$= P_{2n}(P_{2jn-n}C_{2n}+w_2P_{2jn-n-1}C_{2n-1})+w_2P_{2n-1}(P_{2jn-n-1}C_{2n}+w_2P_{2jn-n-2}C_{2n-1})$$
$$\equiv w_2P_{2n-1}(P_{2jn-n-1}C_{2n} + w_2P_{2jn-n-2}C_{2n-1}) \pmod{C_n}$$

by equation 2.1. By Theorem 8, we have

$$w_2P_{2n-1}(P_{2jn-n-1}C_{2n} + w_2P_{2jn-n-2}C_{2n-1}) \equiv 0 \pmod{C_n}. \quad \square$$

**Theorem 16** $M(C_n; \vec{w})|M(P_{2kn-1}; \vec{w})$ *for* $k \geq 1$.

**Proof.** Theorem 8, with $r = n - 1$, shows that this property holds for $k = 1$. Assume that it holds for $1 \leq k \leq j$. By Theorem 4, we have

$$P_{2(j+1)n-1} = P_{2n-1}P_{2jn} + w_2P_{2n-2}P_{2jn-1}$$

which, by our assumption, becomes

$$P_{2(j+1)n-1} \equiv 0 \pmod{C_n}. \quad \square$$

# 3 Fibonacci and Lucas Numbers.

The Fibonacci numbers are defined as follows:

$$F_0 = 1, F_1 = 1, F_p = F_{p-1} + F_{p-2} \quad for \quad p \geq 2.$$

The Lucas numbers are defined as follows:

$$L_1 = 1, L_2 = 3, L_p = L_{p-1} + L_{p-2} \quad for \quad p \geq 3.$$

From Theorem 3, we obtain that $M(P_0; \vec{w}) = 1$ and $M(P_1; \vec{w}) = w_1$. Based on these initial conditions and the recurrence in Equation 1.1, we can prove that

$$M(P_p; (1,1)) = F_p \quad for \quad p \geq 0.$$

In words, the matching polynomials of paths are a polynomial model for the Fibonacci numbers. In Theorem 6, the initial conditions and recurrence for $M(C_p; \vec{w})$ are given. Based on these initial conditions and recurrence, we can prove that

$$M(C_p; (1,1)) = L_p \quad for \quad p \geq 1.$$

In words, the matching polynomials of cycles are a polynomial model for the Lucas numbers.

If we let $w_1 = w_2 = 1$, then Theorem 3 yields the following equation.

$$F_p = \sum_{k=0} \binom{p-k}{k} \quad for \quad p \geq 0 \tag{3.1}$$

Let $Z^+$ denote the positive integers. If we let $w_1 = w_2 = 1$, then Theorem 4 yields the following equation.

$$F_{n+k} = F_n F_k + F_{n-1} F_{k-1} \quad where \quad n, k \in Z^+ \tag{3.2}$$

If we let $w_1 = w_2 = 1$, then Theorem 5 yields the following equation.

$$L_p = F_p + F_{p-2} \quad for \quad p \geq 2 \tag{3.3}$$

If we let $w_1 = w_2 = 1$, then Theorem 13 yields the divisibility relation.

$$F_n | F_{k(n+1)+n} \quad for \quad k \geq 0 \tag{3.4}$$

If we let $w_1 = w_2 = 1$, then Theorem 14 yields the following equation.

$$L_{a+b} = F_a L_b + F_{a-1} L_{b-1} \quad where \quad a, b \in Z^+ \tag{3.5}$$

If we let $w_1 = w_2 = 1$, then Theorem 15 yields the divisibility relation.

$$L_n | L_{2kn+n} \quad where \quad k \in Z^+ \cup \{0\} \tag{3.6}$$

If we let $w_1 = w_2 = 1$, then Theorem 8 yields the following equation.

$$F_r L_{r+1} = F_{2r+1} \quad where \quad r \in Z^+ \tag{3.7}$$

After making some slight adjustments for a different definition of the Fibonacci numbers, we see that Equations 3.1, 3.2 and 3.3 correspond to Problems 4, 7 and 8 on page 204 of [4].

# References

[1] E. J. Farrell, An introduction to matching polynomials, *J. Combinat. Theory* **27B** (1979), 75-86.

[2] E. J. Farrell and S. A. Wahid, Some general classes of comatching graphs, *Internat. J. Math. & Math. Sci.* **10** (1987), 519-524.

[3] F. Harary, *Graph Theory* (Addison-Wesley, Reading, Mass., 1969).

[4] I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An Introduction to the Theory of Numbers*, 5th Ed. (Wiley, New York, 1991).