Kirkman Packing and Covering Designs

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Dedicated to Anne Penfold Street.

Abstract

Černý, Horák, and Wallis introduced a generalization of Kirkman's Schoolgirl Problem to the case where the number of schoolgirls is not a multiple of three; they require all blocks to be of size three, except that each resolution class should contain either one block of size two (when $v \equiv 2 \bmod 3$) or one block of size four (when $v \equiv 1 \bmod 3$). We consider the problem of determining the maximum (resp. minimum) possible number of resolution classes so that any pair of elements (schoolgirls) is covered at most (resp. at least) once.

1 Introduction

Let X be a set of v elements. A *packing* (resp. *covering*) of X is a collection of subsets of X (called *blocks*) such that any pair of distinct elements from X occur together in at most (resp. at least) one block in the collection. A packing or covering is called *resolvable* if its block set admits a partition into parallel classes, each parallel class being a partition of the element set X.

Suppose now that $v \equiv 3 \bmod 6$. A Kirkman Triple System KTS (v) is a collection $\mathcal{T}$ of 3-subsets of X (triples) such that any pair of distinct elements from X occur together in exactly one triple, and such that $\mathcal{T}$ admits a partition into parallel classes. Thus, a KTS (v) is both a resolvable packing and a resolvable covering of a v-set by triples. It is well known that a KTS (v) exists if and only if $v \equiv 3 \bmod 6$.

Kotzig and Rosa [7] posed the problem of determining how 'close' one could come to a Kirkman Triple System when the number v of elements satisfies $v \equiv 0 \bmod 6$; that is, what is the maximum possible number of parallel classes in a resolvable packing of a v-set by triples? It is easy to see that this number cannot exceed $\frac{v}{2} - 1$, and a packing that achieves this bound is called a Nearly Kirkman Triple System NKTS (v).

Theorem 1.1 [7, 3, 2, 10] *There exists an NKTS (v) if and only if $v \equiv 0 \bmod 6$ and $v \geq 18$.*

More recently, Assaf, Mendelsohn, and Stinson [1] posed the problem of determining the minimum possible number of parallel classes in a resolvable covering of a v-set by triples, when $v \equiv 0 \bmod 6$. This number cannot be less than $\frac{v}{2}$.

Theorem 1.2 [1, 8] *There exists a resolvable covering of a v-set by $\frac{v}{2}$ parallel classes of triples if and only if $v \equiv 0 \bmod 6$ and $v \geq 18$.*

Černý, Horák, and Wallis [4] introduced a particular generalization of Kirkman Triple Systems to the case where v is not a multiple of 3. They require all blocks to be of size 3, except that each resolution class should contain either one block of size 2 (when $v \equiv 2 \bmod 3$) or one block of size four (when $v \equiv 1 \bmod 3$). Thus, we define a *Kirkman Packing Design* KPD (v) (resp. *Kirkman Covering Design* KCD (v)) to be a resolvable packing (resp. covering) of a v-set by the maximum (resp. minimum) possible number of resolution classes of this type. It is not difficult to determine upper (resp. lower) bounds on these numbers. Thus, if $v \equiv 2 \bmod 3$ a resolution class covers $v - 1$ pairs, whence a KPD (v) will contain at most $\lfloor \frac{v}{2} \rfloor$ classes, while a KCD (v) will contain at least $\lceil \frac{v}{2} \rceil$ classes. On the other hand, when $v \equiv 1 \bmod 3$ and $v \geq 7$, a resolution class covers $v + 2$ pairs, and so a KPD (v) will contain at most $\left\lfloor \frac{v(v-1)}{2(v+2)} \right\rfloor = \lfloor \frac{v-3}{2} \rfloor$ classes, while a KCD (v) will contain at least $\lfloor \frac{v-3}{2} \rfloor + 1$ classes.

Following [4] we can dispense with the case $v \equiv 2 \bmod 3$ relatively quickly.

Theorem 1.3 *For each $v \equiv 2 \bmod 6$, there is a KPD (v) and a KCD (v) each of which contains $\frac{v}{2}$ resolution classes.*

Proof. Delete one element from a Kirkman Triple System KTS $(v + 1)$. □

Theorem 1.4 *For each $v \equiv 5 \bmod 6, v \geq 17$, there is a KPD (v) containing $\frac{v-1}{2}$ resolution classes and a KCD (v) containing $\frac{v+1}{2}$ resolution classes.*

Proof. Delete one element from a Nearly Kirkman Triple System NKTS $(v+1)$ to get the KPD (v). To get the KCD (v), delete one element from a minimum resolvable covering of a $(v+1)$-set by triples, from Theorem 1.2. □

Remark. The leave (resp. excess) of each of the KPDs (resp. KCDs) in Theorem 1.4 is a near-one-factor.

The remainder of this paper is devoted to the case $v \equiv 1 \bmod 3$.

Suppose that we have a Kirkman Packing Design KPD $(v)$with $\lfloor \frac{v-3}{2} \rfloor$ resolution classes. If $v \equiv 1 \bmod 6$, then the leave will contain $v(v-1)/2 - (v+2)(v-3)/2 = 3$ edges, while if $v \equiv 4 \bmod 6$, then the leave will contain $v(v-1)/2 - (v+2)(v-4)/2 = \frac{v}{2} + 4$ edges. Following [4], we will define a *canonical* Kirkman Packing Design CKPD $(v)$to be a KPD $(v)$with $\lfloor \frac{v-3}{2} \rfloor$ resolution classes such that

(i) if $v \equiv 1 \bmod 6$, then the leave is a triangle (K_3);

(ii) if $v \equiv 4 \bmod 6$, then the leave consists of the vertex-disjoint union of a K_4 and $\frac{v-4}{2}$ edges (K_2s).

Thus, in a canonical KPD, the leave is a disjoint union of cliques (see Proposition 2.1).

Lemma 1.5 [4] *There exists CKPD* $(22+54k)$ *and CKPD* $(25+66k)$ *for any* $k \geq 0$.

Suppose now that we have a Kirkman Covering Design (KCD) $(v)$with $\lfloor \frac{v-3}{2} \rfloor + 1$ resolution classes. If $v \equiv 1 \bmod 6$, then the excess will contain $(v+1)(v-1)/2 - v(v-1)/2 = v-1$ edges, while if $v \equiv 4 \bmod 6$, the excess will contain $(v+2)(v-2)/2 - v(v-1)/2 = \frac{v-4}{2}$ edges. We will define a *canonical* Kirkman Covering Design CKCD $(v)$to be a KCD $(v)$with $\lfloor \frac{v-3}{2} \rfloor + 1$ resolution classes such that

1. if $v \equiv 1 \bmod 6$, then the excess consists of a vertex-disjoint union of $\frac{v-1}{3}$ triangles (K_3s);

2. if $v \equiv 4 \bmod 6$, then the excess consists of a vertex-disjoint union of $\frac{v-4}{2}$ edges (K_2s).

Thus, in a canonical KCD, the excess is a disjoint union of cliques.

Example 1.6 *The following is a CKCD (22).*

Point Set: $(\mathbb{Z}_9 \times \{1,2\}) \cup (\{a\} \times \mathcal{Z}_3) \cup \{\infty\}$

Resolution Classes: develop the class $0_1 1_1 0_2 4_2$ $5_1 1_2 2_2$ $2_1 4_1 3_2$ $\infty 8_1 6_2$ $3_1 7_1 a_0$ $6_1 8_2 a_1$ $5_2 7_2 a_2$ mod 9 (the subscripts on a are evaluated mod 3);

the tenth resolution class is $a_0a_1a_2\infty$ $0_13_16_1$ $1_14_17_1$ $2_15_18_1$ $0_23_26_2$ $1_24_27_2$ $2_25_28_2$.

Note that the pairs at mixed difference 8 between orbits 1 and 2 are covered twice; hence, the excess of the cover consists of 9 disjoint edges, as required. We note for future reference that the elements a_0, a_1, a_2, ∞ of degree zero in the excess from a block in the covering.

Example 1.7 *The following is a CKCD (19).*

Point Set: $(Z_9 \times \{1,2\}) \cup \{\infty\}$

Resolution Classes: develop the class $0_12_10_23_2$ $\infty 1_15_2$ $3_16_18_2$ $5_12_24_2$ $4_17_18_1$ $1_26_27_2$ mod 9.

Note that the pairs at pure difference ± 3 in each of orbits 1 and 2 are covered twice; hence, the excess of the cover consists of 6 disjoint triangles, as required.

We refer the interested reader to [4] for a brief history of, and motivation for, these designs. We are herein concerned with their construction. To this end, we will make essential use of a type of design called a Kirkman frame, which we define in Section 2. In Section 3, we construct packings and coverings for $v \equiv 4$ mod 6, obtaining the following main result.

Theorem 1.8 *There exists a CKPD $(v)$and a CKCD $(v)$for every $v \equiv 4$ mod 6 with $v \geq 22$.*

Then in Section 4, we consider packings and coverings for $v \equiv 1$ mod 6. This appears to be the most challenging fibre of all; we are able to obtain the following result.

Theorem 1.9 *(i) There exists a CKPD $(v)$for every $v \equiv 7$ mod 18 with $v \geq 79$, and for all $v \equiv 1$ mod 6 with $v \geq 3709$;*

(ii) There exists a CKCD $(v)$for every $v \equiv 1$ mod 6 with $v \geq 19$, except possibly for $v = 67$.

We conclude this section with a simple construction for a class of non-canonical covers.

Theorem 1.10 *There is a Kirkman Covering Design KCD (v), with $\frac{v-2}{2}$ resolution classes, for every $v \equiv 4$ mod 6.*

Proof. Take a Kirkman Triple System KTS $(v-1)$, and fix a point x. Now adjoin a new point x' to the system, replacing each triple $\{x, a, b\}$ containing x by the quadruple $\{x', x, a, b\}$. $\square$

Note that the excess of a cover produced by Theorem 1.10 consists of the edge xx' replicated $\frac{v-4}{2}$ times. In the next section, we will give an analogous result for covers with $v \equiv 1$ mod 6 (see Theorem 2.3).

2 Preliminaries

A *group-divisible design* (GDD) is a triple (X, G, B) where X is a set of points, G is a partition of X into *groups*, and B is a collection of subsets of X (blocks) so that any pair of distinct points occur together in either one group or exactly one block, but not both. A K-GDD of type $g_1^{t_1} g_2^{t_2} \dots g_m^{t_m}$ has t_i groups of size $g_i, i = 1, 2, \dots, m$, and $|B_j| \in K$ for every block $B_j \in B$. A *transversal design* TD (k, n) is a $\{k\}$-GDD of type n^k; it is well known that a TD (k, n) is equivalent to $k-2$ mutually orthogonal latin squares of order n. A GDD (X, G, B) is called *resolvable* if its block set B admits a partition into parallel classes, each parallel class being a partition of the point set X.

Proposition 2.1 *A CKPD $(v)$is equivalent to a resolvable $\{3,4\}$-GDD of type $2^{(v-4)/2}4^1$ (when $v \equiv 4$ mod 6), or of type $1^{v-3}3^1$ (when $v \equiv 1$ mod 6), in which each parallel class contains exactly one block of size 4.*

Proof. The leave of a CKPD is a disjoint union of cliques, which we take as the groups of a GDD, and vice-versa. Note that in a GDD with the given parameters the number r of parallel classes is determined by $r \cdot (v+2) = v(v-1)/2 - (\frac{v}{2}+4) \Rightarrow r = \frac{v-4}{2}$ when $v \equiv 4$ mod 6, and by $r \cdot (v+2) = v(v-1)/2 - 3 \Rightarrow r = \frac{v-3}{2}$ when $v \equiv 1$ mod 6. □

A GDD (X, G, B) is called *frame resolvable* if its block set B admits a partition into *holey* parallel classes, each holey parallel class being a partition of $X - G_j$ for some $G_j \in G$. A *Kirkman Frame* is a frame resolvable GDD in which all the blocks have size three. It is a simple consequence of the definition that to each group G_j in a Kirkman frame (X, G, B) there corresponds exactly $\frac{1}{2}|G_j|$ holey parallel classes of triples that partition $X - G_j$. The groups in a Kirkman frame are often referred to as *holes*. Kirkman frames were formally introduced by Stinson [11], who established their spectrum in the case where all the holes have the same size.

Theorem 2.2 [Stinson] *A Kirkman frame of type g^u exists if and only if $u \geq 4, g$ is even and $g(u-1) \equiv 0$ mod 3.*

We now illustrate the main technique that we will be using throughout the remainder of the paper (which is a variant of Stinson's 'Filling in Holes' construction, see [11]) by producing a class of non-canonical covers to complement those of Theorem 1.10.

Theorem 2.3 *There is a Kirkman Covering Design KCD (v), with $\frac{v-1}{2}$ resolution classes, for every $v \equiv 1$ mod 6.*

Proof. For $v = 7$, take the point set $\{\infty, 1, 2, 3, 4, 5, 6\}$, with resolution classes $\{\infty 14,\ 2356\}, \{\infty 25,\ 3164\}$ and $\{\infty 36,\ 1245\}$. Note that the excess consists of three disjoint digons (i.e. two copies each of 14, 25, and 36.)

Now let $v \geq 25$. By Theorem 2.2, we can construct a Kirkman Frame of type $6^{(v-1)/6}$ on the point set $X = \{1, 2, 3, 4, 5, 6\} \times \{j : 1 \leq j \leq (v-1)/6\}$, having holes $G_j = \{1, 2, 3, 4, 5, 6\} \times \{j\}$ for $j = 1, 2, \ldots, (v-1)/6$. To each hole G_j, there corresponds 3 holey parallel classes P_{j1}, P_{j2}, P_{j3} of triples that partition $X - G_j$. Now adjoint a new point ∞ to the frame, and for each hole G_j construct a copy of the foregoing KCD (7) on $G_j \cup \{\infty\}$; then for each $j = 1, 2, \ldots, (v-1)/6$, $\{\infty 1_j 4_j, 2_j 3_j 5_j 6_j\} \cup P_{j1}$, $\{\infty 2_j 5_j, 3_j 1_j 6_j 4_j\} \cup P_{j2}$ and $\{\infty 3_j 6_j, 1_j 2_j 4_j 5_j\} \cup P_{j3}$ yield three resolution classes on $X \cup \{\infty\}$, each of which contains exactly one quadruple, so that we get in all a covering of $X \cup \{\infty\}$ with $3 \cdot (v-1)/6 = (v-1)/2$ such classes, as desired. Note that the excess consists of $(v-1)/2$ vertex-disjoint digons (i.e. two copies each of $1_j 4_j, 2_j 5_j$ and $3_j 6_j$, for $j = 1, 2, \ldots, (v-1)/6$).

For $v = 19$, take the point set $(\mathbb{Z}_9 \times \{1, 2\}) \cup \{\infty\}$ and develop the class $\infty 6_1 6_2\ 5_1 2_1 4_1\ 1_1 8_2 4_2\ 7_1 0_2 3_2\ 0_1 1_2 2_2\ 8_1 3_1 7_2 5_2$ mod 9. Note that pairs at mixed difference 2 are covered three times, whence the excess consists of nine disjoint digons.

Finally, for $v = 13$, there is an edge-decomposition of the complete graph K_{13} into 6 factors, each of which consists of three triangles and a four-cycle (see [6]). We simply replace each four-cycle by a K_4 on the same set of vertices; the following KCD (13) with 6 resolution classes results: $\{168, 45T, 7ED, 0239\}, \{14E, 358, 06D, 297T\}, \{159, 26E, 8TD, 0347\}$, $\{01T, 25D, 469, 378E\}, \{13D, 567, 9TE, 0428\}, \{127, 36T, 05E, 489D\}$ □

Remark. We note that, with the exception of $v = 13$, each of the covers constructed in Theorem 2.3 has as its excess a vertex-disjoint union of digons ($2K_2$s).

Theorems 1.10 and 2.3 together determine the minimum possible number of resolution classes in a KCD $(v)$when $v \equiv 1$ mod 3.

Theorem 2.4 *For each $v \equiv 1$ mod 3, there is a KCD (v)containing $\lfloor \frac{v-1}{2} \rfloor$ resolution classes.*

In applying the 'Filling in Holes' construction illustrated in the proof of Theorem 2.3, we will require Kirkman frames in which the holes are not necessarily all of the same size. To get these, we use the following 'Weighting' construction (see, e.g. Stinson [11]).

Construction 2.5 *Suppose that there is a K-GDD of type $g_1^{t_1} g_2^{t_2} \ldots g_m^{t_m}$ and that for each $k \in K$ there is a Kirkman frame of type h^k. Then there is a Kirkman frame of type $(hg_1)^{t_1} (hg_2)^{t_2} \ldots (hg_m)^{t_m}$.*

Finally, as the 'Filling in Holes' construction will generally involve adjoining more than one infinite point to a Kirkman frame, we will require the notion of an *incomplete* canonical packing (covering). Let $v \equiv w \equiv 1$ or 4 mod 6. A CKPD (v)–CKPD (w) is a triple (X, Y, B) where X is a set of v elements, Y is a subset of X of size w (Y is called the *hole*) and B is a collection of subsets of X (blocks), each of size 3 or 4, such that

(i) $|Y \cap B_i| \leq 1$ for all $B_i \in B$;

(ii) any pair of distinct elements in X occur together either in Y or in at most one block;

(iii) B admits a partition into $\frac{1}{2}(v-w)$ parallel classes on X, each of which contains one block of size 4, and a further $\lfloor \frac{1}{2}(w-3) \rfloor$ holey parallel classes of triples on $X \backslash Y$;

(iv) each element of $X \backslash Y$ is contained in exactly two blocks of size 4.

A CKCD (v)–CKCD (w) is defined similarly, changing 'at most' to 'at least' in Condition (ii), and requiring $\lfloor \frac{1}{2}(w-1) \rfloor$ holey parallel classes of triples on $X \backslash Y$ in Condition (iii), with the further requirement that when $v \equiv w \equiv 1$ mod 6 the excess on $X \backslash Y$ consists of $\frac{1}{3}(v-w)$ vertex-disjoint triangles (Condition (iv) implies only that this excess be 2-regular).

Example 2.6 *The following is a CKPD (34)-CKPD (10).*

$$\begin{aligned} X &= (\mathcal{Z}_{12} \times \{1,2\}) \cup (\{a\} \times \mathcal{Z}_4) \cup \{\infty_1, \infty_2, \ldots \infty_6\} \\ Y &= (\{a\} \times \mathcal{Z}_4) \cup \{\infty_1, \infty_2, \ldots, \infty_6\} \end{aligned}$$

Parallel Classes: develop the class $0_1 2_1 0_2 3_2$ $\infty_1 3_1 8_2$ $\infty_2 5_1 11_2$ $\infty_3 6_1 10_2$ $\infty_4 7_1 2_2$ $\infty_5 8_1 4_2$ $\infty_6 9_1 6_2$ $1_1 4_1 a_0$ $11_1 1_2 a_1$ $5_2 7_2 a_2$ $10_1 9_2 a_3$ mod 12 (the subscripts on a are evaluated mod 4);

Holey Parallel Classes: three classes on $X \backslash Y$ are obtained by developing the triples $0_1 1_1 5_1$ and $0_2 1_2 5_2$ mod 12.

Note that the pairs (on $X \backslash Y$) at pure difference ± 6 in each of orbits 1 and 2 are not covered; hence, the leave on $X \backslash Y$ forms a one-factor, as desired.

Example 2.6A The following is a CKCD (34)-CKCD (10). Take X and Y as in Example 2.6.

Parallel Classes: develop the class $0_1 6_1 0_2 3_2$ $\infty_1 4_1 2_2$ $\infty_2 5_1 4_2$ $\infty_3 7_1 9_2$ $\infty_4 8_1 1_2$ $\infty_5 9_1 10_2$ $\infty_6 10_1 6_2$ $2_1 11_1 a_0$ $1_1 8_2 a_1$ $3_1 7_2 a_2$ $5_2 11_2 a_3$ mod 12 (the subscripts on a are evaluated mod 4);

Holey Parallel Classes: The pure differences used among the parallel classes are ± 3 and ± 6 on orbits 1 and 2. Thus, we get four holey parallel classes of triples on $X \setminus Y$ by constructing on each of $\mathbb{Z}_{12} \times \{1\}$ and $\mathbb{Z}_{12} \times \{2\}$ the blocks of a resolvable TD (3, 4) whose groups are aligned on differences ± 3 and ± 6.

Since the pairs at pure difference ± 6 on each of orbits 1 and 2 are covered twice, the excess on $X \setminus Y$ forms a one-factor, as desired.

Example 2.6B The following is a CKPD (43)-CKPD (13).

$$\begin{aligned} X &= (\mathbb{Z}_{15} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_5) \cup \{\infty_1, \infty_2, \ldots, \infty_8\} \\ Y &= (\{a\} \times \mathbb{Z}_5) \cup \{\infty_1, \infty_2, \ldots, \infty_8\} \end{aligned}$$

Parallel Classes: develop the class $0_1 3_1 0_2 6_2$ $\infty_1 9_1 3_2$ $\infty_2 7_1 5_2$ $\infty_3 10_1 2_2$ $\infty_4 2_1 4_2$ $\infty_5 5_1 10_2$ $\infty_6 8_1 1_2$ $\infty_7 11_1 7_2$ $\infty_8 14_1 13_2$ $6_1 12_1 a_0$ $4_1 8_2 a_1$ $1_1 11_2 a_2$ $13_1 14_2 a_3$ $9_2 12_2 a_4$ mod 15 (the subscripts on a are evaluated mod 5);

Holey Parallel Classes: The parallel classes between them cover all pairs at pure differences ± 3 and ± 6 on each of orbits 1 and 2. Thus, we get five holey parallel classes of triples on $X \setminus Y$ by constructing on each of $\mathbb{Z}_{15} \times \{1\}$ and $\mathbb{Z}_{15} \times \{2\}$ the blocks of a resolvable TD $(3, 5)$ whose groups are aligned on differences ± 3 and ± 6.

As is usual with incomplete designs, the 'missing' subdesign in an incomplete packing (covering) need not exist (e.g. with reference to Example 2.6, the reader will be easily convinced that no CKPD (10) exists). We now illustrate the more general 'Filling in Holes' construction with the following.

Lemma 2.7 *There exists a CKPD (106) and a CKCD (106).*

Proof. We begin with the packing design. Start with a Kirkman Frame of type 24^4, which exists by Theorem 2.2. Adjoin ten (new) infinite points to the frame; call this set of infinite points I. On each of the first three holes of the frame together with I, construct a copy of a CKPD (34)-CKPD (10) (Example 2.6), aligning the hole of the incomplete packing on the points of I. On the fourth hole of the frame, together with I, construct a CKPD (34) (see Appendix). We pair each parallel class in the CKPD (34)-CKPD (10)s, and each of twelve of the parallel classes in the CKPD (34), with a holey parallel class (of triples) in the frame; this yields 48 parallel classes in our packing. The remaining 3 parallel classes are obtained, in turn, by taking the union of a holey parallel class in each CKPD (34)-CKPD (10) with one of the remaining three parallel classes in the CKPD (34). Finally, the leave of this packing consists of the union of the leave of the CKPD (34) with the leaves of each of the incomplete packings, giving us a K_4 and $51K_2s$ spanning the 106 points, as desired. Thus, we have a CKPD (106).

For the covering design, we proceed in like fashion, adjoining ten infinite points to a Kirkman Frame of type 24^4, but filling in holes with (three copies of) a CKCD (34)-CKCD (10) (Example 2.6A) and a CKCD (34) (see Appendix). Again, we get 48 parallel classes in our covering by pairing each parallel class in the CKCD (34)-CKCD (10)s, and each of twelve of the parallel classes in the CKCD (34), with a holey parallel class of triples in the frame. We get 4 more parallel classes by taking, in turn, the union of a holey parallel class in each CKCD (34)-CKCD (10) with one of the remaining four parallel classes in the CKCD (34). Note that the excess of this covering consists of the union of the excess of the CKCD (34) with the excesses of each of the incomplete coverings, yielding 51 vertex-disjoint K_2s, as desired. Thus, we have a CKCD (106). □

As we have stated, the missing subdesign in an incomplete packing (covering) need not exist. If it does, however, then one can 'fill the hole' in the incomplete packing (covering) with a copy of that design, viz:

Proposition 2.8 *If there is a CKPD (v)-CKPD (w) (resp. CKCD (v)-CKCD (w)) and a CKPD (w) (resp. CKCD (w)) then there is a CKPD (v)(resp. CKCD (v)).*

We conclude the section by illustrating some applications of Proposition 2.8.

Lemma 2.9 *There exists a CKPD (70) and a CKCD (70).*

Proof.
We will start by constructing the following (incomplete) CKPD (70)-CKPD (22):

$$\begin{aligned} X &= (\mathbb{Z}_{24} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup \{\infty_1, \infty_2, \ldots, \infty_{19}\}; \\ Y &= (\{a\} \times \mathbb{Z}_3) \cup \{\infty_1, \infty_2, \ldots, \infty_{19}\}. \end{aligned}$$

Parallel Classes: Develop the following class mod 24 (subscripts on a are evaluated mod 3):

$7_1 23_1 10_2 14_2$	$\infty_2 0_1 12_2$	$\infty_7 17_1 13_2$	$\infty_{12} 12_1 7_2$
$16_1 20_1 a_0$	$\infty_3 4_1 2_2$	$\infty_8 6_1 23_2$	$\infty_{13} 14_1 18_2$
$1_2 17_2 a_1$	$\infty_4 21_1 11_2$	$\infty_9 9_1 22_2$	$\infty_{14} 8_1 8_2$
$11_1 19_2 a_2$	$\infty_5 10_1 16_2$	$\infty_{10} 22_1 21_2$	$\infty_{15} 18_1 3_2$
$\infty_1 13_1 5_2$	$\infty_6 19_1 20_2$	$\infty_{11} 5_1 15_2$	$\infty_{16} 2_1 4_2$
	$\infty_{17} 15_1 9_2$	$\infty_{18} 1_1 6_2$	$\infty_{19} 3_1 0_2$

Holey Parallel Classes: The only pure differences on orbits 1 and 2 that are used among the parallel classes are ± 4 and ± 8. Thus, we get 9 holey parallel classes of triples on $X \backslash Y$ by constructing on each of $\mathbb{Z}_{24} \times \{1\}$ and $\mathbb{Z}_{24} \times \{2\}$ the blocks of a resolvable 3-GDD of type 6^4 (see [10]) whose groups are aligned on differences $\pm 4, \pm 8$, and ± 12.

(Since the pairs at pure difference ± 12 on each of orbits 1 and 2 are not covered, the leave on $X \backslash Y$ is a one-factor, as desired.)

Now construct a copy of a CKPD (22) (Lemma 1.5) on the points of Y, pairing off the 9 parallel classes in the CKPD (22) with the 9 holey parallel classes in the CKPD (70)-CKPD (22), yielding a CKPD (70).

For a CKCD (70), we proceed similarly, starting with the following (incomplete) CKCD (70)-CKCD (22):

$$\begin{aligned} X &= (\mathbb{Z}_{24} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_8) \cup \{\infty_1, \infty_2, \ldots, \infty_{14}\}; \\ Y &= (\{a\} \times \mathbb{Z}_8) \cup \{\infty_1, \infty_2, \ldots, \infty_{14}\}. \end{aligned}$$

Parallel Classes: Develop the following class mod 24 (subscripts on a are evaluated mod 8):

$1_1 5_1 0_2 12_2$	$1_2 5_2 a_4$	$\infty_3 4_1 14_2$	$\infty_9 19_1 19_2$
$0_1 12_1 a_0$	$23_1 17_2 a_5$	$\infty_4 14_1 4_2$	$\infty_{10} 3_1 15_2$
$10_1 7_2 a_1$	$20_1 16_2 a_6$	$\infty_5 9_1 18_2$	$\infty_{11} 15_1 8_2$
$7_1 10_2 a_2$	$22_1 6_2 a_7$	$\infty_6 18_1 9_2$	$\infty_{12} 8_1 21_2$
$6_1 22_2 a_3$	$\infty_1 17_1 23_2$	$\infty_7 11_1 13_2$	$\infty_{13} 21_1 2_2$
	$\infty_2 16_1 20_2$	$\infty_8 13_1 11_2$	$\infty_{14} 2_1 3_2$

Holey Parallel Classes: The only pure differences on orbits 1 and 2 that are used among the parallel classes are ± 4 and ± 12. Thus, we get 10 holey parallel classes of triples on $X \backslash Y$ by constructing on each of $\mathbb{Z}_{24} \times \{1\}$ and $\mathbb{Z}_{24} \times \{2\}$ a resolvable 3-GDD of type 6^4 (as in the foregoing packing case) and then constructing one further class of triples using the pairs at pure difference ± 8 in each of the two orbits.

(Since the pairs at pure difference ± 12 in each of orbits 1 and 2 are covered twice, the excess is a one-factor on $X \backslash Y$, as desired.)

Now construct a copy of a CKCD (22) (Example 1.6) on the points of Y, pairing off the 10 parallel classes in the CKCD (22) with the 10 holey parallel classes in the CKCD (70)-CKCD (22), yielding a CKCD (70). □

Lemma 2.10 *There exists a CKPD* (v)*and a CKCD* (v)*for* $v = 82$ *and 88.*

Proof.
Construct a copy of a CKPD (22) (Lemma 1.5) or a CKCD (22) (Example 1.6) on the hole of the appropriate incomplete design, which follows.
CKPD (82)-CKPD (22):

$$X = (\mathbb{Z}_{30} \times \{1,2\}) \cup \{\infty_1, \infty_2, \ldots, \infty_{22}\}, Y = \{\infty_1, \infty_2, \ldots, \infty_{22}\}$$

Parallel Classes: Develop the following class mod 30:

$2_1 22_1 10_2 16_2$	$\infty_2 26_1 29_2$	$\infty_9 14_1 15_2$	$\infty_{16} 29_1 25_2$
$25_1 4_2 24_2$	$\infty_3 19_1 8_2$	$\infty_{10} 18_1 0_2$	$\infty_{17} 28_1 11_2$
$4_1 10_1 14_2$	$\infty_4 21_1 19_2$	$\infty_{11} 5_1 7_2$	$\infty_{18} 7_1 13_2$
$15_1 24_1 27_1$	$\infty_5 8_1 28_2$	$\infty_{12} 17_1 2_2$	$\infty_{19} 16_1 27_2$
$9_2 12_2 21_2$	$\infty_6 9_1 26_2$	$\infty_{13} 23_1 20_2$	$\infty_{20} 1_1 22_2$
$\infty_1 12_1 17_2$	$\infty_7 13_1 5_2$	$\infty_{14} 3_1 3_2$	$\infty_{21} 20_1 6_2$
	$\infty_8 11_1 18_2$	$\infty_{15} 6_1 1_2$	$\infty_{22} 0_1 23_2$

Holey Parallel Classes: The pure differences that are used on orbits 1 and 2 are $\pm 3, \pm 6, \pm 9, \pm 10$, and ± 12. We now get 9 holey parallel classes of triples on $X \backslash Y$ by constructing on each of $\mathbb{Z}_{30} \times \{1\}$ and $\mathbb{Z}_{30} \times \{2\}$ the blocks of a resolvable TD (3, 10) (equivalent to two orthogonal latin squares of order 10), whose groups are aligned on differences $\pm 3, \pm 6, \pm 9, \pm 12$, and ± 15, one of whose parallel classes (which we take to cover pairs at difference ± 10) has been removed.

(Since the pairs at pure difference ± 15 on each of orbits 1 and 2 are not covered, the leave on $X \backslash Y$ is a one-factor, as desired.)

CKCD (82)-CKCD (22):

$$X = (\mathbb{Z}_{30} \times \{1,2\}) \cup \{\infty_1, \infty_2, \ldots, \infty_{22}\}, Y = \{\infty_1, \infty_2, \ldots, \infty_{22}\}$$

Parallel Classes: Develop the following class mod 30:

$25_1 16_1 25_2 1_2$	$\infty_2 24_1 21_2$	$\infty_9 1_1 15_2$	$\infty_{16} 4_1 16_2$
$23_1 3_2 12_2$	$\infty_3 3_1 10_2$	$\infty_{10} 22_1 20_2$	$\infty_{17} 19_1 9_2$
$12_1 18_1 5_2$	$\infty_4 15_1 11_2$	$\infty_{11} 5_1 29_2$	$\infty_{18} 20_1 23_2$
$2_1 17_1 29_1$	$\infty_5 10_1 26_2$	$\infty_{12} 27_1 2_2$	$\infty_{19} 28_1 27_2$
$4_2 7_2 19_2$	$\infty_6 26_1 14_2$	$\infty_{13} 6_1 8_2$	$\infty_{20} 11_1 6_2$
$\infty_1 7_1 28_2$	$\infty_7 9_1 17_2$	$\infty_{14} 13_1 24_2$	$\infty_{21} 8_1 0_2$
	$\infty_8 14_1 18_2$	$\infty_{15} 21_1 22_2$	$\infty_{22} 0_1 13_2$

Holey Parallel Classes: The pure differences that are used on orbits 1 and 2 are $\pm 3, \pm 6, \pm 9, \pm 12$, and ± 15. Thus, we get 10 holey parallel classes of triples on $X \backslash Y$ by constructing on each of $\mathbb{Z}_{30} \times \{1\}$ and $\mathbb{Z}_{30} \times \{2\}$ the blocks of a resolvable TD (3, 10), whose groups are aligned on differences $\pm 3, \pm 6, \pm 9, \pm 12$, and ± 15.

(Since the pairs at pure difference ± 15 on each of orbits 1 and 2 are covered, the excess on $X \backslash Y$ is a one-factor, as desired.)

CKPD (88)-CKPD (22)

$$\begin{aligned} X &= (\mathbb{Z}_{33} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup \{\infty_1, \infty_2, \ldots, \infty_{19}\}; \\ Y &= (\{a\} \times \mathbb{Z}_3) \cup \{\infty_1, \infty_2, \ldots, \infty_{19}\}. \end{aligned}$$

Parallel Classes: Develop the following class mod 33 (subscripts on a are evaluated mod 3):

$30_1 5_1 6_2 18_2$	$\infty_1 16_1 9_2$	$\infty_8 4_1 16_2$	$\infty_{15} 11_1 19_2$
$13_1 32_2 17_2$	$\infty_2 10_1 27_2$	$\infty_9 7_1 23_2$	$\infty_{16} 3_1 10_2$
$12_1 7_2 26_2$	$\infty_3 19_1 22_2$	$\infty_{10} 27_1 21_2$	$\infty_{17} 25_1 12_2$
$24_1 15_1 14_2$	$\infty_4 28_1 24_2$	$\infty_{11} 6_1 31_2$	$\infty_{18} 22_1 13_2$
$18_1 21_1 3_2$	$\infty_5 0_1 2_2$	$\infty_{12} 23_1 29_2$	$\infty_{19} 1_1 11_2$
$29_1 14_1 2_1$	$\infty_6 26_1 15_2$	$\infty_{13} 20_1 25_2$	$8_1 5_2 a_0$
$28_2 1_2 4_2$	$\infty_7 32_1 30_2$	$\infty_{14} 9_1 20_2$	$31_1 17_1 a_1$
			$0_2 8_2 a_2$

Holey Parallel Classes: We get 9 holey parallel classes of triples on $X \backslash Y$ by developing each of the following pairs of triples mod 33: $0_1 1_1 5_1, 0_2 1_2 5_2; 0_1 2_1 13_1, 0_2 2_2 13_2$; and $0_1 7_1 17_1, 0_2 7_2 17_2$. (Since the pairs at mixed difference 0 between orbits 1 and 2 are not covered, the leave on $X \backslash Y$ is a one-factor, as desired.)

CKCD (88)-CKCD (22)

$$\begin{aligned} X &= (\mathbb{Z}_{33} \times \{1,2\}) \cup \{\infty_1, \infty_2, \ldots, \infty_{22}\}; \\ Y &= \{\infty_1, \infty_2, \ldots, \infty_{22}\}. \end{aligned}$$

Parallel Classes: Develop the following class mod 33:

$4_1 28_1 24_2 13_2$	$\infty_1 19_1 2_2$	$\infty_8 23_1 7_2$	$\infty_{15} 32_1 27_2$
$25_1 26_2 11_2$	$\infty_2 24_1 4_2$	$\infty_9 10_1 3_2$	$\infty_{16} 26_1 28_2$
$15_1 23_2 29_2$	$\infty_3 29_1 18_2$	$\infty_{10} 2_1 0_2$	$\infty_{17} 16_1 8_2$
$22_1 0_1 32_2$	$\infty_4 3_1 14_2$	$\infty_{11} 9_1 6_2$	$\infty_{18} 20_1 12_2$
$31_1 1_1 25_2$	$\infty_5 5_1 5_2$	$\infty_{12} 7_1 22_2$	$\infty_{19} 11_1 17_2$
$27_1 6_1 12_1$	$\infty_6 8_1 15_2$	$\infty_{13} 14_1 19_2$	$\infty_{20} 17_1 21_2$
$10_2 1_2 31_2$	$\infty_7 13_1 16_2$	$\infty_{14} 18_1 30_2$	$\infty_{21} 30_1 20_2$
			$\infty_{22} 21_1 9_2$

Holey Parallel Classes: The pure differences that are used on orbits 1 and 2 are $\pm 3, \pm 6, \pm 9, \pm 11, \pm 12$ and ± 15. We now get 10 holey parallel classes of triples on $X \backslash Y$ by constructing on each of $\mathbb{Z}_{33} \times \{1\}$ and $\mathbb{Z}_{33} \times \{2\}$ the blocks of a resolvable TD $(3, 11)$ (equivalent to two orthogonal latin squares of order 11), whose groups are aligned on differences $\pm 3, \pm 6, \pm 9, \pm 12$, and ± 15, one of whose parallel classes (which we take to cover pairs at difference ± 11) has been removed.

(Since the pairs at mixed difference 25 between orbits 1 and 2 are covered twice, the excess on $X \backslash Y$ is a one-factor, as desired.) □

3 Canonical Packings and Coverings for $v \equiv 4 \bmod 6$

In this section, we prove Theorem 1.8.

Theorem 1.8 *There exists a CKPD* (v)*and a CKCD* (v)*for every* $v \equiv 4$ *mod* 6 *with* $v \geq 22$.

Lemma 3.1 *A CKPD* (v)*is equivalent to a CKPD* (v)*-CKPD (4); a CKCD* (v)*-CKCD (4) is equivlaent to a CKCD* (v)*in which the elements of degree zero in the excess form a block (of size 4) in the covering.*

Proof.
In the first case, we identify the elements of the hole in the incomplete packing as the elements that form the K_4 in the leave of the packing; in the second case, identify the elements of the hole in the incomplete covering as the elements of degree zero in the excess of the covering. □

Lemma 3.2 *There exists CKPD* (v)*and CKCD* (v)*for* $v =$ *22, 28, 34, 40, 46, 52, 58 and 64.*

Proof.
For $v = 22$, see Lemma 1.5 and Example 1.6. Packings and coverings for $v = 28, 34, 40, 46, 52, 58$, and 64 are given in the Appendix. □

Remark. Each of the canonical coverings constructed in Lemma 3.2 has the property that the elements of degree zero in the excess form a block (of size 4) in the covering, and so by Lemma 3.1 can be considered as a CKCD (v)-CKCD (4).

Lemma 3.3 *If there is a GDD on s points with block sizes from the set* $\{k \in \mathbb{Z} : k \geq 4\}$ *and group sizes from the set* $\{3,4,5,6,7,8,9,10\}$*, then there is a CKPD* $(6s+4)$ *and a CKCD* $(6s+4)$.

Proof.
Let the given GDD have type $g_1^{t_1} g_2^{t_2} \dots g_m^{t_m}$. Apply Construction 2.5 to this GDD, using 'weight' $h = 6$, to yield a Kirkman frame of type $(6g_1)^{t_1}(6g_2)^{t_2} \dots (6g_m)^{t_m}$. Adjoin four infinite points to the frame and apply 'Filling in Holes' (see e.g., Lemma 2.7), constructing on each hole of size $6g_i$ together with the four infinite points a CKPD $(6g_i + 4)$–CKPD (4) (resp. CKCD $(6g_i + 4)$–CKCD (4)) aligning the hole in the incomplete packing (resp. covering) on the four infinite points; on the last hole of size $6g_m$ together with the four infinite points construct a CKPD $(6g_m + 4)$ (resp. CKCD $(6g_m + 4)$). All the required input designs exist by Lemma 3.1, Lemma

3.2, and the remark following Lemma 3.2. We get a CKPD $(6s+4)$ (resp. CKCD $(6s+4)$), as desired. □

Theorem 3.4 *There exists CKPD* (v) *and CKCD* (v) *for every* $v \equiv 4 \bmod 6$ *with* $v \geq 70$.

Proof.
If $v = 70, 82, 88$ or 106, see Lemmas 2.7, 2.9 and 2.10. For $v = 136$ or 160, proceed as follows. Apply Construction 2.5, using 'weight' $h = 4$, to a 4-GDD of type $6^4 9^1$ (or of type $6^5 9^1$, *see*[10]) to yield a Kirkman frame of type $24^4 36^1$ (or of type $24^5 36^1$). Adjoin four infinite points to the frame and apply 'Filling in Holes,' constructing on each hole of size 24 together with the four infinite points a CKPD (28)−CKPD (4) (resp. CKCD (28)-CKCD (4)) and on the hole of size 36 together with the four infinite points a CKPD (40) (resp. CKCD (40)). For $v = 112$ or 130, proceed similarly, starting with a Kirkman frame of type 18^6 or 18^7 (both of which exist by Theorem 2.2) and adjoining four infinite points, filling in the relevant 22 point designs.

Now let $v = 76$ or $v \geq 94, v \neq 106, 112, 130, 136$ or 160. Write $v = 6s + 4$. Then $s = 12$ or $s \geq 15, s \notin \{17, 18, 21, 22, 26\}$. Apply Lemma 3.3 to the relevant GDD on s points, which we construct as follows. If $s \in \{36, 37, 38, 39, 46, 47\}$ delete the appropriate number of points from a group in either a TD $(5, 8)$ or TD $(6, 8)$. If $s \geq 32$ and $s \neq 36, 37, 38, 39, 46$ or 47, we can write $s = 4n + m$ where $n \geq 7$ is odd and $4 \leq m \leq n$ (e.g. let $m = s \bmod 8 + 4$ and $n = (s - m)/4$). Take a TD $(5, n)$ with a parallel class of blocks and truncate a group to m points. By viewing the resulting parallel class of blocks on the truncated TD as groups, we have produced a $\{4, 5, m, n\}$-GDD of type $4^{n-m} 5^m$ on $4(n - m) + 5m = s$ points, as desired. Finally, if $s \leq 31$ construct the appropriate GDD according to the following table.

s	GDD	Source
12	$4-$GDD of type 3^4	TD $(4, 3)$
15	$4-$GDD of type 3^5	TD $(4, 4)$
16	$4-$GDD of type 4^4	TD $(4, 4)$
19, 20	$\{4, 5\}-$GDD of type $4^4 3^1, 4^5$	TD $(5, 4)$
23, 24, 25	$\{4, 5\}-$GDD of type $5^4 3^1, 5^4 4^1, 5^5$	TD $(5, 5)$
27, 28, 29, 30, 31	$\{4, 5\}-$GDD of type $3^8 r^1$, $r = 3, 4, 5, 6, 7$	resolvable 4-GDD of type 3^8[9]

This completes the proof. □

Theorem 1.8 now follows from Lemma 3.2 and Theorem 3.4.

We conclude this section by considering packings and coverings for $v < 22$. It is easy to see that no CKPD (10) can exist, and with some case analysis, it can be shown that no CKPD (16) exists. There is, however, a (non-canonical) Kirkman Packing Design KPD (16) with 6 resolution classes, as follows (its leave consists of the disjoint union of a 4-cycle and an 8-cycle, on the elements $5, 6, \ldots, 16$):

1 2 5 6	1 3 9 10	1 4 13 14	2 3 15 16	2 4 11 12	3 4 7 8
3 11 13	2 7 13	2 8 9	1 8 11	1 7 15	1 12 16
4 9 15	4 5 16	3 6 12	4 6 10	3 5 14	2 10 14
7 10 12	6 11 14	5 10 15	5 12 13	6 9 16	5 9 11
8 14 16	8 12 15	7 11 16	7 9 14	8 10 13	6 13 15

A Maximum KPD (16)

With respect to coverings, a CKCD (10) does not exist; at time of writing we do not know whether or not there is a CKCD (16). That there are non-canonical coverings of every order $v \equiv 4 \bmod 6$ with the minimum possible number $(v-2)/2$ of resolution classes follows from Theorem 1.10.

Theorem 1.8 and the foregoing KPD (16) together give us the following analogue to Theorem 2.4 concerning the maximum possible number of resolution classes in a KPD $(v)$when $v \equiv 4 \bmod 6$.

Theorem 3.5 *There is a KPD (v)containing $\frac{v-4}{2}$ resolution classes for every $v \equiv 4$ mod 6 with $v \geq 16$.*

4 Canonical Packings and Coverings for $v \equiv 1 \bmod 6$

In this section, we prove Theorem 1.9.

Theorem 1.9

(i) There exists a CKPD $(v)$for every $v \equiv 7$ mod 18 with $v \geq 79$, and for all $v \equiv 1$ mod 6 with $v \geq 3709$.

(ii) There exists a CKCD $(v)$for every $v \equiv 1$ mod 6 with $v \geq 19$, except possibly for $v = 67$.

We make essential use of the following incomplete canonical packing.

Lemma 4.1 *There is a CKPD (25)-CKPD (7).*

Proof.
Let

$$X = ((\mathbb{Z}_3 \times \mathbb{Z}_3) \times \{1,2\}) \cup (\{a\} \times \{(0,0),(0,1),(0,2)\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\},$$

and

$$Y = (\{a\} \times \{(0,0),(0,1),(0,2)\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}.$$

Parallel Classes: Develop the following class mod $(3,3)$:

$$\begin{array}{ll} \infty_1(2,1)_1(1,1)_2 & (0,2)_1(2,0)_1(0,2)_2(1,2)_2 \\ \infty_2(0,1)_1(0,0)_2 & (1,1)_1(1,0)_1a_{(0,0)} \\ \infty_3(0,0)_1(0,1)_2 & (2,2)_2(2,1)_2a_{(0,1)} \\ \infty_4(1,2)_1(2,0)_2 & (2,2)_1(1,0)_2a_{(0,2)} \end{array}$$

The subscripts on a are evaluated modulo the subgroup of $\mathbb{Z}_3 \times \mathbb{Z}_3$ generated by $(1,1)$.

Holey Parallel Classes: There remain two pure differences on each of orbits 1 and 2, which between them generate 2 holey parallel classes of triples on $X \backslash Y$. □

Proposition 4.2 *If there is a CKPD (v)-CKPD (w), then there is a CKCD (v)-CKCD (w).*

Proof.
If (X, Y, B) is a CKPD (v)-CKPD (w), then, since $v \equiv w \equiv 1 \bmod 6$, it follows from the definition that any pair of distinct elements in X, not both of which are in Y, occur in exactly one block in B. Thus, we can label the elements of $X \backslash Y$ (in any way) as $1, 2, \ldots, v-w$ and let $P = \{\{1,2,3\}, \{4,5,6\}, \ldots, \{v-w-2, v-w-1, v-w\}\}$ be a holey parallel class of triples on $X \backslash Y$, whereupon $(X, Y, B \cup P)$ is a CKCD (v)-CKCD (w). □

Corollary 4.3 *There is a CKCD (25)-CKCD (7).*

Proof.
Lemma 4.1 and Proposition 4.2 yield the result. □

Lemma 4.4 *There exists a CKPD (v) for every $v \equiv 7$ mod 18 with $v \geq 79$, and there exists a CKCD (v) for every $v \equiv 7$ mod 18 with $v \geq 25$.*

Proof.
CKCD (25) and CKCD (43) are given in the Appendix. For $v = 61$, apply Proposition 2.8, constructing a CKCD (19) (Example 1.7) on the hole in the following CKCD (61)-CKCD (19):

Point Set: $(\mathbb{Z}_{21} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup \{\infty_1, \infty_2, \ldots, \infty_{16}\}$.

Parallel Classes: Develop the following class mod 21 (subscripts on a are evaluated mod 3):

$1_0 17_1 5_2 6_2$	$\infty_2 18_1 19_2$	$\infty_7 12_1 17_2$	$\infty_{12} 19_1 2_2$
$0_1 20_1 a_0$	$\infty_3 4_1 7_2$	$\infty_8 16_1 8_2$	$\infty_{13} 7_1 13_2$
$6_1 3_2 a_2$	$\infty_4 14_1 1_2$	$\infty_9 2_1 14_2$	$\infty_{14} 9_1 20_2$
$9_2 16_2 a_1$	$\infty_5 11_1 4_2$	$\infty_{10} 13_1 11_2$	$\infty_{15} 8_1 10_2$
$\infty_1 3_1 18_2$	$\infty_6 1_1 0_2$	$\infty_{11} 5_1 12_2$	$\infty_{16} 15_1 15_2$

Holey Parallel Classes: The pure differences (on orbits 1 and 2) used among the parallel classes are ± 1 and ± 7. We get 9 holey parallel classes of triples on $\mathbb{Z}_{21} \times \{1,2\}$ by taking $\{(3i)_j(7+3i)_j(17+3i)_j\colon j=1,2; 0 \le i \le 6\}$ and $\{(3i)_j(10+3i)_j(5+3i)_j : j=1,2; 0 \le i \le 6\}$ together with the seven classes obtained by setting $i = 0, 1, \ldots, 6$ in the following: $\{(3i)_j(3+3i)_j(9+3i)_j, (6+3i)_j(1+3i)_j(4+3i)_j, (12+3i)_j(14+3i)_j(2+3i)_j, (15+3i)_j(19+3i)_j(7+3i)_j, (18+3i)_j(11+3i)_j(5+3i)_j, (13+3i)_j(17+3i)_j(20+3i)_j, (16+3i)_j(10+3i)_j(8+3i)_j : j = 1,2\}$. Between them these nine classes use up all pairs within each of orbits 1 and 2 *except* those pairs at difference ± 1.

Since the pairs at pure difference ± 7 are covered twice on each of orbits 1 and 2 the excess $\mathbb{Z}_{21} \times \{1,2\}$ is a triangle-factor, as desired.

Now let $v \ge 79$. Write $v = 18t+7, t \ge 4$, and adjoin seven infinite points to a Kirkman frame of type 18^t (which exists by Theorem 2.2). Apply 'Filling in Holes' (see e.g., Lemma 2.7), constructing on each of $t-1$ of the holes, together with the seven infinite points, either a CKPD (25)-CKPD (7) (Lemma 4.1) or a CKCD (25)-CKCD (7) (Corollary 4.3), aligning the hole in the incomplete design on the seven infininte points; on the last hole of the frame together with the seven infinite points construct either a CKPD (25) (Lemma 1.5) or a CKCD (25) (see Appendix). A CKPD (v) or CKCD (v) results. □

Lemma 4.5 *There exists a CKCD* (v) *for every* $v \equiv 1$ *or* 13 *mod* 18 *with* $v \ge 19$, *except possibly when* $v = 67$.

Proof.
We begin with $v \equiv 1 \bmod 18$. A CKCD (19) is given by Example 1.7 and CKCD (37) and CKCD (55) are given in the Appendix. For $v \ge 73$, write $v = 18t + 1, t \ge 4$, and adjoin one infinite point to a Kirkman frame of type 18^t, constructing on each hole together with the infinite point a copy of a CKCD (19), identifying the element of degree zero in the excess of the cover with the infinite point; a CKCD (v) results.

Now suppose $v \equiv 13 \bmod 18$. CKCD (31) and CKCD (49) are given in the Appendix. For $v = 85$, construct a CKCD (19) on the hole in the following CKCD (85)-CKCD (19):

Point Set: $(\mathbb{Z}_{33} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup \{\infty_1, \infty_2, \ldots, \infty_{16}\}$.

Parallel Classes: Develop the following class mod 33 (subscripts on a are evaluated mod 3):

$1_1 15_1 21_2 24_2$	$6_1 3_1 27_1$	$\infty_3 12_1 8_2$	$\infty_{10} 14_1 28_2$
$9_1 19_2 0_2$	$16_2 4_2 22_2$	$\infty_4 7_1 14_2$	$\infty_{11} 11_1 6_2$
$29_1 23_2 32_2$	$0_1 18_2 a_0$	$\infty_5 32_1 25_2$	$\infty_{12} 13_1 10_2$
$18_1 31_2 20_2$	$9_2 17_2 a_1$	$\infty_6 2_1 7_2$	$\infty_{13} 30_1 30_2$
$17_1 25_1 29_2$	$10_1 21_1 a_2$	$\infty_7 31_1 15_2$	$\infty_{14} 5_1 13_2$
$16_1 22_1 5_2$	$\infty_1 28_1 26_2$	$\infty_8 24_1 2_2$	$\infty_{15} 4_1 3_2$
$8_1 26_1 27_2$	$\infty_2 20_1 12_2$	$\infty_9 23_1 11_2$	$\infty_{16} 19_1 1_2$

Holey Parallel Classes: We get 9 holey parallel classes of triples on $\mathbb{Z}_{33} \times \{1,2\}$ by developing each of the following pairs of triples mod 33: $0_1 1_1 5_1, 0_2 1_2 5_2; 0_1 2_1 13_1, 0_2 2_2 13_2$ and $0_1 7_1 23_1, 0_2 7_2 23_2$.

Since the pairs at pure difference ± 11 are covered twice on each of orbits 1 and 2 the excess on $\mathbb{Z}_{33} \times \{1,2\}$ is a triangle-factor, as desired.

For $v = 103$, delete all but one point from a block in a 5-GDD of type 4^5 to yield a 4,5-GDD of type $3^4 4^1$. Apply Construction 2.5, using 'weight' $h = 6$, to yield a Kirkman frame of type $18^4 24^1$. Adjoin seven infinite points and apply 'Filling in Holes,' constructing on each hole of size 18 together with the infinite points a CKCD (25)-CKCD (7) (Corollary 4.3), aligning the hole in the incomplete covering on the infinite points, and constructing on the hole of size 24 together with the infinite points a CKCD (31). For $v = 157$, apply Construction 2.5, with $h = 4$, to a 4-GDD of type $6^5 9^1$, (*see*[10]) to yield a Kirkman frame of type $24^5 36^1$; adjoin one infinite point and fill in CKCD (25)s and a CKCD (37), identifying in each case the element of degree zero in the excess of the cover with the infinite point.

Now let $v \equiv 13 \bmod 18$ and $v > 67, v \neq 85, 103, 157$. Write $v = 6s + 1$; then $s \equiv 2 \bmod 3$ and $s \geq 20, s \neq 26$. From (the proof of) Theorem 3.4, there is a GDD on s points with block sizes from the set $\{k \in \mathbb{Z} : k \geq 4\}$ and group sizes from the set $\{3,4,5,6,7,8,9,10\}$. Apply Construction 2.5, using 'weight' $h = 6$, and adjoin one infinite point, filling in the appropriate CKCDs (Example 1.7, Lemma 4.4 and Appendix). A CKCD (v) results. □

Theorem 1.9 (ii) now follows from Lemmas 4.4 and 4.5.

We now work towards establishing a bound for the existence of the packing designs CKPD (v).

Lemma 4.6 *There exists a CKPD* ($6m + 7$) *for every* $m \equiv 3$ *mod* 11.

Proof.
This is part of Lemma 1.5 (i.e. write $m = 11k + 3$). □

Suppose now that $v \equiv 1 \bmod 6, v \geq 3709$. Let $s = \frac{1}{6}(v - 7)$, and select (the unique) $m \in \{11k + 3 : 0 \leq k \leq 11\}$ such that $m \equiv s \bmod 12$. Let

$n = (s-m)/4$; then $n \equiv 0 \bmod 3$ and $n \geq 126 > m$. Take a TD $(5, n)$ and truncate one of its groups to m points. On each of the remaining groups construct a $\{4,7\}$-GDD of type $3^{n/3}$. The result is a $\{4,5,7\}$-GDD of type $3^{4n/3}m^1$. Apply Construction 2.5 to this GDD, using 'weight' $h = 6$, to yield a Kirkman Frame of type $18^{4n/3}(6m)^1$. Adjoin seven infinite points to the frame and apply 'Filling in Holes,' constructing on each hole of size 18 together with the seven infinite points a CKPD (25)-CKPD (7) (Lemma 4.1) and constructing on the hole of size $6m$ together with the seven infinite points a CKPD $(6m+7)$ (Lemma 4.6). The result is a CKPD on $6(4n+m)+7 = 6s+7 = v$ elements; we have thus shown the following.

Theorem 4.7 *There exists a CKPD $(v)$for every $v \equiv 1$ mod 6 with $v \geq$ 3709.*

Theorem 1.9 (i) now follows from Lemma 4.4 and Theorem 4.7. Theorem 1.9 (i) gives us the following analogue to Theorem 3.5 concerning the maximum possible number of resolution classes in a KPD $(v)$when $v \equiv 1 \bmod 6$.

Theorem 4.8 *There is a KPD (v)containing $\frac{v-3}{2}$ resolution classes for every $v \equiv 7$ mod 18 with $v \geq 79$ and every $v \equiv 1$ mod 6 with $v \geq 3709$.*

Thus, those values of $v \equiv 1 \bmod 3$ for which the existence of a CKPD (v) remain open are $v = 55, 61, 67, 73, 85$, and 109, while those values of $v \equiv 1 \bmod 3$ for which the existence of a CKCD (v) remain open are $v = 13, 16$ and 67.

5 Conclusion

We have learned that Colbourn and Ling [5] have considered the problem of determining the maximum possible number of resolution classes in a KPD (v), obtaining Theorem 3.5 as well as the following significant improvement on Theorem 4.8:

Theorem (Colbourn and Ling) *There is a KPD (v) containing $\frac{v-3}{2}$ resolution classes for every $v \equiv 1$ mod 6 with $v > 13$ except possibly for $v \in \{19, 55, 61, 67, 73, 79, 85, 97, 103, 109, 121, 133, 145\}$.*

Moreover, in all of their packings, the leave is a K_3, whence the foregoing is an existence theorem for CKPDs. Now from the first part of Lemma 4.4, we can remove the values $v = 79, 97$ and 133 from the above list of possible exceptions. Moreover, $v = 103$ is taken care of by replacing the covering (input) designs by the analogous packing designs in the construction for the CKCD (103) (see Lemma 4.5), and $v = 121$ is taken care of similarly, starting with a 4, 5-GDD of type $3^5 4^1$. For $v = 145$, apply weight 6 to a 4, 5-GDD of type $5^4 2^1$ and adjoin 13 infinite points, filling in CKPD (43)-CKPD (13)s (Example 2.6B) and a CKPD (25). Since a CKPD (19) does not exist (as noted in [4] and [5]), we have therefore the following result concerning the existence of canonical Kirkman Packing Designs of orders $v \equiv 1 \bmod 6$.

Theorem 5.1 *There is a CKPD (v) for every* $v \equiv 1$ *mod* 6 *with* $v \geq 25$ *except possibly for* $v = 55, 61, 67, 73, 85,$ *and* 109.

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Appendix A: $v \equiv 4 \bmod 6$

We construct the packings and coverings of orders $v \equiv 4 \bmod 6$, $28 \leq v \leq 64$, referred to in Lemma 3.2. In each case, we construct the design with an automorphism of order $(v-4)/2$, on the point set $(\mathbb{Z}_{\frac{v-4}{2}} \times \{1,2\} \cup I$, where I is a set of 4 extra points. For the packings, all differences are covered *except* pure difference $(v-4)/4$ on each of orbits 1 and 2 (when $v \equiv 4 \bmod 12$), or *any* mixed difference between orbits 1 and 2 (when $v \equiv 10 \bmod 12$); the leave therefore consists of a K_4 (on the points of I) together with $(v-4)/2$ disjoint K_2s, as required. For the coverings, *all* differences are covered (including pure difference $(v-4)/4$ on each of orbits 1 and 2 when $v \equiv 4 \bmod 12$) and, when $v \equiv 10 \bmod 12$, some mixed difference between orbits 1 and 2 is covered a second time; the excess therefore consists of $(v-4)/2$ disjoint K_2s, as required. Furthermore, in each covering, the points of I form a block (of size 4) in the covering (see remark following Lemma 3.2).

CKPD (28)
Point Set: $(\mathbb{Z}_{12} \times \{1,2\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$
Parallel Classes: Develop the following class mod 12.

$2_1 6_1 1_2 6_2$	$5_1 10_1 8_2$	$\infty_2 8_1 10_2$
$0_1 1_1 3_1$	$4_1 5_2 9_2$	$\infty_3 9_1 3_2$
$0_2 2_2 11_2$	$\infty_1 7_1 4_2$	$\infty_4 11_1 7_2$

CKCD (28)
Point Set: $(\mathbb{Z}_{12} \times \{1,2\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$
Parallel Classes: Develop the following class mod 12.

$2_1 7_1 2_2 5_2$	$5_1 11_1 4_2$	$\infty_2 8_1 9_2$
$0_1 1_1 3_1$	$4_1 8_2 10_2$	$\infty_3 9_1 11_2$
$0_2 1_2 7_2$	$\infty_1 6_1 3_2$	$\infty_4 10_1 6_2$

The 13th parallel class is $\infty_1 \infty_2 \infty_3 \infty_4$, together with $(0+i)_1(4+i)_1(8+i)_1$ and $(0+i)_2(4+i)_2(8+i)_2$ for $i = 0, 1, 2, 3$.

CKPD (34)
Point Set: $(\mathbb{Z}_{15} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup \{\infty\}$.
Parallel Classes: Develop the following class mod 15.

$2_1 6_1 2_2 7_2$	$5_1 10_1 12_2$	$4_1 11_1 a_0$
$8_1 6_2 14_2$	$7_1 13_1 10_2$	$14_1 8_2 a_2$
$9_1 4_2 13_2$	$0_1 1_1 3_1$	$5_2 9_2 a_1$
	$0_2 1_2 3_2$	$\infty 12_1 11_2$

(Subscripts on a are evaluated mod 3.)

CKCD (34)

Point Set: $(\mathbb{Z}_{15} \times \{1,2\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

Parallel Classes: Develop the following class mod 15.

$2_1 6_1 2_2 8_2$	$7_1 14_1 11_2$	$\infty_1 4_1 12_2$
$8_1 6_2 13_2$	$0_1 1_1 3_1$	$\infty_2 9_1 10_2$
$5_1 11_1 14_2$	$0_2 1_2 3_2$	$\infty_3 12_1 4_2$
	$10_1 5_2 9_2$	$\infty_4 13_1 7_2$

The 16th parallel class is $\infty_1 \infty_2 \infty_3 \infty_4$, together with $(0+i)_1(5+i)_1(10+i)_1$ and $(0+i)_2(5+i)_2(10+i)_2$ for $i = 0,1,2,3,4$.

CKPD (40)

Point Set: $(\mathbb{Z}_{18} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup \{\infty\}$.

Parallel Classes: Develop the following class mod 18.

$2_1 17_1 4_2 6_2$	$3_1 14_1 15_1$	$5_1 7_1 16_2$
$0_1 13_1 13_2$	$0_2 1_2 8_2$	$6_1 10_1 a_0$
$11_1 3_2 9_2$	$8_1 2_2 5_2$	$12_1 11_2 a_1$
$4_1 7_2 12_2$	$1_1 9_1 15_2$	$10_2 14_2 a_2$
$\infty 16_1 17_2$		

(Subscripts on a are evaluated mod 3.)

CKCD (40)

Point Set: $(\mathbb{Z}_{18} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup \{\infty\}$.

Parallel Classes: Develop the following class mod 18.

$9_1 17_1 10_2 12_2$	$4_1 13_1 11_2$	$0_2 9_2 14_2$
$12_1 3_2 4_2$	$0_1 1_1 15_2$	$2_1 6_1 a_0$
$16_1 6_2 16_2$	$7_1 14_1 13_2$	$1_2 8_2 a_1$
$15_1 2_2 17_2$	$5_1 8_1 10_1$	$3_1 7_2 a_2$
$\infty 11_1 5_2$		

(Subscripts on a are evaluated mod 3.) The 19th parallel class is $\infty a_0 a_1 a_2$, together with $(0+i)_1(6+i)_1(12+i)_1$ and $(0+i)_2(6+i)_2(12+i)_2$ for $i = 0,1,2,3,4,5$.

CKPD (46)

Point Set: $(\mathbb{Z}_{21} \times \{1,2\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

Parallel Classes: Develop the following class mod 21.

$0_1 16_1 9_2 17_2$	$5_1 13_1 20_2$	$18_1 3_2 8_2$
$\infty_1 19_1 19_2$	$2_1 8_1 18_2$	$1_1 3_1 10_1$
$\infty_2 6_1 5_2$	$7_1 17_1 15_2$	$11_1 14_1 15_1$
$\infty_3 9_1 0_2$	$12_1 14_2 16_2$	$2_2 11_2 12_2$
$\infty_4 20_1 4_2$	$4_1 1_2 7_2$	$6_2 10_2 13_2$

CKCD (46)

Point Set: $(\mathbb{Z}_{21} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup \{\infty\}$.

Parallel Classes: Develop the following class mod 21.

$20_1 4_1 20_2 0_2$	$19_1 16_1 5_2$	$12_1 14_2 10_2$
$\infty 1_1 15_2$	$18_1 7_1 12_2$	$17_1 7_2 2_2$
$11_1 3_2 a_0$	$3_1 5_1 11_2$	$13_1 17_2 4_2$
$8_2 6_2 a_1$	$0_1 15_1 18_2$	$10_1 19_2 9_2$
$8_1 9_1 a_2$	$2_1 6_1 14_1$	$1_2 13_2 16_2$

(Subscripts on a are evaluated mod 3.) The 22nd parallel class is $\infty a_0 a_1 a_2$, together with $(0+i)_1(7+i)_1(14+i)_1$ and $(0+i)_2(7+i)_2(14+i)_2$ for $i = 0,1,2,3,4,5,6$.

CKPD (52)

Point Set: $(\mathbb{Z}_{24} \times \{1,2\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

Parallel Classes: Develop the following class mod 24.

$7_1 21_1 12_2 15_2$	$1_1 23_1 3_2$	$4_1 4_2 11_2$	$12_1 16_1 17_1$
$\infty_1 22_1 19_2$	$11_1 18_1 0_2$	$14_1 17_2 23_2$	$2_1 10_1 13_1$
$\infty_2 6_1 22_2$	$3_1 9_1 2_2$	$19_1 9_2 20_2$	$13_2 14_2 18_2$
$\infty_3 20_1 8_2$	$0_1 15_1 10_2$	$5_1 1_2 16_2$	$5_2 7_2 21_2$
$\infty_4 8_1 6_2$			

CKCD (52)

Point Set: $(\mathbb{Z}_{24} \times \{1,2\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

Parallel Classes: Develop the following class mod 24.

$22_1 4_1 19_2 14_2$	$19_1 9_1 7_2$	$23_1 8_2 17_2$	$0_1 7_1 3_1$
$\infty_1 12_1 11_2$	$13_1 11_1 15_2$	$5_1 6_2 13_2$	$14_1 15_1 2_1$
$\infty_2 18_1 0_2$	$17_1 8_1 22_2$	$10_1 3_2 21_2$	$16_2 4_2 18_2$
$\infty_3 6_1 9_2$	$1_1 20_1 20_2$	$16_1 12_2 23_2$	$1_2 2_2 5_2$
$\infty_4 21_1 10_2$			

The 25th parallel class is $\infty_1 \infty_2 \infty_3 \infty_4$, together with $(0+i)_1(8+i)_1(16+i)_1$ and $(0+i)_2(8+i)_2(16+i)_2$ for $i = 0,1,\ldots,7$.

CKPD (58)

Point Set: $(\mathbb{Z}_{27} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup \{\infty\}$.

Parallel Classes: Develop the following class mod 27.

$17_1 9_1 3_2 18_2$	$19_1 3_1 22_2$	$15_1 5_2 0_2$	$21_1 16_1 7_1$
$\infty 23_1 16_2$	$22_1 25_1 20_2$	$12_1 17_2 11_2$	$26_1 11_1 1_1$
$5_1 4_1 a_0$	$6_1 2_1 12_2$	$14_1 1_2 21_2$	$7_2 15_2 25_2$
$10_1 6_2 a_1$	$18_1 24_1 26_2$	$13_1 2_2 4_2$	$10_2 13_2 14_2$
$23_2 9_2 a_2$	$0_1 20_1 24_2$	$8_1 8_2 19_2$	

(Subscripts on a are evaluated mod 3.)

CKCD (58)

Point Set: $(\mathbb{Z}_{27} \times \{1,2\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

Parallel Classes: Develop the following class mod 27.

$18_1 21_1 1_2 9_2$	$8_1 16_1 24_2$	$13_1 15_2 12_2$	$26_1 24_1 20_1$
$\infty_1 10_1 8_2$	$23_1 3_1 26_2$	$2_1 21_2 22_2$	$5_1 15_1 0_1$
$\infty_2 6_1 23_2$	$14_1 25_1 19_2$	$19_1 20_2 3_2$	$13_2 18_2 11_2$
$\infty_3 1_1 5_2$	$11_1 12_1 25_2$	$17_1 0_2 14_2$	$2_2 17_2 6_2$
$\infty_4 7_1 7_2$	$9_1 22_1 4_2$	$4_1 16_2 10_2$	

The 28th parallel class is $\infty_1 \infty_2 \infty_3 \infty_4$, together with $(0+i)_1(9+i)_1(18+i)_1$ and $(0+i)_2(9+i)_2(18+i)_2$ for $i = 0, 1, \ldots, 8$.

CKPD (64)

Point Set: $(\mathbb{Z}_{30} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup \{\infty\}$.

Parallel Classes: Develop the following class mod 30.

$5_1 24_1 25_2 0_2$	$7_1 2_1 11_2$	$20_1 13_1 6_2$	$3_1 16_2 15_2$
$\infty 23_1 21_2$	$25_1 17_1 2_2$	$18_1 10_2 23_2$	$12_1 26_2 14_2$
$0_1 26_1 a_0$	$9_1 15_1 9_2$	$10_1 13_2 20_2$	$22_1 8_1 21_1$
$8_2 24_2 a_1$	$14_1 4_1 1_2$	$11_1 19_2 22_2$	$19_1 1_1 28_1$
$6_1 5_2 a_2$	$29_1 27_1 18_2$	$16_1 4_2 12_2$	$3_2 27_2 29_2$
			$17_2 28_2 7_2$

(Subscripts on a are evaluated mod 3.)

CKCD (64)

Point Set: $(\mathbb{Z}_{30} \times \{1,2\}) \cup (\{a\} \times \mathbb{Z}_3) \cup \{\infty\}$

Parallel Classes: Develop the following class mod 30.

$8_1 24_1 8_2 23_2$	$1_1 26_1 18_2$	$20_1 28_1 9_2$	$3_1 28_2 0_2$
$\infty 12_1 20_2$	$19_1 2_1 15_2$	$5_1 26_2 3_2$	$29_1 5_2 6_2$
$14_1 10_1 a_0$	$0_1 11_1 4_2$	$27_1 21_2 2_2$	$9_1 21_1 6_1$
$11_2 25_2 a_1$	$7_1 13_1 16_2$	$15_1 1_2 27_2$	$16_1 18_1 25_1$
$17_1 7_2 a_2$	$22_1 23_1 24_2$	$4_1 22_2 14_2$	$19_2 10_2 13_2$
			$12_2 29_2 17_2$

(Subscripts on a are evaluated mod 3.)

The 31th parallel class is $\infty a_0 a_1 a_2$, together with $(0+i)_1(10+i)_1(20+i)_1$ and $(0+i)_2(10+i)_2(20+i)_2$ for $i = 0, 1, \ldots, 9$.

Appendix B: $v \equiv 1 \bmod 6$

We construct the coverings of orders $v \equiv 1 \bmod 6$, $25 \leq v \leq 55$, referred to by Lemmas 4.4 and 4.5. Covers of order $v \equiv 7 \bmod 12$ are constructed with an automorphism of order $(v-1)/2$ on the point set $(\mathbb{Z}_{\frac{v-1}{2}} \times \{1,2\}) \cup \{\infty\}$; in each case pairs at pure difference $\pm(v-1)/6$ in each of orbits 1 and 2 are covered *twice*, creating the desired excess of $(v-1)/3$ vertex-disjoint triangles. On the other hand, covers of order $v \equiv 1 \bmod 12$ are constructed with an automorphism of order $(v-1)/3$ on the point set $(\mathbb{Z}_{\frac{v-1}{3}} \times \{1,2,3\}) \cup \{\infty\}$; the excess in each case is created by repeating some mixed difference $d(i,j)$ between orbits i and j, for $(i,j) = (1,2), (2,3)$, and $(1,3)$, where $d(1,2) + d(2,3) \equiv d(1,3) \bmod (v-1)/3$.

CKCD (25)

Point Set: $(\mathbb{Z}_8 \times \{1,2,3\}) \cup \{\infty\}$.

Parallel Classes: Develop each of the following two classes mod 8 (the first base class generates a short orbit of four parallel classes).

$0_2 4_2 0_3 4_3$	$\infty 0_1 4_1$		$2_1 4_1 0_2 6_3$	$\infty 6_2 2_3$
$1_1 2_2 3_2$	$5_1 6_2 7_2$	and	$0_1 5_2 3_2$	$2_2 7_3 1_3$
$1_2 2_3 3_3$	$5_2 6_3 7_3$		$7_1 5_3 0_3$	$3_1 6_1 3_3$
$1_3 2_1 3_1$	$5_3 6_1 7_1$		$5_1 7_2 4_2$	$1_1 1_2 4_3$

(Note that pairs at mixed differences 2 (between orbits 1 and 2), 4 (between orbits 2 and 3), and 6 (between orbits 1 and 3) are covered twice; hence, the excess consists of 8 disjoint triangles, as desired.)

CKCD (31)

Point Set: $(\mathbb{Z}_{15} \times \{1,2\}) \cup \{\infty\}$.

Parallel Classes: Develop the following class mod 15.

$3_1 13_1 9_2 3_2$	$14_1 8_2 1_2$	$10_1 8_1 11_2$
$\infty 4_1 2_2$	$11_1 6_2 4_2$	$7_1 12_1 6_1$
$0_1 12_2 7_2$	$1_1 5_2 0_2$	$5_1 2_1 9_1$
		$10_2 13_2 14_2$

(Note that pairs at pure difference ± 5 in each of orbits 1 and 2 are covered twice; hence, the excess consists of 10 disjoint triangles, as desired.)

CKCD (37)

Point Set: $(\mathbb{Z}_{12} \times \{1,2,3\}) \cup \{\infty\}$.

Parallel Classes: Develop each of the following two classes mod 12 (the first base class generates a short orbit of six parallel classes).

$0_2\,6_2\,0_3\,6_3$	$\infty\,0_1\,6_1$		$3_1\,10_1\,3_2\,6_3$	$\infty\,2_2\,11_3$
$1_1\,2_2\,3_3$	$7_1\,8_2\,9_3$		$9_1\,11_2\,6_2$	$8_1\,0_2\,8_3$
$4_1\,3_2\,2_3$	$10_1\,9_2\,8_3$	and	$7_1\,2_3\,0_3$	$11_1\,9_2\,4_3$
$2_1\,3_1\,5_1$	$8_1\,9_1\,11_1$		$2_1\,10_2\,8_2$	$6_1\,1_2\,5_3$
$1_2\,4_2\,5_2$	$7_2\,10_2\,11_2$		$5_2\,3_3\,10_3$	$5_1\,7_2\,9_3$
$1_3\,4_3\,5_3$	$7_3\,10_3\,11_3$		$0_1\,4_1\,1_3$	$1_1\,4_2\,7_3$

(Note that pairs at mixed differences 2 (between orbits 1 and 2), 3 (between orbits 2 and 3), and 5 (between orbits 1 and 3) are covered twice; hence, the excess consists of 12 disjoint triangles, as desired.)

CKCD (43)

Point Set: $(\mathbb{Z}_{21} \times \{1,2\}) \cup \{\infty\}$.

Parallel Classes: Develop the following class mod 21.

$20_1\,5_1\,12_2\,19_2$	$8_2\,5_2\,9_2$	$9_1\,6_2\,17_2$
$\infty\,19_1\,13_2$	$13_1\,16_1\,14_2$	$3_1\,17_1\,20_2$
$18_1\,14_1\,2_1$	$0_1\,4_2\,2_2$	$1_1\,12_1\,7_2$
$7_1\,15_1\,8_1$	$11_1\,16_2\,0_2$	$6_1\,4_1\,15_2$
$18_2\,11_2\,3_2$	$10_1\,10_2\,1_2$	

(Note that pairs at pure difference ± 7 are covered twice on each of orbits 1 and 2; hence, the excess consists of 14 disjoint triangles, as desired.)

CKCD (49)

Point Set: $(\mathbb{Z}_{16} \times \{1,2,3\}) \cup \{\infty\}$.

Parallel Classes: Develop each of the following two classes mod 16 (the first base class generates a short orbit of eight parallel classes).

$0_2\,8_2\,0_3\,8_3$	$\infty\,0_1\,8_1$		$9_1\,12_1\,4_2\,13_3$	$\infty\,15_2\,7_3$
$1_1\,2_2\,3_3$	$9_1\,10_2\,11_3$		$5_1\,10_2\,1_2$	$11_1\,5_3\,11_3$
$4_1\,6_2\,7_2$	$12_1\,14_2\,15_2$		$2_1\,12_2\,9_2$	$6_1\,6_2\,12_3$
$4_2\,6_3\,7_3$	$12_2\,14_3\,15_3$	and	$4_1\,8_2\,13_2$	$13_1\,11_2\,8_3$
$4_3\,6_1\,7_1$	$12_3\,14_1\,15_1$		$3_2\,15_3\,1_3$	$0_1\,14_2\,3_3$
$3_1\,5_1\,10_1$	$11_1\,13_1\,2_1$		$2_2\,14_3\,9_3$	$15_1\,5_2\,4_3$
$3_2\,5_2\,9_2$	$11_2\,13_2\,1_2$		$10_1\,14_1\,6_3$	$8_1\,7_2\,2_3$
$2_3\,5_3\,9_3$	$10_3\,13_3\,1_3$		$7_1\,1_1\,0_3$	$3_1\,0_2\,10_3$

(Note that pairs at mixed differences 14 (between orbits 1 and 2), 12 (between orbits 2 and 3), and 10 (between orbits 1 and 3) are covered twice; hence, the excess consists of 16 disjoint triangles, as desired.)

CKCD (55)

Point Set: $(\mathbb{Z}_{27} \times \{1,2\}) \cup \{\infty\}$.

Parallel Classes: Develop the following class mod 27.

$9_1 20_1 1_2 19_2$	$8_1 14_1 17_1$	$21_1 10_2 16_2$
$\infty 23_1 21_2$	$7_1 19_1 24_1$	$22_1 7_2 12_2$
$2_1 16_1 25_1$	$0_2 11_2 20_2$	$13_1 6_1 6_2$
$15_1 22_2 24_2$	$5_2 8_2 9_2$	$0_1 1_1 3_2$
$11_1 2_2 17_2$	$4_1 15_2 25_2$	$3_1 5_1 18_2$
$26_1 4_2 23_2$	$12_1 13_2 26_2$	$10_1 18_1 14_2$

(Note that the pairs at pure difference ± 9 are covered twice on each of orbits 1 and 2; hence, the excess consists of 18 disjoint triangles.)