Some Results on the Structure of Finite Nilpotent Algebras over fields of prime characteristic

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ABSTRACT. Let M be a finite dimensional commutative nilpotent algebra over a field K of prime characteristic p. It has been conjectured that $dimM \geq p\ dimM^{(p)}$ where $M^{(p)}$ is the subalgebra of M generated by x^p, $x \in M$, [2]. This was proved (by Eggert) in the case $dimM^{(p)} \leq 2$ in 1971. This result was extended to the noncommutative case in 1994 [8]. Not only is this conjecture important in its own right but it was shown (by Eggert) that a proof of the above conjecture would result in a complete classification of the group of units of finite commutative ring of characteristic p, with an identity. In this short paper we obtain a proof of Eggert's conjecture in case $dimM^{(p)} = 3$.

Introduction

An algebra M is said to be nilpotent if $M^n = 0$ for some integer $n \geq 1$. (Recall that if $j \geq 1$ is an integer, then M^j is the subalgebra of M generated by all monomials of degree j in the elements of M.) If n is the least integer such that $M^n = 0$, then it is called the nilpotency index of M. Since M is a K–algebra it is of course a K–vector space, and the additive quotients M^j/M^{j+1} inherit a vector space structure from M. We use the notation d_j to stand for $dim_K(M^j/M^{j+1})$. We note that, since M is nilpotent, if $M^j \neq 0$, then, M^{j+1} is a proper subalgebra of M^j and so $d_j \neq 0$. Moreover $dimM = \sum_{j=1}^{n-1} d_j$.

Suppose X is an algebra generating set for M. We denote by $N_j(X)$ the set of monomials of degree j in the elements of X and let $M_j(X) = N_j(X) \setminus M^{j+1}$. Where there is no possibility of confusion, we will write M_j for $M_j(X)$ and unless explicitly stated we work with the same generating set X.

It will be convenient to consider the following relation on the elements of M_j. We will say that σ_1, $\sigma_2 \in M_j$ are similar if

$$K\sigma_1 + M^{j+1} = K\sigma_2 + M^{j+1}.$$

It is easy to see that this defines an equivalence relation on M_j and we will use use the notation $\sigma_1 \sim \sigma_2$ to denote that σ_1 and σ_2 are similar in this context.

Suppose $N \subseteq M$. By $< N >$ we mean the linear span (over K) of N in M. If N is a singleton, $N = \{x\}$, say, we write Kx instead of $< N >$.

Finally, we recall that a field, K, of characteristic p is called perfect if the map $x \rightarrow x^p$ is an automorphism of K. This says that every element of K has a *pth* root in K.

Results

Our theorem will follow from a series of lemmas on the structure of nilpotent algebras. As none of these results appear in the literature, we will give complete proofs. To avoid repetition, throughout Lemmas 1–7 we will assume that M is a finite dimensional commutative nilpotent algebra over a field K, of prime characteristic, p, and that X is an algebra generating set for M.

Lemma 1. *Suppose $d_k = dim(M^k/M^{k+1}) = 1$ for some $k \geq 1$. If $M^{k+1} \neq 0$ then there exists an $x \in X$ such that $M^k =< x^k, x^{k+1}, x^{k+2} \ldots >$. In particular, $d_j \leq 1$ for every $j \geq k$.*

Proof: Since $d_k = 1$,

$$M^k = Kxx'_2 \ldots x'_k + M^{k+1}$$

for some x, x'_i in X, $2 \leq i \leq k$. If we write $m' = x_2x_3 \ldots x_k$, then

$$M^{k+1} = M^kM \subseteq xm'M + M^{k+2} \subseteq xM^k + M^{k+2} \subseteq Kx^2m' + M^{k+2}. \quad (1)$$

Choose i maximal, $1 \leq i \leq k$ such that M^k is of the form $M^k = Kx^im + M^{k+1}$ and $m = x_1x_2 \ldots x_{k-i}$ a product of $k - i$ elements of X. If $i \neq k$, equation (1) yields

$$M^{k+1} = Kx^{i+1}m + M^{k+2}.$$

By hypothesis $M^{k+1} \neq 0$, so $x^{i+1}m = x^{i+1}x_1x_2 \ldots x_{k-i} \in M_{k+1}$ and because m is not an empty product $x^{i+1}x_1x_2 \ldots x_{k-i-1} \in M_k$. But $d_k = 1$, so $M^k = Kx^{i+1}x_1x_2 \ldots x_{k-i-1} + M^{k+1}$, contradicting the maximality of i. It follows that $i = k$ and $M^k = Kx^k + M^{k+1}$.

Now using equation (1) we see by induction that $M^{k+j} = Kx^{k+j} + M^{k+j+1}$ for all $j \geq 0$ and hence $M^{k+j} =< x^{k+j}, x^{k+j+1} \ldots >$. In particular, $M^k =< x^k, x^{k+1}, \ldots >$ and $d_l \leq 1$ for all $l \geq k$. This proves the lemma. □

Lemma 2. *If $d_k = dim(M^k/M^{k+1}) = 2$ for some $k \geq 2$, then $d_{k+1} \leq 2$.*

Proof: By hypothesis, there exist elements $xm_1 = xx_2 \dots x_k$, $ym_2 = yy_2 \dots y_k \in M_k$ such that

$$M^k = Kxm_1 + Kym_2 + M^{k+1} \tag{2}$$

Hence

$$\begin{aligned} M^{k+1} = M^k M \subseteq xm_1 M + ym_2 M + M^{k+2} \subseteq xM^k + yM^k + M^{k+2} \\ \subseteq Kx^2 m_1 + Kxym_1 + Kxym_2 + Ky^2 m_2 + M^{k+2}. \end{aligned} \tag{3}$$

If $x = y$ we see from (3) that

$$M^{k+1} \subseteq Kx^2 m_1 + Ky^2 m_2 + M^{k+2}. \tag{4}$$

Thus $d_{k+1} \leq 2$.

Suppose there exists $\gamma \in M_{k-1}$ such that $xy\gamma \in M_{k+1}$. Since$k - 1 \geq 1$, there exists $xy\gamma_0 \in M_k$. But then by substituting $Kxy\gamma_0 + M^{k+1}$ for $Kxm_1 + M^{k+1}$ or $Kym_2 + M^{k+1}$ in (2) we would again have $x = y$ so that $d_{k+1} \leq 2$.

It follows that we need only consider the case where for every $\gamma \in M_{k-1}$, $xy\gamma \in M^{k+2}$. In this case (3) becomes

$$M^{k+1} \subseteq Kx^2 m_1 + Ky^2 m_2 + M^{k+2}$$

so that again $d_{k+1} \leq 2$, proving the lemma. □

Lemma 3. *Suppose that $l \geq j$ are positive integers, and that α, $\gamma \in M_j$ satisfy $\alpha \sim \gamma$. Let $\beta \in M_{l-j}$, then if either $\alpha\beta$, or $\gamma\beta$ is in M_l then $\alpha\beta \sim \gamma\beta$ (in M_l.)*

Proof: The proof is an immediate consequence of the definition of the relation $\sim$. □

Lemma 4. *Let j and k be positive integers with $k \leq j-1$. Suppose $x \in X$ and all monomials of the form $x^k\alpha$, $\alpha \in M_{j-k}$ form a single equivalence class in M_j. Then all elements of the form $x^k\alpha_1 \in M_{j+1}$, $\alpha_1 \in M_{j+1-k}$ also form a single equivalence class in M_{j+1}.*

Proof: Suppose that $x^k\alpha_1$, $x^k\beta_1 \in M_{j+1}$ with α_1, $\beta_1 \in M_{j+1-k}$. Since $k \leq j-1$, it follows that $j+1-k \geq 2$. Hence α_1 and β_1 have factorisations $\alpha_1 = r\alpha_2$, $\beta_1 = s\beta_2$ such that $x^k\alpha_2$, $x^k\beta_2 \in M_j$.

Since $j - k \geq 1$ we can further factorise α_2 and β_2 and obtain $\alpha_2 = t\alpha_3$ $\beta_2 = u\beta_3$ for some t, $u \in M_1$. (If $j - k = 1$ we take α_3 and β_3 to be the empty word in X.) Now $x^k\alpha_2 \sim x^k\beta_2$ and so $x^k t\alpha_3 \sim x^k u\beta_3$. By Lemma 3 since $x^k\alpha_1 \in M_{j+1}$, we have

$$x^k\alpha_1 = x^k t\alpha_3 r \sim x^k u\beta_3 r.$$

In particular we see that $x^k\beta_3 r \in M_j$ and $x^k r\alpha_3 \in M_j$, and so it follows again by Lemma 3, that

$$x^k\beta_1 = x^k s\beta_3 u \sim x^k r\alpha_3 u \sim x^k r\beta_3 u.$$

Then putting the above two relations together we get that $x^k\alpha_1 \sim x^k\beta_1$ as required. □

Lemma 5. *Suppose that $d_k = d_{k+1} = 2$ for some $k \geq 2$, and there exist elements $xx_2\dots x_k$, $xy_2\dots y_k \in M_k$ for which $M^k = Kxx_2\dots x_k + Kxy_2\dots y_k + M^{k+1}$. Then there exists $y \in X \setminus \{x\}$ such that $M^j =< x^j, x^{j+1}, x^{j+2}, \dots,\ x^{j-1}y, x^jy, x^{j+1}y \dots >$ for every $j \geq k$.*

Proof: In view of (2) and (4) it suffices to show that

$$M^k/M^{k+1} =< x^k + M^{k+1},\ x^{k-1}y + M^{k+1} > .$$

If $M^k = Kx\alpha + Kx\beta + M^{k+1}$ then arguing as in (2) we see that

$$M^{k+1} = Kx^2\alpha + Kx^2\beta + M^{k+2}.$$

If $k = 2$ then $d_3 = 2$ and so $x^2 \notin M^3$ and therefore is in M_2 and we can replace $x\alpha + M^3$ or $x\beta + M^3$ with $x^2 + M^3$ and have a basis for M^2/M^3 of the required type.

We may assume therefore that $k \geq 3$.Then α and β are elements of M_{k-1} and $k - 1 \geq 2$. We now show that

$$M^k/M^{k+1} =< x^{k-1}\gamma + M^{k+1}, x^{k-1}\mu + M^{k+1} > \tag{5}$$

To this end, as in Lemma 1, select a basis $x^j\gamma_1 + M^{k+1}, x^j\mu_1 + M^{k+1}$ for M^k/M^{k+1} with j as large as possible. Then because $d_{k+1} = 2$, $x^{j+1}\gamma_1 + M^{k+2}$, $x^{j+2}\mu_1 + M^{k+2}$ form a basis for M^{k+1}/M^{k+2}. If $j < k-1$, Lemma 4 implies there exist linearly independent elements $x^{j+1}\gamma + M^{k+1}$ and $x^{j+1}\mu + M^{k+1}$ in M^k/M^{k+1} contradicting the maximality of j. It follows that $j = k - 1$ and we have established (5).

Since $d_{k+1} = 2$, it follows that $x^k\gamma + M^{k+2}$ and $x^k\mu + M^{k+2}$ are linearly independent elements of M^{k+1}/M^{k+2} and so $x^k \notin M^{k+1}$. It follows that $x^k \in M_k$ and so we can replace one of the basis elements of M^k/M^{k+1} above with $x^k + M^{k+1}$ to produce a basis of the required form. This proves the lemma. □

In Lemma 5 we looked at the case where $d_k = d_{k+1} = 2$ and where two distinct equivalence classes of $\sim$ in M_k contained elements with a common factor - namely x. We now analyse the case where this does not occur. We will restrict our attention only to those cases required for the proof of our main theorem.

Lemma 6. *Suppose that M is a nilpotent algebra and that $X = \{x, y, z\}$ is an algebra generating set for M. Suppose that for $k \geq 2$ that $d_k = d_{k+1} = 2$. Suppose further that no class of M_k contains elements of the form $x^l y^m z^{k-m-l}$ with at least two of l, m, $k - m - l$ positive. Then one of the elements of X, x say, will satisfy the conditions $x^k \in M^{k+1}$ and $xM^k \subseteq M^{k+2}$.*

Proof: Under the conditions of the lemma (and since $d_k = 2$) we may assume without loss of generality that

$$M^k = Ky^k + Kz^k + M^{k+1}$$

Again it is easy to see from (2), (3), (4), and the hypothesis of the lemma that

$$M^{k+1} = Ky^{k+1} + Kz^{k+1} + M^{k+2}.$$

Also $xM^k \subseteq M^{k+2}$. Note that the argument here depends crucially on the fact that $k \geq 2$. Now since $x^k \in M^k$ it follows that

$$x^k = \lambda y^k + \gamma z^k + m$$

for some λ and $\gamma \in K$ and some $m \in M^{k+1}$. Multiplying by y gives

$$x^k y = \lambda y^{k+1} + \gamma z^k y + my$$

But since $k \geq 2$ all the above elements apart from y^{k+1} are contained in M^{k+2}. It follows that $\lambda = 0$. A similar argument using multiplication by z gives that $\gamma = 0$. But then $x^k \in M^{k+1}$ and the lemma is proved. □

There remains one further case to consider. This is the case where M is generated by the set $X = \{x, y, z\}$, $k \geq 2$, $d_k = d_{k+1} = 2$ and M_k is partitioned into two equivalence classes by $\sim$, one consisting of exactly one monomial say y^k, the second containing an element of the form $x^i z^{k-i}$ with $i \geq 1$ and $k - i \geq 1$. We now prove our final lemma.

Lemma 7. *Suppose that M satisfies the conditions above, then for every $l \geq k$ such that $d_l = 2$, $x^i z^{l-i} \sim x^j z^{l-j}$ for every i, j $0 \leq i, j \leq l$. In particular x^l and z^l will always lie in the same equivalence class of M_l*

Proof: If $x^{i+1} z^{k-i}$ and $x^i z^{k-i+1}$ are both in M^{k+2} then we would not have $d_{k+1} = 2$. Without loss of generality we may assume that $x^{i+1} z^{k-i} \in M_{k+1}$. Then clearly $x^{i+1} z^{k-i-1} \in M_k$ and so

$$x^{i+1} z^{k-i-1} \sim x^i z^{k-i}.$$

But then from Lemma 3 we see that

$$x^{i+1}z^{k-i} \sim x^i z^{k-i+1}$$

and so $x^{i-1}z^{k-i+1} \in M_k$. Hence $x^{i-1}z^{k-i-1} \sim x^i z^{k-i}$ and again by Lemma 3 we have that $x^{i-1}z^{k-i} \sim x^i z^{k-i+1}$. We have therefore that

$$x^{i-1}z^{k-i+1} \sim x^i z^{k-i} \sim x^{i+1}z^{k-i-1}$$

and that

$$x^i z^{k-i+1} \sim x^{i+1}z^{k-i} \sim x^{i-1}z^{k-i+2}$$

The lemma follow by induction. □

We now have all the necessary facts gathered to prove our main theorem. The proof follows from the seven lemmas above and a number of basic facts from linear algebra.

Let M be a nilpotent algebra over a field of prime characteristic and let $M^{(p)}$ denote the subalgebra of M generated by x^p, $x \in M$. We now prove the main result of this paper:

Theorem. *Let M be a finite dimensional commutative nilpotent algebra over a perfect field of prime characteristic p, If $dimM^{(p)} = 3$, then $dimM \geq pdimM^{(p)} = 3p$.*

Proof: Let us assume the theorem is false and let M be a counterexample of least dimension. Then clearly $M^{3p} = 0$. Let x^p, y^p and z^p be a set of linearly independent elements of $M^{(p)}$ and consider the subalgebra A generated by x, y and z. Then since $dimA^{(p)} = 3$, it follows from the minimality of our counterexample that $A = M$ and $X = \{x, y, z\}$ is an algebra generating set for M.

Now $M^{3p} = 0$. We consider separately the cases $M^{2p} \neq 0$ and $M^{2p} = 0$.

Suppose first that $M^{2p} \neq 0$. We now show that this forces $M^{2p+1} = 0$. For if not, then it follows from Lemmas 1 and 2 that $d_{p-1} = 1$. (If $d_{p-1} \geq 2$ then $dimM \geq 2(p-1)+p+2 = 3p$ a contradiction.) But then since $M^p \neq 0$ by Lemma 1 relabelling if necessary, $M^{p-1} =< x^i, i \leq i \leq n >$ where n is an integer greater than $2p+1$. In particular $x^{2p} \neq 0$ and $x^{2p} \in M^{(p)}$. The nilpotency of M gives immediately that x^p and x^{2p} are linearly independent in $M^{(p)}$ and so since $dimM^{(p)} = 3$ we may assume (relabelling if necessary) that $M^{(p)}$ is generated as a subspace by x^p, x^{2p}, and y^p. Thus again using minimality, we may assume that $X = \{x, y\}$ generates M. Now $yx^{p-2} \in M^{p-1}$, hence $yx^{p-2} = \sum_{i=p-1}^{n} \lambda_i x^i$. Hence $(y - \sum_{i=p-1}^{n} \lambda_i x^{i-p+2})x^{p-2} = 0$. Put $y_0 = y - \sum_{i=p-1}^{n} \lambda_i x^{i-p+2}$ then $y_0^p = y^p - \lambda_{p-1}^p x^p - \lambda_p^{2p} x^{2p}$. Again it is clear that y_0^p, x^p, x^{2p} generate $M^{(p)}$. Moreover $y_0 x^{p-2} = 0$. Arguing as above, x and y_0 will generate M. Let $y = y_0$. But then $y^p \in M^{p-1}y = 0$

which is a contradiction. We have now established that if $M^{2p} \neq 0$ then $M^{2p+1} = 0$.

Let this be the case. Then another easy counting argument using Lemmas 1 and 2 shows that $d_p = 1$, and so $M^p =< x^j, j \geq p >$. As in the previous paragraph we can choose y with $yx^{p-1} = 0$. But then since $y^{p-1}x \in M^p = Mx^{p-1}$ we have that $y^p x = 0$. Then if $y^p = \sum_{i=p}^{2p} \lambda_i x^i$ on multiplying by x we obtain the relation $0 = \sum_{i=p}^{2p} \lambda_i x^{i+1}$. As the x^i are linearly independent,(since M is nilpotent), we have $\lambda_i = 0$, for $p \leq i \leq 2p-1$. But then $y^p = \lambda_{2p} x^{2p}$ a contradiction. It follows therefore that $M^{2p} = 0$.

Assume now that $M^{2p} = 0$. Then in particular $d_1 = 3$ for otherwise x^p, y^p and z^p would be linearly dependent mod M^{2p}, and therefore (since $M^{2p} = 0$) linearly dependent over K; impossible. Let $k \geq 2$ be the least integer for which $d_k < 3$. (Note that $2 \leq k \leq p-1$ as otherwise counting dimensions would give that $dimM \geq 3 + 3(p-2) + 3 = 3p$) If $p = 2$ then $dimM = d_1 + dimM^2 \geq 3 + 3 = 6 = 3p$. It follows that $p \geq 3$.

By Lemma 2, since $k \leq p-1$ we have that $d_p \leq 2$. If $d_p = 1$, then since $M^{p+1} \neq 0$ we have from Lemma 1 that $M^p =< x^i, p \leq i \leq n >$ with $n < 2p$. Now $yx^{p-1} \in M^p$ and so $yx^{p-1} = \sum_{i=p}^{n} \lambda_i x^i$. Similarly $zx^{p-1} = \sum_{i=1}^{p} \mu_i x^i$. Hence

$$(y - \sum_{i=p}^{n} \lambda_i x^{i-p+1})x^{p-1} = (z - \sum_{i=p}^{n} \mu_i x^{i-p+1})x^{p-1} = 0.$$

If we put $y_0 = y - \sum_p^n \lambda_i x^{i-p+1}$ and $z_0 = z - \sum_p^n \mu_i x^{i-p+1}$ then arguing exactly as in the second paragraph x_0, y_0 and z_0 will generate M. Let $y = y_0$ and $z = x_0$. Then from the above we have that $yx^{p-1} = zx^{p-1} = 0$.

Now $z^{p-1}x \in Mx^{p-1}$ and so $z^p x = 0$. Similarly $y^p x = 0$. Now $z^p = \sum_{i=p}^{n} \lambda_i x^i$ and so on multiplying this equation across by x and using the fact that $z^p x = 0$ we have that $0 = \sum_p^n \lambda_i x^{i+1}$. But then $z^p = \lambda_n x^n$. A similar argument shows that $y^p = \mu_n x^n$. But then y^p and z^p would be linearly independent- a contradiction. It follows therefore that $d_p = 2$. Since $k \leq p-1$ it follows from Lemma 2 that $d_{p-1} = 2$.

Suppose now that

$$M^{p-1} =< x\alpha + M^p, x\beta + M^p >$$

where $\alpha, \beta \in M_{p-2}$. Then by Lemma 5 we have that $M^{p-1} =< x^j, x^{j-1}y, j \geq p-1 >$. Suppose $x^n \neq 0, x^{n+1} = 0$ and $x^m y \neq 0$, $x^{m+1}y = 0$. Now

$$zx^{p-2} = \sum_{i=p-1}^{n} \lambda_i x^i - \sum_{i=p-2}^{m} \mu_i x^i y.$$

Hence

$$(z - \sum_{i=p-1}^{n} \lambda_i x^{i-p+1} - \sum_{i=p-2}^{m} \mu_i x^{i-p+2} y) x^{p-2} = 0.$$

Put $z_0 = z - (\sum_{i=p-1}^{n} \lambda_i x^{i-p+1} + \sum_{i=p-2}^{m} \mu_i x^{i-p+2} y)$. Then as above $X = \{x, y, z_0\}$ generates M. Let $z = z_0$. Then $z^{p-1} \in M^{p-1}$ and so $z^p = 0$ which gives yet another contradiction.

It remains finally to consider the case where no two distinct equivalence classes of M_k contain elements with common factors. Suppose first that no element of M_k has two distinct factors. By Lemma 6 we have say that $x^k \in M^{k+1}$ and $xM^k \subseteq M^{k+2}$. But then

$$x^p \in x^{p-k} M^{k+1} \subseteq M^{k+1+2(p-k)}.$$

Thus $M^{k+1+2(p-k)} = M^{2p-k+1} \neq 0$. But then since $d_i \geq 3,\ 1 \leq i \leq k-1$ and $d_i = 2$ for $k \leq i \leq p$ and $d_i \geq 1$ for $p+1 \leq i \leq 2p-k+1$, we have that

$$dim M \geq 3(k-1) + 2(p-k+1) + 2p - k + 1 - (p+1) + 1 = 3p$$

a contradiction.

We finally need only consider the case where M_k consists of precisely two distinct equivalence classes and where one of these classes consists of a single element y^k , and the other has an element of the form $x^i z^{k-i}$ with $i \geq 1$ and $k - i \geq 1$. Thus

$$M^k \setminus M^{k+1} =< y^k + M^{k+1},\ x^i z^{k-i} + M^{k+1} >$$

Note that the condition on the equivalence classes here imply that

$$x^i y^j z^l \in M^{k+1} \qquad i + j + l = k \tag{6}$$

for $j \geq 1$ and at least one of i and j greater than one. By lemma 7 we must have

$$x^i z^{l-i} \sim x^j z^{l-j}, \qquad 1 \leq i, j \leq l. \tag{7}$$

Now since $d_p = 2$ it follows from (6) and (7) that

$$M^p \setminus M^{p+1} =< y^p + M^{p+1},\ x^p + M^{p+1} > \tag{8}$$

It follows from (7) that $x^p \sim z^p$, and so $z^p = \gamma x^p + m,\quad m \in M^{p+1}$. Let $z_0 = z - \lambda x$ where $\lambda^p = \gamma$. (Note that λ exists since K is perfect.) Then $z_0^p = z^p - \gamma x^p$. Then as above $X' = \{x, y, z_0\}$ generates M. Moreover $z_0^p = m \in M^{p+1}$.

We have just shown that if X satisfies the hypothesis of Lemma 7, then we can change the generators to a new set $X' = \{x, y, z_0\}$ still with the property that $\{x^p, y^p, z_0^p\}$ is a basis for $M^{(p)}$, but also satisfying $z_0^p \in M^{p+1}$. But now X' cannot satisfy the conditions of Lemma 7, for part of the conclusion of that lemma is that $t^p \in M_p$ for every $t \in X'$ and so in partiular $z_0^p \notin M^{p+1}$.

Hence the set X' satisfies the hypothesis of either Lemma 5 or Lemma 6 and either way, as we have already seen, this gives a contradiction.

Since there are no other cases to be considered, the theorem is now proved. □

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