On One-Designs

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Dedicated to Anne Penfold Street

1 Introduction

In 1945, modern combinatorics was in its infancy. Design theory had achieved a start with the pioneering work of Fisher and Yates, followed by Bose. In graph theory, there was the book of König, and Tutte had begun his fundamental work, but the book by Berge was still a decade in the future. And it was in 1944-1945 that Donald Coxeter gave a graduate course [3], at the University of Toronto, on "Configurations".

Coxeter's first illustration of a configuration was the Desargues Configuration (see Figure 1) of two triangles (2,3,4) and (5,6,7) in perspective from the point 1.

The lines in the figure can be listed as 126, 145, 137, 240, 349, 056, 579, 238, 089, 678, and the figure can be called a (10,10,3,3) configuration, that is, a figure with 10 points, 10 lines, 3 lines through each point, 3 points on each line. In general, we use the notation (v, b, r, k), where v = number of points, b = number of lines (or blocks), r = replication number of each point, k = number of points in a block.

A configuration can also be represented as a regular graph (points in the graph are joined if they lie on a line in the configuration). Thus the Desargues Configuration gives a 6-regular graph on 10 points (see Figure 2).

The idea of a configuration leads to the definition of a *One-Design* as a system comprising v points (or varieties), b blocks (or lines), r being the replication number for points, k being the number of points in a block. By counting the number of elements in the design array in two ways, we get the fundamental result $bk = rv$.

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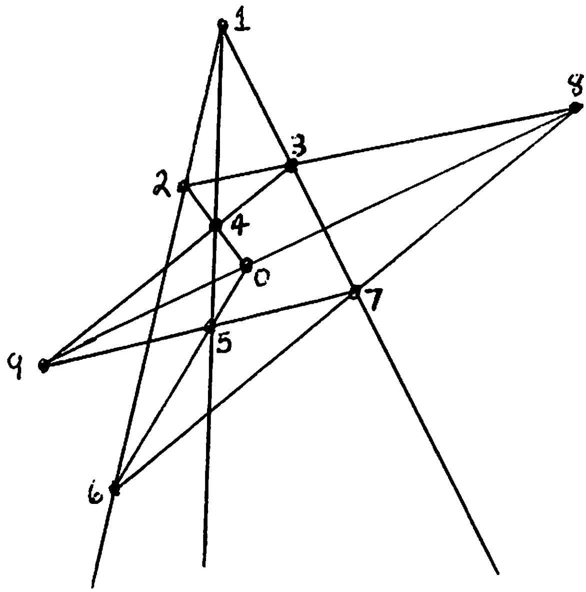

Figure 1. The Desargues Configuration

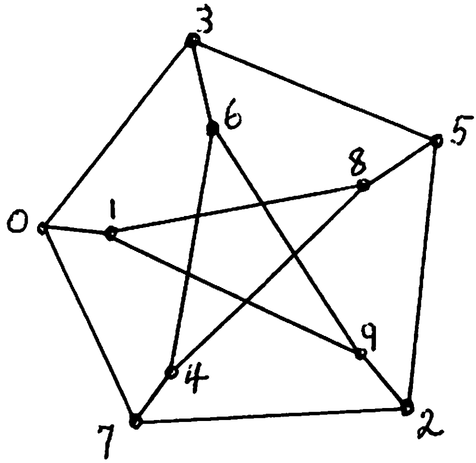

Figure 2. Complement of the Desargues Graph

The lines of the one-design need not be achievable in the Euclidean plane. Thus, the design (8,8,3,3) was used as an illustration in Coxeter's course. It can be written down as (124), (235), (346), (457), (568), (671), (782), (813), but can not be represented by lines in the Euclidean plane (figure 3 shows a Euclidean representation with 7 ordinary Euclidean lines together with one curved "line"). Of course this configuration is easily representable in the projective plane with 13 points by deleting one point and a line not through this point to leave 8 triples and 4 pairs; the 8 triples form the design (8,8,3,3).

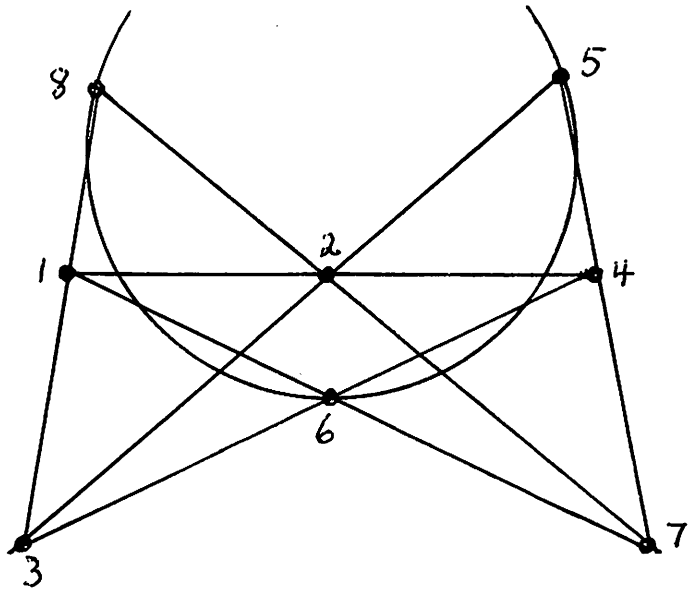

Figure 3. The One-Design (8,8,3,3)

The graph of the (8,8,3,3) configuration is the complement of the graph in Figure 4.

Figure 4. Complement of the graph of (8,8,3,3)

The complete quadrilateral (4,6,3,2) is the simplest one-design in which the numbers of points and blocks differ (see Figure 5).

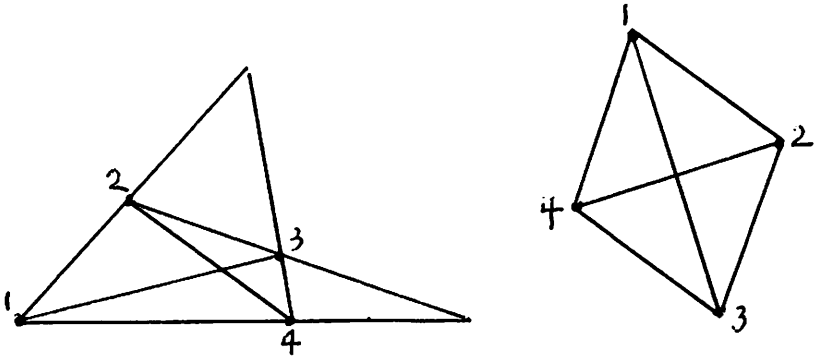

Figure 5. The complete quadrilateral and its graph

Another favourite configuration that appeared frequently in Coxeter's course was the Pappus configuration (9,9,3,3) shown in Figure 6.

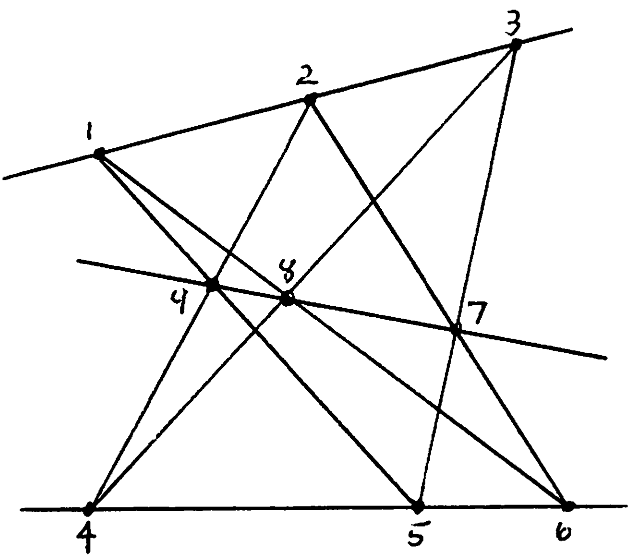

Figure 6. The Pappus Configuration

The Pappus Configuration can be listed as (123), (456), (789), (159), (267), (348), (168), (249), (357). The complement of its graph is shown in Figure 7. It comprises three disjoint triangles.

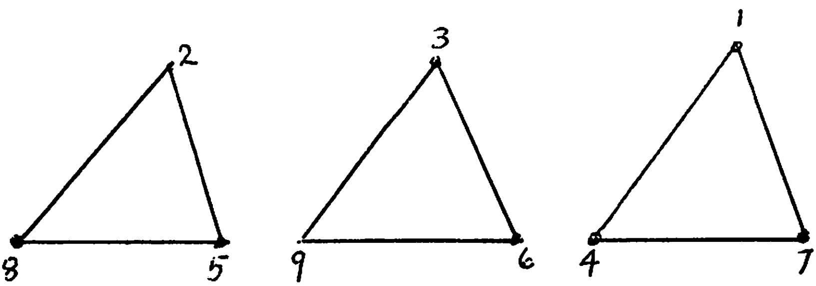

Figure 7. Complement of the Pappus Graph

2 Construction of One-Designs

A well known modern combinatorics text states "One-Designs are very easy to construct, and therefore have little interest." [1]

Certainly, the construction of any one-design is simple. Suppose we start from the design with parameters (v, b, r, k) where, of course, $bk = rv$. The result $bk = rv$ is a sufficient condition for the existence of the design. For let $v = dV$, $k = dK$, where d is the g.c.d. of v and k, and where K and V are relatively prime. Then $bK = rV$, and it follows that $b = mV$, $r = mK$. We now give a numerical example of the construction of a one-design (the generalization is obvious).

Problem. To construct (16,24,9,6).

Step 1. $V = 8$, $K = 3$ $(d = 2)$.
First, construct the symmetric one-design (8,8,3,3). This is easily done by a difference method; start from any initial triple and cycle, modulo 8. Say we use (125) and get the other blocks as (236), (347), (458), (561), (672), (783), (814).

Step 2. Write down these 8 blocks twice more to give a total of 24 blocks in a one-design with parameters (8,24,9,3).

Step 3. Replace each symbol by d symbols (here, $d = 2$) to give blocks of length 6 and a design (16,24,9,6). The blocks are now $(1_1, 1_2, 2_1, 2_2, 5_1, 5_2)$, $(2_1, 2_2, 3_1, 3_2, 6_1, 6_2)$, $(3_1, 3_2, 4_1, 4_2, 7_1, 7_2)$, $(4_1, 4_2, 5_1, 5_2, 8_1, 8_2)$, et cetera.

As a second illustration, let us construct the one-design (24,16,10,15). Here $d = 3$, $V = 8$, $K = 5$. Also $m = 2$.

So we start from (8,8,5,5); it can be constructed by cycling the block (1 2 3 4 5), modulo 8. Then repeat these 8 blocks ($m = 2$) to give (8,16,10,5). Finally, expand each block [the initial block becomes] $1_1 1_2 1_3$ $2_1 2_2 2_3$ $3_1 3_2 3_3$ $4_1 4_2 4_3$ $5_1 5_2 5_3$ to end up with (24,16,10,15).

Finally, look at the example (20,15,3,4). Here $d = 4$, $V = 5$, $K = 1$, $m = 3$. The initial one-design is merely (5,5,1,1) and it comprises 5 singletons (1), (2), (3), (4), (5). Repeat these 3 times to give 15 singletons forming the design (5,15,3,1). Expand each singleton 4 times to end up with (i_1, i_2, i_3, i_4), where $i = 1, 2, 3, 4, 5$, and each block appears 3 times; this produces (20,15,3,4).

Norman Biggs, in his book on Discrete Mathematics [2], gives a proof due to David Billington, in the journal *Discrete Mathematics*, that it is possible to obtain a design without repeated blocks, in the special case that you impose a restriction on the total number of blocks. This type of restriction is possibly mathematically pleasing, but it is unrealistic in many practical applications (as in the statistical design of experiments).

3 The Neumann-Praeger Designs

Although general one-designs may lack interest, as claimed in the quotation at the beginning of Section 2, Cheryl Praeger and Peter Neumann have recently [4] obtained interesting results on special one-designs. We shall call one-designs in which all blocks meet in one or more points by the name Neumann-Praeger designs. The results of Neumann and Praeger indicate that it is certainly worthwhile to consider one-designs that satisfy additional conditions.

First, we note that we always consider *incomplete blocks* in one-designs, that is, $k < v$. Since $bk = rv$, it follows that $r < b$. We first prove a simple numerical lemma about Neumann-Praeger designs.

Look at any specific block of an NP design. The elements of this block occur a total of $k(r-1)$ times in other blocks. If this specific block meets all of the other $b-1$ blocks, then

$$b - 1 \leq k(r-1),$$

and equality only occurs if the specific block has exactly one element in common with each of the other blocks. It follows that

$$k \geq \frac{b-1}{r-1} > \frac{b-1}{r-1} - \frac{(b-r)}{r(r-1)} = \frac{b}{r}.$$

Since $k > \frac{b}{r}$, and $\frac{b}{r} = \frac{v}{k}$, we have

$$k > \frac{v}{k},$$

whence $k^2 > v$, that is, $v < k^2$.

The Neumann-Praeger Theorem strengthens this result. We state the theorem as follows.

Theorem. (Neumann-Praeger) *If one has a one-design (v, b, r, k) in which each pair of blocks intersect in one or more varieties, then*

$$v \leq k^2 - k + 1.$$

In the next section, we give the simple and instructive proof of this result.

4 Proof of the Neumann-Praeger Theorem

Take a specific element a in the design. It occurs in r blocks of the design and is absent from $b - r$ blocks of the design.

Now let λ_{ax} be the number of blocks in the design that contain a and x, where x is another element (this is just the number of pairs ax that appear). Clearly

$$\sum_{x \neq a} \lambda_{ax} = r(k-1),$$

since there are $k - 1$ choices for x in each of the r blocks containing a.

Now it is a piece of folklore that most combinatorial results come from counting some objects in two different ways. We next count the number of block pairs that contain x, subject to the restriction that one block of the pair comes from the λ_{ax} blocks containing a, the other block of the pair comes from the $r - \lambda_{ax}$ blocks that do not contain a. The total number of such pairs is $\sum_{x \neq a} \lambda_{ax}(r - \lambda_{ax})$ and this number must be greater that or equal to $r(b-r)$, since each of the r blocks must be represented for some x, and similarly each of the $b-r$ blocks must be represented, for some x. [For future reference, we note that equality only occurs if each block containing a meets each of the blocks not containing a in precisely one point; in short, unless two blocks are identical, we can choose a in one block and not in the other, and thus have the block intersection equal to unity.]

We thus have the inequality

$$\begin{aligned} r(b-r) &\leq \lambda_{ax}(r - \lambda_{ax}) \\ &= r \sum \lambda_{ax} - \sum \lambda_{ax}^2 \\ &= r^2(k-1) - \sum \lambda_{ax}^2. \end{aligned}$$

Now the variance inequality, for any variable t whose mean value is $\bar{t}$, can be applied.

$$\begin{aligned}\sum_t (t-\bar{t})^2 &= \sum t^2 - 2\bar{t}\sum t + n\bar{t}^2 \\ &= \sum t^2 - 2\frac{\sum t}{n}\sum t + n\frac{(\sum t)^2}{n^2} \\ &= \sum t^2 - \frac{(\sum t)^2}{n} \geq 0.\end{aligned}$$

Thus $\sum t^2 \geq \frac{(\sum t)^2}{n}$, in general. So

$$\begin{aligned}r(b-r) &\leq r^2(k-1) - \frac{(\sum \lambda_{ax})^2}{v-1} \\ &= r^2(k-1) - \frac{r^2(k-1)^2}{v-1}.\end{aligned}$$

Hence $b - r \leq r(k-1)\ (1 - \frac{k-1}{v-1})$, whence

$$(b-r)(v-1) \leq r(k-1)(v-k).$$

But $\frac{b}{r} = \frac{v}{k}$ and so

$$\frac{b-r}{r} = \frac{v-k}{k} = \alpha.$$

Thus $\alpha r(v-1) \leq r(k-1)\alpha k$, whence

$$\begin{aligned}v - 1 &\leq k(k-1), \\ v &\leq k^2 - k + 1.\end{aligned}$$

This is the Neumann-Praeger Theorem.

The result is sharp, since clearly v is equal to $k^2 - k + 1$ for a finite projective geometry.

5 When is $v = k^2 - k + 1$?

If we have equality, then the variance inequality employed must be an equality and so each λ_{ax} is equal to its mean value, that is, λ_{ax} is a fixed number T. Then

$$T = \frac{\sum \lambda_{ax}}{v-1} = \frac{r(k-1)}{k^2-k} = \frac{r}{k} = \frac{b}{v}.$$

We have thus established that we have a balanced incomplete block design, since λ_{ax} is a constant T. The design has parameters v, $b = Tv$, $r = Tk$, k, $\lambda = T$.

We now revert to the fact, previously noted, that any two blocks that are not identical intersect in a single element. Write down the usual intersection equations for an arbitrarily selected base block (x_i = number of blocks intersecting the base block in i points). Then

$$x_1 + x_k = b - 1 = Tv - 1 = Tk^2 - Tk + T - 1$$

$$x_1 + kx_k = k(r-1) = k(Tk-1)$$

It follows that

$$(k-1)x_k = Tk - T + 1 - k = (T-1)(k-1).$$

Thus any base block is identical with $T-1$ other blocks; in short, the BIBD (v, Tv, Tk, k, T) is made up of v non-identical blocks, each repeated T times.

Thus the design is a T-multiple of the design $(v, v, k, k, 1)$, where $v = k^2 - k + 1$; this design is just $PG(2, k-1)$. We state this result as a corollary to the Neumann-Praeger Theorem: If all blocks in a 1-design intersect, and if v achieves its upper bound of $k^2 - k + 1$, then the design comprises T copies of $PG(2, k-1)$, and so is a 2-design (a BIBD) in which every pair occurs exactly T times.

6 The Second Neumann-Praeger Theorem

Neumann and Praeger also considered the case where any two blocks of the 1-design intersect in at least s points ($s \geq 1$). In that case, Section 4 is easily modified to give $\sum_{x \neq a} \lambda_{ax} = r(k-1)$, $r(b-r)s \leq r^2(k-1) - \sum \lambda_{ax}^2$. The algebra then proceeds as before to produce the result

$$\begin{aligned} s(b-r)(v-1) &\leq r(k-1)(v-k), \\ s(v-1) &\leq k(k-1), \\ v-1 &\leq \frac{k(k-1)}{s}, \\ v &\leq 1 + \frac{k(k-1)}{s}. \end{aligned}$$

Again, if equality occurs, then $v = 1 + \frac{k(k-1)}{s}$, and λ_{ax} is a fixed number $T = \frac{r(k-1)}{v-1} = \frac{sr}{k} = \frac{sb}{v}$. Also, either any two blocks are identical or else they meet in exactly s points. So we again have a BIBD, and it has parameters v, $b = \frac{Tv}{s}$, $r = \frac{Tk}{s}$, k, $\lambda = T$.

The intersection equations of Section 4 now become

$$x_1 + x_k = \frac{Tv}{s} - 1 = \frac{T}{s}(k^2 - k + 1) - 1,$$

$$x_1 + k_{x_k} = k(\frac{Tk}{s} - 1).$$

So $(k-1)x_k = \frac{T}{s}k - \frac{T}{s} + 1 - k = (k-1)(\frac{T}{s} - 1)$. Hence, any block appears a total of $x_k + 1 = \frac{T}{s}$ times. This give the Generalized Neumann-Praeger Theorem. If a 1-design is such that each block meets all other blocks in s points $(s \geq 1)$, then $v \leq 1 + \frac{k(k-1)}{s}$, and the quantity v achieves its upper bound only when the design is a $\frac{T}{s}$-multiple of the BIBD with parameters (v, v, k, k, s), where $v = 1 + \frac{k(k-1)}{s}$.

7 Conclusion

The Neumann-Praeger Theorems indicate that, although 1-designs may be too general to posess many properties, restricted classes of 1-designs can be extremely interesting.

References

[1] T. Beth, D. Jungnickel, and H. Lenz, *Design Theory*, Bibliographisches Institut - Wissenschaftsverlag, Mannheim - Wien - Zürich, 1985.

[2] Norman Biggs, *Discrete Mathematics*, Clarendon Press, Oxford, 1985.

[3] H.S.M. Coxeter, Graduate course on "Configurations", University of Toronto, 1944–1945.

[4] Peter M. Neumann and Cheryl E. Praeger, An inequality for Tactical Configurations, *Bull. London Math. Soc.* **28** (1996), 471–475.