A New Upper Bound Formula for Two Color Classical Ramsey Numbers

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ABSTRACT. The two color Ramsey number $R(k,l)$ is the smallest integer p such that for any graph G on p vertices either G contains a K_k or $\overline{G}$ contains a K_l, where $\overline{G}$ denotes the complement of G. A new upper bound formula is given for two color Ramsey numbers. For example, we get $R(7,9) \leq 1713$, $R(8,10) \leq 6090$ etc.

The problem of determining the Ramsey numbers is known to be very difficult and so we are often satisfied with partial results, e. g. upper or lower bounds.

An $(m,n;p)$-graph is a graph with order p which has no K_m and no $\overline{K}_n$ as a subgraph. If p is unspecified, the graph will be called an (m,n)-graph. The Ramsey number $R(m,n)$ is the smallest integer p such that for any graph with order p, either G contains a K_m or $\overline{G}$ contains a K_n. It is easy to see that $R(m,n) = p$ iff the largest (m,n)-graph has $p-1$ vertices. In

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this paper, we will use $N(K_i)$ ($N(\overline{K}_j)$ resp.) to denote the number of K_i ($\overline{K}_j$ resp.) in G.

Theorem 1. *For any (m,n)-graph G, the following inequalities must hold:*

$$(s+1)N(K_{s+1}) \leq N(K_s)[R(m-s,n)-1] \quad 0<s<m-1, \quad (1.1)$$

$$(t+1)N(\overline{K}_{t+1}) \leq N(\overline{K}_t)[R(m,n-t)-1] \quad 0<t<n-1. \quad (1.2)$$

In particular for any (n,n)-graph, then

$$N(K_{n-1})+N(\overline{K}_{n-1}) \leq N(K_{n-2})+N(\overline{K}_{n-2}). \quad (1.3)$$

Proof: By the definition of Ramsey number, for any K_s, there are at most $R(m-s,n)-1$ vertices in G which form a K_{s+1} in combination with the K_s. Otherwise there exists either a K_m or a $\overline{K}_n$ as a subgraph of G, a contradiction. On the other hand, for any K_{s+1}, it contains exactly $s+1$ K_s. Hence (1.1) is true.

Similarly, we can prove that (1.2) is true.

Using (1.1), (1.2) and $R(2,n)=R(n,2)=n$, it is easy to prove that (1.3) is true. □

Note that (1.3) and the following facts:

$$\begin{cases} N(K_2)+N(\overline{K}_2)=\frac{1}{2}p(p-1)>2p=N(K_1)+N(\overline{K}_1) & \text{if } p>R(3,3)-1=5, \\ N(K_2)+N(\overline{K}_2)\leq N(K_1)+N(\overline{K}_1) & \text{if } p\leq R(3,3)-1=5; \end{cases}$$

$$\begin{cases} N(K_3)+N(\overline{K}_3)=\frac{1}{6}p(p-1)(p-2)-\frac{1}{2}\sum_i d_i(p-1-d_i) & \\ \quad \geq \frac{1}{6}p(p-1)(p-2)-\frac{1}{8}p(p-1)^2>N(K_2)+N(\overline{K}_2) & \text{if } p>R(4,4)-1=17, \\ N(K_3)+N(\overline{K}_3)\leq N(K_2)+N(\overline{K}_2) & \text{if } p\leq R(4,4)-1=17. \end{cases}$$

where $\{d_1,d_2,\ldots,d_p\}$ is a degree sequence of G.

So, we raise a conjecture as follows:

Conjecture 2: Let n and p be natural numbers, $p>R(n,n)-1$. Then

$$N(K_{n-1})+N(\overline{K}_{n-1}) > N(K_{n-2})+N(\overline{K}_{n-2}).$$

Now, let $s=t=l$ and $p=R(m,n)-1$, by Theorem 1, then

$$2N(K_2) \leq N(K_1)[R(m-1,n)-1], \quad (2.1)$$

$$2N(\overline{K}_2) \leq N(\overline{K}_1)[R(m,n-1)-1]. \quad (2.2)$$

Thus we have $p(p-1)=2(N(K_2)+N(\overline{K}_2)) \leq p[R(m-1,n)+R(m,n-1)-2]$. i.e. $R(m,n)\leq R(m-1,n)+R(m,n-1)$.

Note that when $R(m-1,n)$, $R(m,n-1)$ are even, then $R(m,n) \leq R(m-1,n)+R(m,n-1)-1$. In fact, if $R(m,n)$ is odd, clearly it is true. If $R(m,n)$ is even, i.e. $N(K_1)=N(\overline{K}_1)$ is odd, thus, by Theorem 1, $2N(K_2) \leq N(K_1)[R(m-1,n)-1]-1$ and $2N(\overline{K}_2) \leq N(\overline{K}_1)[R(m,n-1)-1]-1$. Therefore we have $p(p-1) = 2[N(K_2)+N(\overline{K}_2)] \leq p[R(m-1,n)+R(m,n-1)-2]-2$, i.e. it is also true.

For the case $s=t=2$, we can obtain a deeper result.

Theorem 3. *Let $a+1$, $b+1$ and $c+1$ be upper bounds on $R(m-2,n)$, $R(m,n-2)$ and $R(m-1,n)$ respectively. If $p=R(m,n)-1 \geq 2c+1+\frac{1}{3}(b-a)$ and $m \leq n$, then*

$$R(m,n) \leq \frac{1}{2}(b+3c+5) + \frac{1}{2}\sqrt{(b+3c+3)^2-8-4a-4(1+c)(3c+b-a)}.$$

Proof: Let $s=t=2$, $p=R(m,n)-1$ and G be an $(m,n;p)$-graph by Theorem 1, we have that $3N(K_3)+3N(\overline{K}_3) \leq aN(K_2)+bN(\overline{K}_2)$. Note that $N(K_3)+N(\overline{K}_3) = \frac{1}{6}p(p-1)(p-2) - \frac{1}{2}\sum_i d_i(p-1-d_i)$ and $N(K_2)+N(\overline{K}_2) = \frac{1}{2}p(p-1)$, thus we have

$$p(p-1)(p-2-a) \leq \sum_i (p-1-d_i)(3d_i+b-a) \qquad (\star)$$

Let $f(d)=(p-1-d)(3d+b-a)$. There is an unique maximum value of $f(d)$ at $d_0 = \frac{1}{6}(3p-3-b+a)$. Since G is $(m,n;p)$-graph and $p \geq 2c+1+\frac{1}{3}(b-a)$, $d_i \leq c \leq d_0$. Thus $f(d_i) \leq f(c)$. We substitute $f(c)$ for $f(d_i)$ in $(\star)$. Hence $p(p-1)(p-2-a) \leq p(p-1-c)(3c+b-c)$, and then $[p-\frac{1}{2}(b+3c+3)]^2 \leq \frac{1}{4}(b+3c+3)^2-2-a-(1+c)(3c+b-a)$.

This completes the proof. □

Corollary.([1], Theorem 2.4) $R(n,n) \leq 4R(n-2,n)+2$.

Proof: Let G be an $(m,n;p)$-graph, where $p=R(n,n)-1$. Since $f(d_i) \leq f(d_0) = \frac{1}{12}(3p-3+b-a)^2$, $a=b$ and $(\star)$, $p(p-1)(p-2-a) \leq \frac{1}{12}(3p-3)^2p$. i.e. $p \leq 5+4a$. Let $a=R(n-2,n)-1$. Thus we have $R(n,n) \leq 4R(n-2,n)+2$. □

m \ n	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	43
4		18	25	41	61	84	115	149
5			49			216	316	442
6				165		495	780	1171
7					540			2826
8						1870		
9							6625	

Table 1.
Known nontrivial values and some upper bounds for $R(m,n)$

Using Theorem 3 and Table 1 in [2], we can obtain Table 2 as follows:

m \ n	5	6	7	8	9	10
5		87	143			
6			298			
7				1031	1713	
8					3583*	6090
9						12715*

* Using $R(m,n) \leq R(m-1,n) + R(m,n-1)$.

Table 2. Some new upper bounds for $R(m,n)$

References

[1] F.R.K. Chung and C.M. Grinstead, A survey of bounds for classical Ramsey numbers, *J. Graph Theory* 7 (1983), 25–37.

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