Embedding Partial Even-Cycle Systems Into Resolvable Maximum Packings

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Dedicated to Anne Penfold Street.

1 Introduction

An *m-cycle system* of *order n* is a pair (S, C), where C is a collection of edge disjoint m-cycles which partition the edge set of K_n (the complete undirected graph on n vertices) with vertex set S. In what follows we will consider even-cycle systems only; i.e., $m = 2k$. The obvious necessary conditions for the existence of a $2k$-cycle system of order n are:

(1) n is odd, and

(2) $n(n-1)/4k$ is an integer.

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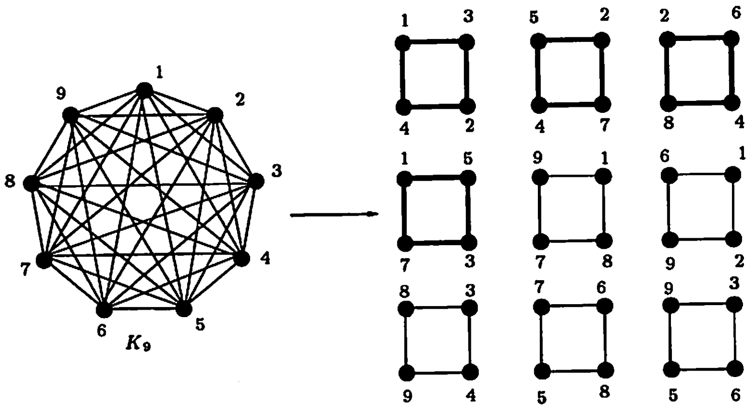
Example 1.1 **(A 4-cycle system of order 9.)**



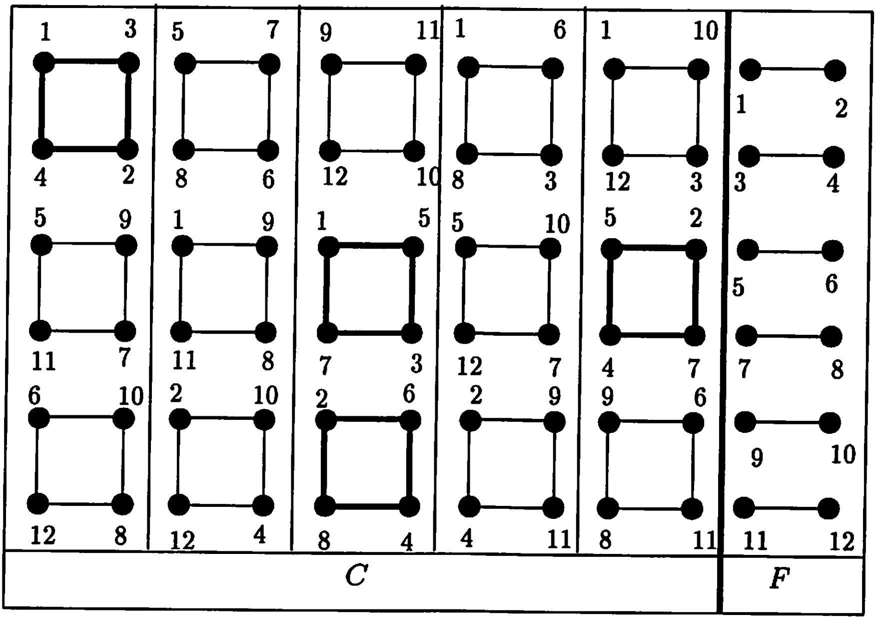
As far as the authors are aware, whether or not these necessary conditions are also sufficient remains an open problem. However, the focus of this paper is not concerned with the spectrum problem for $2k$-cycle systems, so this is of little concern here.

Now if (S, C) is a $2k$-cycle system of order n, since n is odd, C cannot contain a parallel class of cycles (i.e., a collection of vertex disjoint $2k$-cycles whose vertices partition S). The way to get around this, of course, is with maximum packings.

A *packing* of K_{2n} with $2k$-cycles is a triple (S, C, L), where C is a collection of edge disjoint $2k$-cycles which partitions $E(K_{2n})\backslash L$, where S is the vertex set of K_{2n} and L is the collection of edges *not* belonging to any of the cycles in C. The *number* $2n$ is called the *order* of the packing (S, C, L) and the collection of unused edges L is called the *leave*. If $|L|$ is as small as possible, that is, if $|C|$ is as large as possible, (S, C, L) is called a *maximum* packing. A bit of reflection shows that the leave of any packing (maximum or otherwise) of K_{2n} with $2k$-cycles must contain at least $n/2$ edges, and so the *smallest possible leave* is a 1-factor. In everything that follows, by a maximum packing of K_{2n} with $2k$-cycles we will *always* mean a maximum packing (S, C, F) where F is a 1-factor.

If (S, C, F) is a maximum packing of order $2n$ and k divides n, then it is possible for C to contain a parallel class of $2k$-cycles. If C can be partitioned into $n - 1$ parallel classes (each containing n/k $2k$-cycles) then (S, C, F) is said to be *resolvable*.

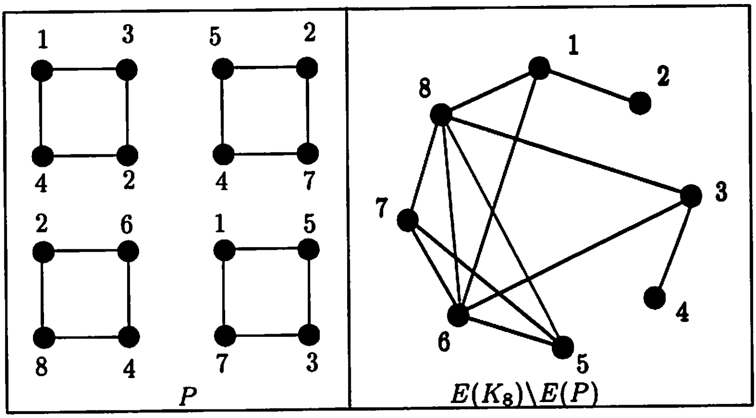
Example 1.2 (A resolvable maximum packing (S, C, F) of K_{12} with 4-cycles.)



The obvious necessary condition for the existence of a resolvable maximum packing of K_{2n} with $2k$-cycles is: n/k is an integer. If k is even, this necessary condition is sufficient. However, in the case where k is odd, the spectrum problem remains unsettled. As with the existence problem for $2k$-cycle systems, since we are not concerned with the spectrum problem, not knowing the entire spectrum of resolvable $2k$-cycle systems presents no difficulty in what follows.

A *partial* $2k$-cycle system of order n is a pair (X, P), where P is a collection of edge disjoint $2k$-cycles of the edge set of K_n with vertex set X. The difference between a partial $2k$-cycle system and a (complete) $2k$-cycle system is that in a partial system the cycles do not necessarily include all of the edges of K_n.

Example 1.3 (A partial 4-cycle system (X, P) of order 8.)



Now clearly the graph $E(K_8)\backslash E(P)$ CANNOT be decomposed into 4-cycles. Therefore we can ask whether or not it is possible to *embed* (X, P) in a 4-cycle system. That is, does there exist a 4-cycle system (S, C) such that $X \subseteq S$ and $P \subseteq C$? Inspection shows that the partial 4-cycle system of order 8 in Example 1.3 is embedded in the 4-cycle system of order 9 in Example 1.1. In general, the partial $2k$-cycle system (X, P) is *embedded* in the $2k$-cycle system (S, C) provided $X \subseteq S$ and $P \subseteq C$.

Now Example 1.3 illustrates the easily believable fact that, in general, a partial $2k$-cycle system cannot necessarily be *completed* to a $2k$-cycle system. That is to say, if (X, P) is a partial $2k$-cycle system of order n, $E(K_n)\backslash E(P)$ cannot necessarily be decomposed into edge disjoint $2k$-cycles. Hence the problem arises of finding an algorithm which will embed a partial $2k$-cycle system in a $2k$-cycle system. Additionally, we would like the containing $2k$-cycle system to be as small as possible. The best general results to date on embedding partial $2k$-cycle systems are that for fixed k and large n, a partial $2k$-cycle system of order n can be embedded in a $2k$-cycle system of order approximately kn [5]. For a history of the embedding of partial cycle systems the interested reader is referred to [4].

The partial $2k$-cycle system (X, P) is said to be embedded in the resolvable maximum packing (S, C, F) provided $X \subseteq S$ and $P \subseteq C$. Inspection shows that the partial 4-cycle system of order 8 in Example 1.3 is embedded in the resolvable maximum packing of order 12 in Example 1.2.

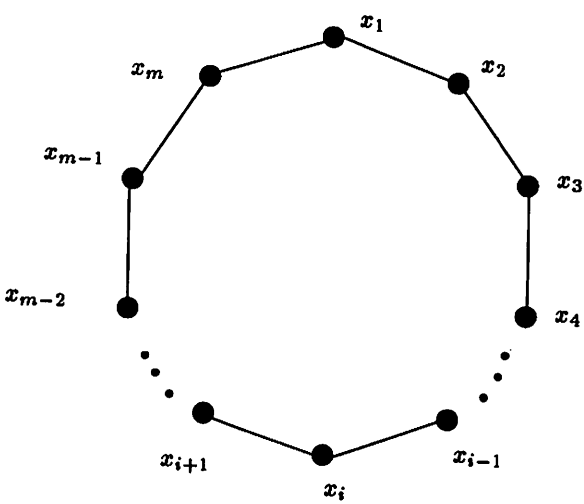
To date there is no work on embedding partial $2k$-cycle systems into resolvable maximum packings. This is not quite accurate, since an easy application of Richard Wilson's Theorem [8] shows this can be done for "sufficiently large" n. A more accurate statement is that there are no "small" embedding results. The object of this paper is to give a "small" embedding, with respect to "sufficiently large", of partial $2k$-cycle systems into resolvable maximum packings. In particular, we will give a quadratic embedding with respect to the size of the partial system.

Additionally, we will do this so that partial parallel classes are "preserved". That is, if (X, P) is a partial $2k$-cycle system and $\pi = \{\pi_1, \pi_2, \ldots, \pi_s\}$ is any partition of P into partial parallel classes, then the containing system (S, C, F) will have the property that there is a resolution $\pi = \{\pi_1^*, \pi_2^*, \ldots, \pi_t^*\}$, $t \geq s$, of C into parallel classes so that $\pi_i \subseteq \pi_i^*$, $i = 1, 2, 3, \ldots, s$.

Example 1.4 (preserving partial parallel classes.) In Example 1.3 take $\pi_1 = \{(1, 3, 2, 4)\}$, $\pi_2 = \{(1, 5, 3, 7), (2, 6, 4, 8)\}$, and $\pi_3 = \{(5, 2, 7, 4)\}$. Then (X, P) is embedded in the resolvable maximum packing of order 12 in Example 1.2 and inspection shows that the partial parallel classes π_1, π_2, and π_3 are preserved.

2 Preliminaries

We will collect together in this section the principal ingredients necessary for the constructions in Sections 3 and 4. In what follows we will denote the cycle



by any cyclic shift of $(x_1, x_2, x_3, \ldots, x_m)$ or $(x_1, x_m, x_{m-1}, \ldots, x_2)$. We will begin with an embedding result for partial idempotent commutative quasigroups due to Allan Cruse [2].

A *partial* idempotent ($x^2 = x$) quasigroup is a partial quasigroup $(P, \circ)$ with the additional requirement that $x \circ x$ *is defined* for every $x \in P$ *and*

$x \circ x = x$. In other words, the word "partial" quantifies products of the form $x \circ y$ where $x \neq y$. A *partial* idempotent commutative ($x^2 = x, xy = yx$) quasigroup is a partial idempotent quasigroup $(P, \circ)$ with the additional requirement that if $x \circ y$ is defined, then so is $y \circ x$ and furthermore $x \circ y = y \circ x$. The partial quasigroup $(P, \circ_1)$ is embedded in the quasigroup $(Q, \circ_2)$ if and only if $P \subseteq Q$ and $x \circ_1 y = x \circ_2 y$ for all x, $y \in P$ for which $x \circ_1 y$ is defined.

Theorem 2.1 (Allan Cruse [2]) *A partial $x^2 = x$, $xy = yx$ quasigroup of order n can be embedded in a $x^2 = x$, $xy = yx$ quasigroup of order t for every ODD $t \geq 2n + 1$.* □

Theorem 2.2 ([1, 3]) *If k is ODD, there exists a resolvable maximum packing of K_{2km} with $2k$-cycles for every positive integer m.* □

We will need one more result before proceeding to the constructions in Sections 3 and 4. The following well-known theorem is due to Dominique Sotteau [7].

Theorem 2.3 (D. Sotteau [7]) *The complete bipartite graph $K_{x,y}$ can be partitioned into $2k$-cycles if and only if* (i) *x and y are even;* (ii) *$x \geq k$, $y \geq k$; and* (iii) *$2k \mid xy$.* □

Corollary 2.4 *If $2k \equiv 0$ (mod 4), $K_{2k,2k}$ can be resolvably partitioned into $2k$-cycles.*

Proof: Let X be a set of size k and let $\pi(i, j)$ be a partition of $K_{k,k}$ with parts $X \times \{i\}$ and $X \times \{j\}$ into $2k$ cycles (Sotteau's Theorem). Since $|X| = k$, each $2k$-cycle is a parallel class of $K_{k,k}$. Then $K_{2k,2k}$ with parts $X \times \{1,2\}$ and $X \times \{3,4\}$ can be partitioned into parallel classes by piecing together the cycles in $\pi(1,3)$ and $\pi(2,4)$ and the cycles in $\pi(1,4)$ and $\pi(2,3)$. □

For technical reasons (Corollary 2.4 is true for $2k \equiv 0$ (mod 4) only) we will need different constructions for $2k \equiv 0$ (mod 4) and $2k \equiv 2$ (mod 4). We will handle the easiest case $2k \equiv 0$ (mod 4) in Section 3 followed by the more difficult case $2k \equiv 2$ (mod 4) in Section 4.

3 $2k \equiv 0$ (mod 4)

We will first give a construction for resolvable maximum packings of K_{2n} with $2k$-cycles, followed by an embedding algorithm.

The 0 (mod 4) resolvable maximum packing construction. Let (X, C_1, F_1) be a resolvable maximum packing of K_{2k} with $2k$-cycles and

$(Q, \circ)$ a commutative quasigroup of order $2n$ such that $x \circ x = e$ for all $x \in Q$ (unipotent). (This is equivalent to a 1-factorization of K_{2n} with vertex set Q.) Set $S = Q \times X$ and define a collection C of $2k$-cycles of K_{4kn} with vertex set S as follows:

(1) For each $a \in Q$, let $(\{a\} \times X, \{a\} \times C_1, \{a\} \times F_1)$ be a copy of (X, C_1, F_1) and place the cycles of $\{a\} \times C_1$ in C; and

(2) for each pair $a \neq b \in Q$, let $\pi(a, b)$ be a resolvable partition of the complete bipartite graph $K_{2k,2k}$, with parts $\{a\} \times X$ and $\{b\} \times X$ into $2k$-cycles (Corollary 2.4). Place the cycles of $\pi(a, b)$ in C.

Then (S, C, F), where $F = \{\{i\} \times f \mid f \in F_1, i \in Q\}$, is a maximum packing of K_{4kn} with $2k$-cycles. The following is a resolution of C into parallel classes of $2k$-cycles:

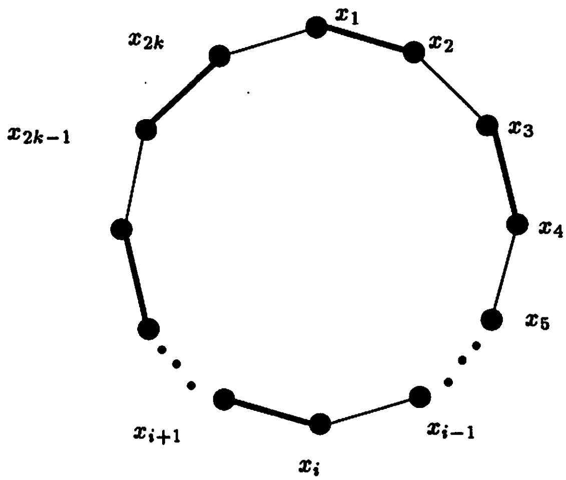
(a) The $2k$-cycles in (1) can be pieced together to form $k - 1$ parallel classes of K_{4kn}.

(b) For each $a \in Q$, $a \neq e$, the parallel classes in the $\pi(x, y)$s such that $x \circ y = y \circ x = a$ can be pieced together to form k parallel classes of K_{4kn}. Denote any such resolution by $\pi(a)$.

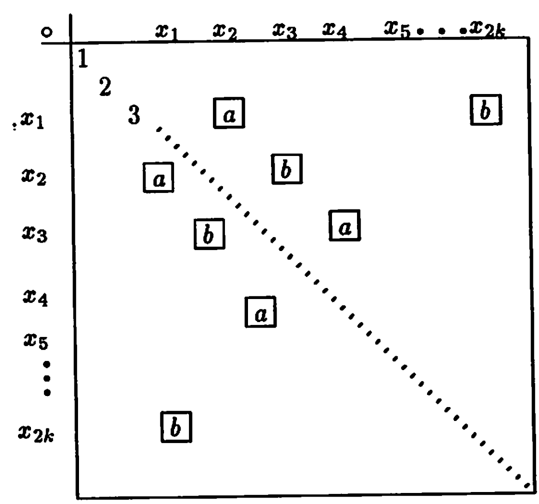
Combining the parallel classes in (a) and (b) gives a total of $2kn - 1$ parallel classes of K_{4kn} (which is exactly the correct number). □

Before plunging into an algorithm for embedding partial $2k$-cycle systems in resolvable maximum packings we point out that we will NOT keep track of the SIZE of the containing system. We will postpone this until Section 5. There is a good reason for doing this! The algorithm is tedious enough without worrying about bounds as we go along.

The 0 (mod 4) embedding algorithm. Let (Y, P) be a partial $2k$-cycle system ($2k \equiv 0 \pmod 4$). For each cycle $c = (x_1, x_2, x_3, \ldots, x_{2k-1}, x_{2k}) \in P$ let $t(c)$ be a set of size 2 such that $t(c) \cap Y = \phi$ and such that the sets $t(c)$, all $c \in P$, are pairwise disjoint. Let $Y^* = Y \cup \{t(c) \mid c \in P\}$. Define a (partial) binary operation "$\circ$" on Y^* as follows: (i) $x \circ x = x$, all $x \in Y^*$; and (ii) if $c \in P$ and $t(c) = \{a, b\}$, partition c into 2 sets of alternate edges A and B; i.e., $(x_i, x_{i+1}) \in A$ if and only if $(x_{i+1}, x_{i+2}) \in B$. If $(z, w) \in A$, define $z \circ w = w \circ z = a$ and if $(z, w) \in B$, define $z \circ w = w \circ z = b$.



$$\begin{cases} A = \{(x_1, x_2), (x_3, x_4), \cdots, (x_{2k-1}, x_{2k})\} \\ B = \{(x_2, x_3), (x_4, x_5), \cdots, (x_{2k}, x_1)\} \end{cases}$$



Then $(Y^*, \circ)$ is a partial idempotent commutative quasigroup and as such can be embedded in a (complete) idempotent commutative quasigroup $(Q^*, \circ)$ by Cruse's Theorem 2.1. Extend $(Q^*, \circ)$ to a quasigroup $(Q = Q^* \cup \{e\}, \circ)$ defined by (i) $x \circ y = y \circ x$, all $x \neq y \in Q^*$, and (ii) $x \circ x = e$ and $x \circ e = e \circ x = x$, all $x \in Q$. Then $(Q, \circ)$ is a unipotent commutative quasigroup and so is equivalent to a 1-factorization of $K_{|Q|}$ with vertex set Q.

Now use $(Q, \circ)$ in the 0 (mod 4) resolvable maximum packing construction to obtain a resolvable maximum packing (S, C, F). If $c = (x_1, x_2, x_3, \ldots, x_{2k}) \in P$ and $t(c) = \{a, b\}$, the k parallel classes in $\pi(a)$ contain the edges belonging to $c\pi(a) = \{\pi(x_i, x_{i+1}) \mid (x_i, x_{i+1}) \in c \text{ and } x_i \circ x_{i+1} = x_{i+1} \circ x_i = a\}$. A similar statement is true for $\pi(b)$. It follows that each of $\pi(a) \backslash c\pi(a)$ and $\pi(b) \backslash c\pi(b)$ partitions $S \backslash (\{x_1, x_2, \ldots, x_{2k}\} \times X)$ into k parallel classes.

Now define a collection cX of $4k^2$ $2k$-cycles with vertex set $\{x_1, x_2, \ldots, x_{2k}\} \times X$ as follows. For each $i, j \in X$ (i and j not necessarily distinct) place the cycle $((x_1, i), (x_2, j), (x_3, i), (x_4, j), (x_5, i), (x_6, j), \ldots, (x_{2k-1}, i), (x_{2k}, j))$ in cX. Then the edge set of cX is the same as the edge set of $c\pi(a) \cup c\pi(b)$. Further, cX contains $2k$ copies of the cycle c; namely $((x_1, i), (x_2, i), (x_3, i), \ldots, (x_{2k}, i))$, all $i \in X$. Additionally cX can be partitioned into $2k$ parallel classes as follows: Let $(X, \bullet)$ be an idempotent quasigroup (any idempotent quasigroup will do) and place the cycles $((x_1, i), (x_2, j), \ldots)$ and $((x_1, v), (x_2, s), \ldots)$ in the same parallel class if and only if $i \bullet j = v \bullet s$. Denote this resolution by $\pi(cX)$. Partition $\pi(cX)$ into two sets of k parallel classes $\pi_a(cX)$ and $\pi_b(cX)$. Then each of $(\pi(a) \backslash c\pi(a)) \cup \pi_a(cX)$ and $(\pi(b) \backslash c\pi(b)) \cup \pi_b(cX)$ consists of k parallel classes of K_{4kn}, each collection containing k disjoint copies of the cycle c. Finally, if $c_1 \neq c_2 \in P$, the edge sets of $\pi(c_1 X)$ and $\pi(c_2 X)$ are disjoint and so the above substitutions can be done for every $t(c) = \{a, b\}$. The result is a resolvable maximum packing (S, C^*, F) with $2k$-cycles containing $2k$ disjoint copies of the partial $2k$-cycle system (Y, P). □

Lemma 3.1 *Let $2k \equiv 0$ (mod 4). A partial $2k$-cycle system can be embedded in a resolvable maximum packing with $2k$-cycles which preserves partial parallel classes.*

Proof: Let (X, P) be a partial $2k$-cycle system and $\{\pi_1, \pi_2, \ldots, \pi_q\}$ a partition of P into partial parallel classes. Let $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_q, b_q\}$ be q disjoint 2-element sets, each disjoint from X. Modify the construction of the partial quasigroup $(Y^*, \circ)$ by taking $t(c) = \{a_i, b_i\}$ if and only if $c \in \pi_i$. Since the $2k$-cycles in each partial parallel class are disjoint, $(Y^*, \circ)$ is still a quasigroup and the 0 (mod 4) embedding algorithm places the cycles in π_i in the same parallel class of K_{4kn}. □

4 $2k \equiv 2$ (mod 4)

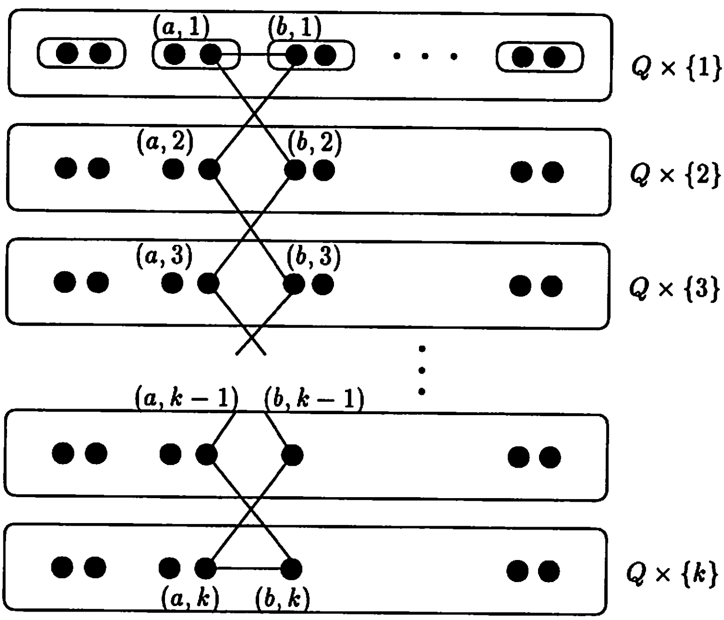
The format of this section is exactly the same as Section 3. However, the construction and embedding algorithm are slightly different, due to the fact that Corollary 2.4 handles only the case where $2k \equiv 0$ (mod 4).

The 2 (mod 4) resolvable maximum packing construction. Let (X, C_1, F_1) be a resolvable maximum packing of K_{2k} with $2k$-cycles and

$(Q, \circ)$ a commutative quasigroup of order $2n \geq 6$ with holes $H = \{h_1, h_2, \ldots, h_n\}$ of size 2. (See [6].) Let $S = Q \times K$, $K = \{1, 2, 3, \ldots, k\}$, and define a collection C of $2k$-cycles of K_{2kn} with vertex set S as follows:

(1) For each hole $h \in H$, let $(h \times K, C_1(h), F_1(h))$ be a copy of (X, C_1, F_1) and place the cycles of $C_1(h)$ in C (there are $k-1$ such $2k$-cycles).

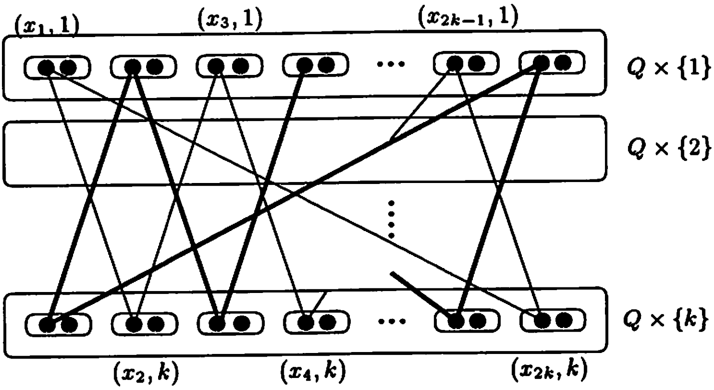
(2) Let (K, C_2) be a decomposition of K_k into $(k-1)/2$ k-cycles (remember that k is odd). We can assume the cycle $(1, 2, 3, \ldots, k) \in C_2$. For each pair a, $b \in Q$, a and b in different holes of H, place the $2k$-cycle $((\mathbf{a}, \mathbf{1}), (\mathbf{b}, \mathbf{1}), (a, 2), (b, 3), (a, 4), \ldots, (a, k-1), (\mathbf{b}, \mathbf{k}), (\mathbf{a}, \mathbf{k}), (b, k-1), (a, k-2), \ldots (b, 2))$ in C.



(3) For each cycle $(x_1, x_2, x_3, \ldots, x_k) \neq (1, 2, 3, \ldots, k)$, and each pair $a, b \in Q$, a and b in different holes of H, place the $2k$-cycle $((a, x_1), (b, x_2), (a, x_3), (b, x_4), \ldots, (a, x_k), (b, x_1), (a, x_2), (b, x_3), \ldots, (a, x_{k-1}), (b, x_k))$ in C.

(4) Let (Q, C_3, H) be a resolvable maximum packing of K_{2n} with $2k$-cycles, where H is the collection of holes of the quasigroup $(Q, \circ)$. (See Theorem 2.2.) For each $i \in \{1, 2, 3, \ldots, k\} \backslash \{1, k\}$, let $(Q \times \{i\}, C_3 \times \{i\}, H \times \{i\})$ be a copy of (Q, C_3, H) and place the cycles of $C_3 \times \{i\}$ in C.

(5) For each $2k$-cycle $(x_1, x_2, x_3, \ldots, x_{2k}) \in C_3$, place the TWO $2k$-cycles $((x_1, 1), (x_2, k), (x_3, 1), (x_4, k), (x_5, 1), (x_6, k), \ldots, (x_{2k-1}, 1)), (x_{2k}, k))$ and $((x_1, k), (x_2, 1), (x_3, k), (x_4, 1), (x_5, k), (x_6, 1), \ldots, (x_{2k-1}, k),$ $(x_{2k}, 1))$ in C.



Then (S, C, F), where $F = \{h \times \{i\} \mid h \in H, i \in K\}$, is a maximum packing of K_{2kn} with $2k$-cycles.

It is a bit trickier than the 0 (mod 4) case to see that (S, C, F) is resolvable, but not much more. The following is a resolution of C into parallel classes:

(a) Let $h = \{a, b\} \in H$ and $h(a) = \{\{x, y\} \mid x \circ y = y \circ x = a$ and x and y belong to different holes of $H\}$; the same definition for $h(b)$. Each pair $\{x, y\} \in h(a)$ gives $(k-1)/2$ parallel classes of $K_{k,k}$ with parts $\{x\} \times K$ and $\{y\} \times K$. (These are type (2) and (3) $2k$-cycles.) Denote by $h^*(a)$ the collection of all such parallel classes. Now piece together these parallel classes to form $(k-1)/2$ parallel classes $\pi h^*(a)$ of $S \backslash (h \times K)$. We can now piece together any $(k-1)/2$ parallel classes of type (1) with the $(k-1)/2$ parallel classes of $\pi h^*(a)$ to obtain $(k-1)/2$ parallel classes $\pi(a)$ of K_{2kn}. We can do this so that one of the parallel classes of $\pi(a)$ contains *all* type (2) $2k$-cycles. Using the other $(k-1)/2$ parallel classes of type (1) with $\pi h^*(b)$ gives an additional $(k-1)/2$ parallel classes $\pi(b)$ of K_{2kn}. As with $\pi(a)$, we can do this so that one of the parallel classes of $\pi(b)$ contains *all* of the type (2) $2k$-cycles. This gives a total of $k-1$ parallel classes of K_{2kn}. This uses up all $2k$-cycles of types (1), (2), and (3).

(b) Each parallel class in (4) induces a parallel class in (5). Combining these two classes partitions the remaining $2k$-cycles into parallel classes.

Combining the parallel classes in (a) and (b) partitions C into parallel classes of K_{2kn}. □

The 2 (mod 4) embedding algorithm. Let (Y, P) be a partial $2k$-cycle system ($2k \equiv 2 \pmod 4$) of order n. Define a *partial* idempotent commutative quasigroup $(Y^*, \circ)$ as in the embedding for $2k \equiv 0 \pmod 4$. Now embed $(Y^*, \circ)$ in an idempotent commutative quasigroup $(Q^*, \circ)$ of order km where m is odd (see Cruse's Theorem 2.1 [2]) and take the direct product of $(Q^*, \circ)$ with a quasigroup of order 2 to obtain a commutative quasigroup $(Q, \circ)$ of order $2km$ with holes H of size 2. Now use $(Q, \circ)$ in the resolvable maximum packing construction to obtain a resolvable maximum packing (S, C, F). If $c = (x_1, x_2, x_3, \ldots, x_{2k}) \in P$ and $t(c) = \{a, b\}$, we can construct one of the parallel classes π_1 of $\pi(a)$ to contain all of the type (2) $2k$-cycles defined by a, and we can construct one of the parallel classes π_2 of $\pi(b)$ to contain all of the type (2) $2k$-cycles defined by b. Denote by A the type (2) $2k$-cycles of π_1 defined by the edges $(x, y) \in c$ where $x \circ y = y \circ x = a$, and by B the type (2) $2k$-cycles of π_2 defined by the edges $(z, w) \in c$ where $z \circ w = w \circ z = b$. Since A and B are defined by alternate edges of the $2k$-cycle $c = (x_1, x_2, x_3, \ldots, x_{2k})$ it follows that each of $\pi_1 \backslash A$ and $\pi_2 \backslash B$ partitions $S \backslash (\{c_1, c_2, \ldots, c_{2k}\} \times K)$ into a parallel class.

We now partition the edge set of $A \cup B$ into two parallel classes A^* and B^* (each with vertex set $\{c_1, c_2, \ldots, c_{2k}\} \times K$) as follows:

(1) A^* contains the cycle $((x_1, 1), (x_2, 1), (x_3, 1), \ldots, (x_{2k}, 1))$ as well as the cycles $((x_1, i), (x_2, i+1), (x_3, i), (x_4, i+1), \ldots, (x_{2k-1}, i), (x_{2k}, i+1))$ and $((x_1, i+1), (x_2, i), (x_3, i+1), (x_4, i), \ldots, (x_{2k-1}, i+1), (x_{2k}, i))$ for $i \in \{2, 4, 6, \ldots, k-1\}$; and

(2) B^* contains the cycle $((x_1, k), (x_2, k), (x_3, k), \ldots, (x_{2k}, k))$ as well as the cycles $((x_1, j), (x_2, j+1), (x_3, j), \ldots, (x_{2k}, j+1))$ and $((x_1, j+1), (x_2, j), (x_3, j+1), \ldots, (x_{2k}, j))$ for $j \in \{1, 3, 5, 7, \ldots, k-2\}$.

Since A and A^* as well as B and B^* are mutually balanced, that is, they contain exactly the same edges, $(\pi_1 \backslash A) \cup A^*$ is a parallel class of $K_{|S|}$ containing a copy of c and $(\pi_2 \backslash B) \cup B^*$ is a parallel class of $K_{|S|}$ containing a copy of c. If $c_1 \neq c_2 \in P$, the edge sets of type (2) in the resolvable maximum packing construction are disjoint and so the above substitutions can be done for each cycle $c \in P$. The resulting collection C^* of $2k$-cycles gives a resolvable maximum packing (S, C^*, F) with $2k$-cycles containing TWO disjoint copies of the partial $2k$-cycle system (Y, P). □

Lemma 4.1 *Let $k \equiv 2 \pmod 4$. A partial $2k$-cycle system can be embedded in a resolvable maximum packing with $2k$-cycles which preserves partial parallel classes.*

Proof: The proof is identical to the proof of Lemma 3.1. □

5 The size of the embedding

We give an upper bound on the size of the containing systems in Lemmas 3.1 and 4.1.

The 2k ≡ 0 (mod 4) case. In the 0 (mod 4) embedding algorithm let (Y, P) be a partial $2k$-cycle system of order n. Then $|P| \le n(n-1)/4k$ and so $Y^* = Y \cup \{t(c) \mid c \in P\}$ has size $|Y^*| \le n + n(n-1)/2k$. The use of Cruse's Theorem gives an idempotent commutative quasigroup $(Q^*, \circ)$ of order $|Q^*| \le 2n + n(n-1)/k + 1$, and so $|Q| \le 2n + n(n-1)/k + 2$. The 0 (mod 4) resolvable maximum packing construction gives a resolvable maximum packing (S, C, F) followed by the resolvable maximum packing (S, C^*, F) of order $|S| \le 2k(2n+n(n-1)/k+2) = 2n^2+2n(2k-1)+4k$. □

The 2k ≡ 2 (mod 4) case. This is identical to the $2k \equiv 0$ (mod 4) case up through the construction of the partial idempotent quasigroup $(Y^*, \circ)$ of order $n + n(n-1)/2k$. By Cruse's Theorem we can embed $(Y^*, \circ)$ in an idempotent commutative quasigroup of EVERY odd order $\ge 2(n + n(n-1)/2k) + 1$. Let m be the *smallest* odd positive integer such that $km \ge 2n + n(n-1)/k + 1$. A simple calculation shows that $2n + n(n-1)/k + k + 1 \ge km$. Let $(Q^*, \circ)$ be an idempotent commutative quasigroup of order km containing $(Y^*, \circ)$. Then the quasigroup $(Q, \circ)$ in the 2 (mod 4) embedding algorithm has order $2km$ and so the resolvable maximum packing construction gives a resolvable maximum packing (S, C, F) followed by the resolvable maximum packing (S, C^*, F) of order $2n^2 + 2n(2k-1) + 2k(k+1)$.

Theorem 5.1 *A partial $2k$-cycle system of order n can be embedded in a resolvable maximum packing with $2k$-cycles of order (i) $2n^2+2n(2k-1)+4k$ if $k \equiv 0$ (mod 4), and order (ii) $2n^2 + 2n(2k-1) + 2k(k+1)$ if $k \equiv 2$ (mod 4). Both embeddings preserve partial parallel classes.* □

Final remarks. The authors are certain that the above bounds are not best possible. It is not clear exactly what the best possible bounds are. However, both bounds are quadratic which is certainly much better than an "existence embedding" which shows only that a finite embedding is possible.

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