The Number of 4-Cycles in 2-Factorizations of K_n

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Dedicated to Anne Penfold Street.

1 Introduction

A *2-factor* of the complete undirected graph K_n is a collection of vertex disjoint cycles which partition the vertex set of K_n. A *2-factorization* of K_n is a partition of the edge set of K_n into 2-factors. More formally, a 2-factorization of K_n is a pair (X, F), where F is a collection of edge disjoint 2-factors which partition the edge set of K_n with vertex set X. The *number* n is called the *order* of the 2-factorization (X, F).

Of course, a 2-factorization of K_n exists if and only if n is odd and in this case the number of 2-factors in $(n-1)/2$.

Now a smallest cycle in K_n is a 3-cycle and a largest cycle is a Hamilton cycle (= a cycle of length n). The most extensively studied 2-factorizations are Kirkman triple systems (= all cycles have length 3) and Hamilton decompositions (= all cycles have length n). It is well-known that Kirkman triple systems exist precisely when $n \equiv 3 \pmod 6$ [8] and Hamilton decompositions exist for all odd n [6].

Quite recently I. Dejter, F. Franek, E. Mendelsohn, and A. Rosa [3] looked at the problem of constructing 2-factorizations of K_n containing a specified number of 3-cycles. Modulo a few exceptions they give a complete solution for $n \equiv 1$ or $3 \pmod 6$, while the problem for $n \equiv 5 \pmod 6$ remains open.

The purpose of this paper is to attack the same problem for 4-cycles. We need to be more specific. Let $F = \{F_1, F_2, F_3, \ldots, F_{(n-1)/2}\}$ be a 2-factorization of K_n, and denote by x_i the number of 4-cycles belonging to F_i. A simple calculation shows that

$$\max \sum x_i \le \begin{cases} (n-1)(n-5)/8, n \equiv 1 \pmod 4, \\ (n-1)(n-3)/8, n \equiv 3 \pmod 4. \end{cases}$$

The existence of a Hamilton decomposition of K_n shows that $\min \sum x_i = 0$. For each $n \ge 5$ set

$$FC(n) = \begin{cases} \{0, 1, 2, \ldots, (n-1)(n-5)/8\}, & \text{if } n \equiv 1 \pmod 4; \text{ and} \\ \{0, 1, 2, \ldots, (n-1)(n-3)/8\}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

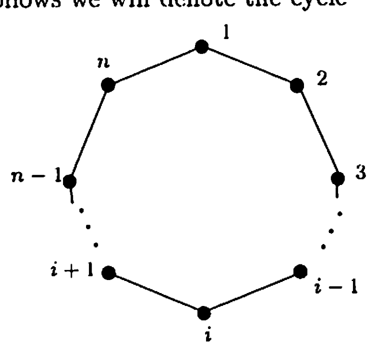
Let $Q(n) = \{x \mid \sum x_i = x$ for some 2-factorization of $K_n\}$.

We give a *complete solution* of the problem of constructing 2-factorizations of K_n with a specified number of 4-cycles by showing that:

$$\begin{cases} (1) & Q(5) = \{0\}, \\ (2) & Q(7) = \{0, 1, 3\}, \\ (3) & Q(9) = \{0, 1, 2, 3\}, \text{ and} \\ (4) & Q(n) = FC(n), \text{ all odd } n \ge 11. \end{cases}$$

We will organize our results into six sections, the first being a general recursive construction followed by a section for each of $n \equiv 1, 3, 5$, and $7 \pmod 8$, followed by a summary.

Finally, in what follows we will denote the cycle



by any cyclic shift of $(1,2,3,\ldots,n)$ or $(1,n,n-1,n-2,\ldots,3,2)$.

2 The qv+m Construction

The following construction is used repeatedly in what follows.

The qv + m Construction Let $k \geq 3$, $Q = \{1,2,3,\ldots,q\}$, $V = \{1,2,3,\ldots,2t = v\}$, and $(Q,\circ)$ a commutative quasigroup with holes H of not necessarily the same size. (See [5].) Further, let Z be a set of odd size m and set $S = Z \cup (Q \times V)$. Let h^* be any hole in H, which we will call the *initial hole* and let $(Z \cup (h^* \times V), T(h^*))$ be any 2-factorization of $K_{|h^*|v+m}$. For each hole h, $h \neq h^*$, let $(Z \cup (h \times V), T(h))$ be a 2-factorization of $K_{|h|v+m}$ containing a sub-2-factorization $(Z, Z(h))$ of order m.

Now if $g \in H$ is any hole, including the initial hole h^*, *regardless* of the size of the hole g, if we take $(|g|v)/2$ 2-factors of $K_{|g|v+m}$ we will always have $(m-1)/2$ 2-factors remaining. We will use this important fact in what follows.

We construct a collection of 2-factors T of K_{qv+m} with vertex set S as follows:

(1) For each $a \in h^*$ let $\pi(a) = \{\{x,y\} \mid x \circ y = y \circ x = a;\ x \neq y$ and $\{x,y\} \cap h^* = \phi\}$. Now $K_{v,v}$ has exactly $v/2$ 2-factors and so the bipartite graphs $K_{v,v}$ with parts $\{x\} \times V$ and $\{y\} \times V$, all $\{x,y\} \in \pi(a)$, can be pieced together to produce $v/2$ 2-factors of $E(K_{qv+m} \backslash K_{|h^*|v+m})$, where $K_{|h^*|v+m}$ has vertex set $Z \cup (h^* \times V)$. Running over all $a \in h^*$ gives a total of $(|h^*|v)/2$ such 2-factors. Now choose any $(|h^*|v)/2$ 2-factors of $T(h^*)$ and piece these together with the above $(|h^*|v)/2$ 2-factors to obtain $(|h^*|v)/2$ 2-factors of K_{qv+m}. Place these 2-factors in T.

(2) For each hole $h \in H$, $h \neq h^*$, construct $(|h|v)/2$ 2-factors of K_{qv+m} as in (1) using the 2-factors which do NOT contain cycles belonging to the sub-2-factorization $(Z, Z(h))$ of $(Z \cup (h \times V), T(h))$. Place these 2-factors of K_{qv+m} in T.

(3) Piece together the remaining $(m-1)/2$ 2-factors of $T(h^*)$ along with the remaining $(m-1)/2$ 2-factors of each $T(h)$, $h \neq h^*$, making sure to *delete* the cycles belonging to the sub-2-factorization $(Z, Z(h))$ from each of the remaining 2-factors in each $T(h)$. Place these 2-factors in T.

The union of the 2-factors in (1), (2), and (3) gives a total of $(|h^*|v)/2 + \sum_{h \in H\backslash\{h^*\}}(|h|v)/2 + (m-1)/2 = (qv+m-1)/2$ 2-factors which form a 2-factorization of K_{qm+v} with vertex set S.

Corollary 2.1 *The $qv+m$ Construction gives a 2-factorization of K_{qv+m} containing exactly $c+\sum_{h\in H\setminus\{h^*\}} S(h)+\sum|xy|$ 4-cycles, where $c \in Q(|h^*|v+m)$, $S(h) =$ the number of 4-cycles in $T(h)\setminus Z(h)$ (see (2)), and $|xy| =$ the number of 4-cycles in the 2-factorization of $K_{v,v}$ with parts $\{x\} \times V$ and $\{y\} \times V$.*

With the $qv + m$ Construction and Corollary 2.1 in hand, we proceed to the cases $n \equiv 1, 3, 5$, and 7 mod (8).

3 $n \equiv 1 \pmod 8$

This is the most tedious case to handle for the simple reason that $Q(9) = FC(9)\setminus\{4\} = \{0, 1, 2, 3\}$.

Lemma 3.1 $Q(9) = \{0, 1, 2, 3\}$.

Proof: To begin with the nonexistence of a solution to the Oberwolfach Problem $OP(9; 4, 5)$ [1] shows that $4 \notin Q(9)$.

(i) $0 \in Q(9)$: Take a Kirkman triple system of order 9. (We will use this in Section 5.)

(ii) $1 \in Q(9)$: $(1,2,3,4,5)(6,7,8,9), (1,6,2,7,3,8,5,9,4), (5,6,4,7,1,8,2,9,3), (1,3,6,8,4,2,5,7,9)$.

(iii) $2 \in Q(9)$: $(1,2,3,4,5)(6,7,8,9), (1,6,2,8,3)(4,7,5,9), (1,4,2,5,8,6,3,9,7), (8,1,9,2,7,3,5,6,4)$.

(iv) $3 \in Q(9)$: $(1,2,3)(4,5,6,7,8,9), (1,6,4,7,5)(3,8,9,3), (1,8,5,3,7)(2,6,9,4), (1,3,6,8,4)(2,5,9,7)$.

Lemma 3.2 $Q(17) = FC(17)$.

Proof: The complete graph K_{17} can be decomposed into 4 copies of the Piotrowski graph (see [2, 4, 7]) each of which is the union of two Hamilton cycles. Each Piotrowski graph can also be decomposed into two 2-factors of types (i) $4 + 13$ and $3 + 14$, or (ii) $4 + 4 + 9$ and $3 + 4 + 10$, or (iii) $4+4+4+5$ and $3+4+4+6$. In other words, each Piotrowski graph can be decomposed into two 2-factors so that the total number of 4-cycles in these two 2-factors is $0, 1, 3$, or 5. Combining the four disjoint Piotrowski graphs independently gives $\{0, 1, 2, 3, \ldots, 20\}\setminus\{17, 19\} \subseteq Q(17)$.

The following solution to the Oberwolfach Problem $OP(17; 4, 4, 4, 5)$ shows that $24 \in Q(17)$: $(0,1,8,9)(2,5,11,7)(3,13,10,15)(\infty,6,4,12,14)$ (mod 16).

Now replace the two 2-factors

$$(0,1,8,9)(2,5,11,7)(3,13,10,15)(\infty,6,4,12,14)$$

and

$$(1,2,9,10)(3,6,12,8)(4,14,11,0)(\infty,7,5,13,15)$$

in the above 2-factorization with the two 2-factors

$$(0,1,8,9)(2,5,11,7)(3,6,4,12,14,\infty,15,10,13)$$

and

$$(1,2,9,10)(4,14,11,0)(3,8,12,6,\infty,7,5,13,15).$$

This decreases the number of 4-cycles by two and so $22 \in Q(17)$.

At this point we have shown that $FC(17)\backslash\{17,19,21,23\} \subseteq Q(17)$.

Take K_{17} to have vertex set $(Z_5 \times \{1,2,3\}) \cup \{A,B\}$. Define a 2-factor F by

$$\begin{aligned} F &= ((0,1),(0,3),(0,2),(2,1),(2,2))((11,1),(2,3),(1,2),(4,3)) \\ &\quad ((4,1),(3,3),(4,2),(1,3))(A,(3,1),B,(3,2)). \end{aligned}$$

If $x \in Z_5$, denote by $F + x$ the 2-factor of K_{17} obtained from F by adding x (mod 5) to the first coordinates of the ordered pairs belonging to F. The complement C of $F \cup (F+1) \cup (F+2) \cup (F+3) \cup (F+4)$ consists of two components one of which has vertex set $(Z_5 \times \{3\}) \cup \{A,B\}$ and is isomorphic to K_7, and the other has vertex set $Z_5 \times \{1,2\}$ and is isomorphic to the graph with vertex set Z_{10} consisting of the 30 edges of lengths 1, 2 and 4. For simplicity we will describe the following two 2-factorization of this latter component of the graph C using the symbols $\{1,2,3,4,5,6,7,8,9,10\}$: (i) (1, 3, 2, 10)(4, 6, 8, 9, 7, 5), (1, 2, 6, 7)(3, 4, 8, 10, 9, 5), (1, 5, 6, 10, 4, 2, 8, 7, 3, 9) and (ii) (1, 3, 2, 10)(4, 6, 8, 9, 7, 5), (5, 6, 10, 9)(1, 2, 8, 4, 3, 7), (1, 5, 3, 9)(2, 4, 10, 8, 7, 6). Taking into account that $Q(7) = \{0,1,3\}$ (see Section 6), we can now combine the 15 4-cycles in the 2-factors $F, F+1, F+2, F+3$, and $F+4$ with $x+y$ 4-cycles, where $x \in \{2,3\}$ and $y \in \{0,1,3\}$. This gives $\{17,19,21\} \subseteq Q(17)$, leaving only the value 23 in doubt. The following example (by computer construction) takes care of 23:

$$\left\{ \begin{array}{l} (0,1,2,3)(4,5,6,7)(8,9,10,11)(12,13,14,15,16), \\ (0,2,4,6)(1,3,5,7)(8,10,12,14)(9,11,15,13,16), \\ (0,4,1,5)(2,6,3,7)(8,12,9,15)(10,13,11,14,16), \\ (0,7,8,13)(1,6,9,14)(2,5,10,14)(3,4,12,11,16), \\ (0,8,1,9)(2,10,3,11)(4,13,5,14)(6,12,15,7,16), \\ (0,10,1,12)(4,9,5,15)(6,11,7,14)(2,13,3,8,16), \\ (0,11,4,16)(1,13,6,15)(2,8,5,12)(3,9,7,10,14), \\ (1,11,5,16)(4,8,6,10)(7,12,3,15,0,14,2,9,13). \end{array} \right.$$

Combining all of the above cases shows that $Q(17) = FC(17)$.

Lemma 3.3 $Q(25) = F(25)$.

Proof: In the $qv + m$ Construction take $q = 6$, $v = 4$, and $m = 1$; and use a commutative quasigroup of order 6 with holes of size 2 (see [5]). Since $K_{4,4}$ can be 2-factored into 0 or 4 4-cycles, Corollary 2.1 gives $FC(25)\backslash\{48, 49, 60\} \subseteq Q(25)$. To handle the values 58 and 59 we again use the $qv + m$ Construction, this time with $q = 12$, $v = 2$, and $m = 1$; and a commutative quasigroup of order 12 with one hole of size 4 (the initial hole) and the remaining holes of size 2. This quasigroup is easy to construct. Take a transversal design with 3 groups of size 4 and blocks of size 3, add a common point to each group, delete any point other than the "added point", and define an idempotent commutative quasigroup on the two blocks of size 5 and each of the blocks of size 3. Finally, to handle the value 60, take $q = 12$, $v = 2$, and $m = 1$ in the $qv + m$ Construction and use a commutative quasigroup of order 12 with holes of size 2 (see [5]).

Lemma 3.4 $Q(33) = FC(33)$.

Proof: Use the $qv + m$ Construction with $q = 8$, $v = 4$, and $m = 1$; and a commutative quasigroup of order 8 with holes of size 2. In view of Corollary 2.1 $FC(33)\backslash\{109, 110, 111, 112\} \subseteq Q(33)$. The values $109, 110, 111, 112$ are handled exactly as in the case $n = 25$, first by using a commutative quasigroup of order 16 with exactly one hole of size 4 (the initial hole) and the remaining holes of size 2, and then a commutative quasigroup of order 16 with all holes of size 2. The first quasigroup can be constructed from a pairwise balanced design (PBD) of order 17 with one block of size 5 and the remaining blocks of size 3 [5] by deleting a point from the block of size 5. The second quasigroup is constructed in [5].

Lemma 3.5 $Q(41) = FC(41)$.

Proof: Take $q = 10$, $v = 4$, and $m = 1$ in the $qv + m$ Construction along with a quasigroup of order 10 with holes of size 2. This gives $FC(41)\backslash\{176, 177, 178, 179, 180\} \subseteq Q(41)$. To handle these values use the $qv + m$ Construction with $q = 20$, $v = 2$, and $m = 1$, and a commutative quasigroup of order 20 with one hole of size 6 (the initial hole) and the remaining holes of size 2. This quasigroup can be constructed by taking a Steiner triple system of order 21 with a subsystem of order 7 and deleting a point belonging to the subsystem of order 7.

Lemma 3.6 *$Q(n) = FC(n)$ for all $n \equiv 1 \pmod 8$, $n \geq 49$.*

Proof: We split the proof into two parts: $n \equiv 1 \pmod{16}$ and $n \equiv 9 \pmod{16}$.

$\mathbf{n \equiv 1}$ (mod 16). Write $n = 16k+1$. Take $q = 2k \geq 6$, $v = 8$, and $m = 1$ in the $qv+m$ Construction along with a commutative quasigroup of order $2k$ with holes of size 2. Trivially $K_{8,8}$ can be 2-factored into 0 or 16 4-cycles and so an easy counting argument shows that $FC(n) = Q(n)$ (Corollary 2.1).

$\mathbf{n \equiv 9}$ (mod 16). Write $n = 16k+9$. In order to handle this case we will need two special 2-factorizations of K_{25}: one containing a sub-2-factorization of order 9 with 56 4-cycles none of which belong to the sub-2-factorization; and one containing a sub-2-factorization of order 9 with 0 4-cycles. Both are constructed in Lemma 3.3. Using the $qv+m$ Construction with $q = 2k$, $v = 8$, $m = 1$ and the special 2-factorization of K_{25} above, Corollary 2.1 gives $FC(n) = Q(n)$.

Combining these two cases completes the proof.

4 $n \equiv 3 \pmod 8$

We begin with the cases $n = 11$ and 19 followed by a construction for all $n \equiv 3 \pmod 8$, $n \geq 27$.

Lemma 4.1 $Q(11) = FC(11)$.

Proof: We break the proof into five parts.

(i) To begin with we will denote by $G_{i,j}$ the subgraph of K_{11}, with vertex set Z_{11} consisting of all edges of length i and j. The subgraph $G_{1,2}$ of K_{11} can be decomposed into two 2-factors of type $11, 11$; or $11, 7+4$; or $11, 4+4+3$. Since $G_{3,4}$ is isomorphic to $G_{1,2}$ this gives $\{0,1,2,3,4\} \subseteq Q(11)$. (We also note that there exists a solution to the Oberwolfach Problem $OP(11;3,8)$ which is a 2-factorization of K_{11} with zero 4-cycles in which each 2-factor contains a cycle of length 3. We will use this 2-factorization in Lemma 4.3.)

(ii) There exist solutions to the Oberwolfach Problem $OP(11;4,7)$ and $OP(11;3,4,4)$ which gives $\{5,10\} \subseteq Q(11)$ [1].

(iii) Take K_{11} to have vertex set $(Z_4 \times \{1,2\}) \cup \{A,B,C\}$ and let $F = (A,(0,1),B,(1,2))$
$(C,(3,1),(2,1),(0,2),(1,1),(2,2),(3,2))$. If $x \in Z_4$, denote by $F+x$ the 2-factor of K_{11} obtained from F by adding $x \pmod 4$ to first coordinates of the ordered pairs belonging to F. Then $\{F+x \mid x \in Z_4\} \cup \{(A,B,C)((0,1),(2,1),(2,2),(0,2))((1,1),(3,1),(3,2),(1,2))$ is a 2-factorization of K_{11} containing *exactly* 6 4-cycles.

(iv) Take K_{11} to have vertex set $(Z_5 \times \{1,2\}) \cup \{\infty\}$ and let

$$F = (\infty,(0,1),(3,2))((1,1),(0,2),(2,2),(1,2))((3,1),(4,1),(3,1),(4,2)).$$

Then $\{F + x \mid x \in Z_5\}$ is a 2-factorization of K_{11} containing 10 4-cycles. The union of F and $F + 2$ can be decomposed into two 2-factors as follows:

(a) $((3,1),(2,2),(4,2),(3,2))(\infty,(0,1),(1,1),(1,2),(4,1),(2,1),(0,2))$,
and $(\infty,(3,2),(0,1),(1,2),(2,2),(0,2),(1,1),(4,1),(3,1),(4,2),(2,1)))$.
(b) $((0,1),(1,1),(4,1),(1,2))(\infty,(0,2),(2,1),(4,2),(2,2),(3,1),(3,2))$,
and $((1,1),(0,2),(2,2),(1,2))(\infty,(0,1),(3,2),(4,2),(3,1),(4,1),(2,1))$.

This reduces the number of 4-cycles by 3 and 2 respectively. Hence $\{7, 8\} \subseteq Q(11)$.

(v) Finally, the 2-factorization of K_{11} given by

$$\begin{cases} (1,2,3,4)(5,6,7,8,9,10,11), \\ (1,8,3,9)(2,6,4,11),(,7,10), \\ (1,3,5)(2,7,4,9)(6,10,8,11), \\ (2,4,10)(1,7,3,11)(5,8,6,9), \\ (1,6,3,10)(2,5,4,8)(7,9,11) \end{cases}$$

shows that $9 \in Q(11)$.

Combining all of the above cases shows that $Q(11) = FC(11)$.

Lemma 4.2 $Q(19) = FC(19)$.

Proof: In the $qv + m$ Construction take $q = 3$, $v = 6$, and $m = 1$; and use an idempotent commutative quasigroup of order 3 (= a commutative quasigroup with holes of size 1). It is an easy exercise to construct 2-factorizations of $K_{6,6}$ containing 0, 3, or 9 4-cycles. Since $Q(7) = \{0, 1, 3\}$ (see Section 6), Corollary 2.1 gives $FC(19) \backslash \{35\} \subseteq Q(19)$. The value 35 is handled by the following example (by computer):

$$\begin{cases} (0,1,2,3)(4,5,6,7)(8,9,10,11)(12,13,14,15)(16,17,18), \\ (0,2,4,6)(1,3,5,7)(8,10,12,14)(9,16,11,17)(13,15,18), \\ (0,4,1,5)(2,6,3,7)(8,12,9,13)(10,16,15,17)(11,14,18), \\ (0,7,8,15)(1,6,9,11)(2,5,10,13)(3,16,12,18)(4,14,17), \\ (0,8,1,9)(2,10,3,11)(4,12,17,13)(5,15,6,18)(7,14,16), \\ (0,10,1,12)(3,14,6,17)(4,8,5,16)(7,13,11,15)(2,9,18), \\ (0,11,4,18)(1,14,10,15)(2,8,3,12)(5,9,7,17)(6,13,16), \\ (0,13,5,14)(1,16,2,17)(3,9,4,15)(6,8,18,10)(7,11,12), \\ (0,16,8,17)(2,14,9,15)(5,11,6,12)(1,13,3,4,10,7,18). \end{cases}$$

Combining the above cases completes the proof.

Lemma 4.3 *$Q(n) = FC(n)$ for all $n \equiv 3$ (mod 8) ≥ 27.*

Proof: Write $n = 8k+3$. Take $q = 2k \geq 6$, $v = 4$, and $m = 3$ in the $qv+m$ Construction and use a commutative quasigroup of order $2k$ with holes of size 2 and a pair of 2-factorizations of K_{11} one containing a sub-2-factorization of order 3 with 10 4-cycles, and one containing a sub-2-factorization of order 3 with 0 4-cycles. Both are constructed in Lemma 4.1. Corollary 2.1 now gives $Q(n) = FC(n)$.

5 $n \equiv 5 \pmod 8$

Trivially, $Q(5) = \{0\}$. We handle the cases 13 and 21 separately, followed by a construction for all $n \equiv 5 \pmod 8$, $n \geq 29$.

Lemma 5.1 $Q(13) = FC(13)$.

Proof: The complete graph K_{13} can be written as the union of 3 Piotrowski graphs (see [2, 4, 7]) each of which is the union of two Hamilton cycles. Each Piotrowski graph can also be decomposed into two 2-factors of types (i) $4+9$ and $3+10$, or (ii) $4+4+5$ and $3+4+6$. In other words each Piotrowski graph can be decomposed into two 2-factors so that the total number of 4-cycles in these two 2-factors in $0, 1$, or 3. Combining the three Piotrowski graphs independently gives $\{0,1,2,3,4,5,6,7,9\} \subseteq Q(13)$.

The $qv+m$ Construction with $q = 3$, $v = 4$, and $m = 1$ using an idempotent commutative quasigroup of order 3 (= a commutative quasigroup with holes of size 1) gives $\{0,4,8,12\} \subseteq Q(13)$.

At this point we have $FC(13) \backslash \{10,11\} \subseteq Q(13)$.

Now take K_{13} to have vertex set $(Z_5 \times \{1,2\}) \cup \{A,B,C\}$. Then $F = (A,(0,1),B,(0,2))$
$(C,(2,1),(3,1),(4,2))((1,1),(4,1),(2,2),(1,2),(3,2))$ is a two factor of K_{13}. The complement of $F \cup (F+1) \cup (F+2) + (F+3) + (F+4) \pmod 5$ is a 2-factor of type $3+10$. Hence $10 \in Q(13)$.

Finally the following 2-factorization shows that $11 \in Q(13)$.

$$\left\{\begin{array}{l}(1,8,10)(2,11,6,4,7,12)(3,9,5,13),\\(1,13,2,10,7)(3,6,5,12)(4,8,11,9),\\(1,6,2,9,12)(3,10,4,11)(5,7,13,8),\\(1,9,10,5,11)(2,7,3,8)(4,12,6,13),\\(1,2,3,4,5)(6,7,8,9)(10,11,12,13),\\(1,3,5,2,4)(6,8,12,10)(7,9,13,11).\end{array}\right.$$

Lemma 5.2 $Q(21) = FC(21)$.

Proof: Use the $qv+m$ Construction with $q = 3$, $v = 6$, and $m = 3$, an idempotent commutative quasigroup of order 3, and a pair of 2-factorizations of K_{11} each containing a sub-2-factorization of order 3, one containing

0 4-cycles and the other 3 4-cycles. (See Lemma 3.1.) As pointed out in Lemma 4.2, there exists 2-factorizations of $K_{6,6}$ containing $0, 3$, or 9 4-cycles. Corollary 2.1 shows that $FC(21)\backslash\{37, 38, 39, 49\} \subseteq Q(21)$. The values $37, 38$, and 39 are handled with the $qv + m$ Construction with $q = 10, v = 2$, and $m = 1$ using a commutative quasigroup of order 10 with one hole of size 4 (the initial hole) and the remaining holes of size 2. To construct such a quasigroup, delete a point from the block of size 5 of a PBD of order 11 with one block of size 5 and the remaining blocks of size 3. (See [5].) The value 40 is obtained by using the $qv + m$ Construction with $q = 10, v = 2$, and $m = 1$ and commutative quasigroup of order 10 with holes of size 2.

Lemma 5.3 *$Q(n) = F(n)$ for all $n \equiv 5 \pmod 8) \geq 29$.*

Proof: Write $n = 8k + 5$. Inspection of the 2-factorizations in Lemma 5.1 gives a 2-factorization of K_{13} with a sub-2-factorization of order 5 containing 12 4-cycles and a 2-factorization of K_{13} with a sub-2-factorization of order 5 containing 0 4-cycles. In the $qv + m$ Construction take $q = 2k$, $v = 4$, and $m = 3$, and use a commutative quasigroup of order $2k$ with holes of size $2k$. Corollary 2.1 completes the proof.

6 $n \equiv 7 \pmod 8$

We begin with the cases 7 and 15 followed by a construction for all $n \equiv 7 \pmod 8$, $n \geq 31$.

Lemma 6.1 $Q(7) = \{0, 1, 3\}$.

Proof: It is an easy exercise to show that $2 \notin Q(7)$. The following example shows that $Q(7) = \{0, 1, 3\}$.

(i) $0 \in Q(7)$: Take a Hamilton decomposition of K_7.

(ii) $1 \in Q(7) : (1, 2, 3)(4, 6, 5, 7), (1, 4, 3, 6, 7, 2, 5), (1, 6, 2, 4, 5, 3, 7)$.

(iii) $3 \in Q(7) : (1, 2, 7)(3, 5, 4, 6), (1, 3, 4)(2, 5, 7, 6), (1, 5, 6)(2, 3, 7, 4)$.

Lemma 6.2 $Q(15) = FC(15)$.

Proof: The following 2-factorization of K_{15} containing 0 4-cycles contains a sub-2-factorization of order 7:

$$\left\{\begin{array}{l}(9,10,11,12,13,14,15)(1,2,3)(4,5,6,7,8)\\(9,11,13,15,10,12,14)(1,4,6)(2,7,3,8,5)\\(9,12,15,11,14,10,13)(1,5,7)(2,4,3,6,8)\\(1,8,9,2,10,6,11,3,12,5,13,4,14,7,15)\\(2,6,9,3,10,5,11,4,12,7,13,1,14,8,15)\\(3,5,9,4,10,7,11,1,12,8,13,2,14,6,15)\\(4,7,9,1,10,8,11,2,12,6,13,3,14,5,15).\end{array}\right.$$

If we replace the sub-2-factorization of order 7 in the above 2-factorization with the 2-factorization (ii) in Lemma 6.1 the result is a 2-factorization of order 15 containing exactly one 4-cycle; i.e., $1 \in Q(15)$.

We now use the $qv + m$ Construction with $q = 3$, $v = 4$, and $m = 3$ with an idempotent commutative quasigroup of order 3 (= all holes of size 1) and the 2-factorizations (ii) and (iii) of order 7 in Lemma 6.1. Corollary 2.1 gives $FC(15)\backslash\{0, 1, 20\} \subseteq Q(15)$. Note (for use in Lemma 6.3) that the 2-factorization containing 18 4-cycles is constructed by taking 4 4-cycles on each of the three copies of $K_{4,4}$, a Hamilton decomposition of K_7 on the initial hole, and the 2-factorization (iii) of K_7 on the two holes other than the initial hole. Combined with the preceding two examples this result gives $FC(15)\backslash\{20\} \in Q(15)$.

The following example shows that $20 \in Q(15)$:

$$\left\{\begin{array}{l}(1,11,8)(2,4,3,10)(6,7,9,14)(5,12,13,15)\\(1,12,9)(2,5,3,11)(7,9,4,15)(6,13,14,10)\\(1,13,4)(2,6,3,12)(8,9,5,10)(7,14,15,11)\\(1,14,5)(2,7,3,13)(9,4,6,11)(8,15,10,12)\\(1,15,6)(2,8,3,14)(4,5,7,12)(9,10,11,13)\\(1,7,10)(2,3,15,19)(4,11,12,14)(5,6,8,13)\\(1,2,15,12,6,9,3)(4,7,13,10)(5,8,14,11)\end{array}\right.$$

Combining all of the above results completes the proof.

Lemma 6.3 *$Q(n) = FC(n)$ for all $n \equiv 7$ (mod 8) ≥ 31.*

Proof: Write $n = 8k + 7$. Inspection of the 2-factorizations in Lemma 6.2 gives a 2-factorization of K_{15} with a sub-2-factorization of order 7 containing 0 4-cycles and a 2-factorization of K_{15} with a sub-2-factorization of order 7 containing 18 4-cycles none of which belong to the sub-2-factorization of order 7. Now use the $qv + m$ Construction with $q = 2k$, $v = 4$ and $m = 7$ with a commutative quasigroup of order $2k$ with holes of size 2 and the above pair of 2-factorizations of K_{15}. An easy application of Corollary 2.1 shows that $Q(n) = FC(n)$.

7 Summary

We summarize the results in this paper with the following theorem.

Theorem 7.1 $Q(5) = \{0\}$, $Q(7) = \{0,1,3\}$, $Q(9) = \{0,1,2,3\}$, *and* $Q(n) = FC(n)$ *for all odd* $n \geq 11$. □

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