Open packing in graphs

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ABSTRACT. Let G be a graph and let S be a subset of vertices of G. The open neighborhood of a vertex v of G is the set of all vertices adjacent with v in G. The set S is an open packing of G if the open neighborhoods of the vertices of S are pairwise disjoint in G. The lower open packing number of G, denoted $\rho_L^o(G)$, is the minimum cardinality of a maximal open packing of G while the (upper) open packing number of G, denoted $\rho^o(G)$, is the maximum cardinality among all open packings of G. In this paper, we present theoretical and computational results for the open packing numbers of a graph.

1 Introduction

In this paper, we follow the notation of [2]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. The open neighborhood of $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. For a set S of vertices, we define the open neighborhood $N(S) = \cup_{v \in S} N(v)$, and the closed neighborhood $N[S] = N(S) \cup S$. For each $v \in V$, we let $N_2(v)$ denote the set of all vertices at distance exactly 2 from v in G; that is, $N_2(v) = \{u \in V \mid d(u,v) = 2\}$. For a set S of vertices, we define $N_2(S) = \cup_{v \in S} N_2(v)$.

*Research supported in part by the University of Natal and the South African Foundation for Research Development.

A *packing* of a graph G is a set of vertices whose closed neighborhoods are pairwise disjoint. Equivalently, a *packing* of a graph G is a set of vertices whose elements are pairwise at distance at least 3 apart in G. The *lower packing number* of G, denoted $\rho_L(G)$, is the minimum cardinality of a maximal packing of G while the *(upper) packing number* of G, denoted $\rho(G)$, is the maximum cardinality among all packings of G. The packing number of a graph has been studied in [1, 3, 4, 5], and elsewhere.

In this paper we study the concept of open packings in graphs. Let S be a subset of vertices of G. The set S is an open packing of G if the open neighborhoods of the vertices of S are pairwise disjoint in G. The *lower open packing number* of G, denoted $\rho_L^o(G)$, is the minimum cardinality of a maximal open packing of G while the *(upper) open packing number* of G, denoted $\rho^o(G)$, is the maximum cardinality among all open packings of G.

2 The open packing number

2.1 Bounds on the open packing number

Any maximal open packing of a graph contains at least one vertex, so $\rho^o(G) \geq 1$ for all graphs G. That there exist graphs with open packing number equal to 1 is evident from the complete graph K_n on n vertices. For $n \geq 2$, let T_n be a star $K_{1,n-1}$ on n vertices. Then $\rho^o(T_n)/n = 2/n \to 0$ as $n \to \infty$.

Proposition 1 *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families of subsets of V for some graph G with each F in $\mathcal{F}_2$ containing some F' in $\mathcal{F}_1$. Then the maximum number of disjoint sets from $\mathcal{F}_1$ is greater than or equal to the maximum number of disjoint sets from $\mathcal{F}_2$. In particular, $\rho^o(G) \geq \rho(G)$.*

In what follows in this subsection, we investigate upper bounds on the open packing number. We begin with the following proposition.

Proposition 2 *Let $G = (V, E)$ be a graph of order $n \geq 2$ with degree sequence $d_1, d_2, \ldots, d_n$ where $d_1 \leq d_2 \leq \cdots \leq d_n$. Then*

$$\rho^o(G) \leq \max\{k \,|\, d_1 + \cdots + d_k \leq n\}.$$

Proof: Let $S = \{v_1, v_2, \ldots, v_k\}$ be an open packing of G. Then $N(v_i) \cap N(v_j) = \emptyset$ for $1 \leq i < j \leq k$. Thus $|\cup_{i=1}^k N(v_i)| = \sum_{i=1}^k deg\, v_i \geq d_1 + d_2 + \cdots + d_k$. On the other hand, $\cup_{i=1}^k N(v_i) \subseteq V$, so $|\cup_{i=1}^k N(v_i)| \leq n$. □

As an immediate corollary, we have the following upper bound on the open packing number of a graph in terms of its order and minimum degree.

Corollary 1 *If G is a graph of order n with minimum degree δ, then $\rho^o(G) \le n/\delta$.*

That the bound in Corollary 1 is sharp, may be seen as follows. For each positive integer δ, we construct a δ-regular graph G_δ of order n that contains an open packing of cardinality n/δ. For $\delta = 1$, take $G_\delta \cong mK_2$. Then G_1 is a 1-regular graph of order $n = 2m$ whose entire vertex set forms an open packing. Thus G contains an open packing of cardinality $n = n/\delta$. For $\delta = 2$, let G_δ be the $4m$-cycle $v_1, v_2, \dots, v_{4m}, v_1$. Then G_2 is a 2-regular graph of order $n = 4m$ with $S = \{v_i \mid i = 1 \text{ or } 2\,(mod 4)\}$ an open packing of G_2 of cardinality $n/2 = n/\delta$.

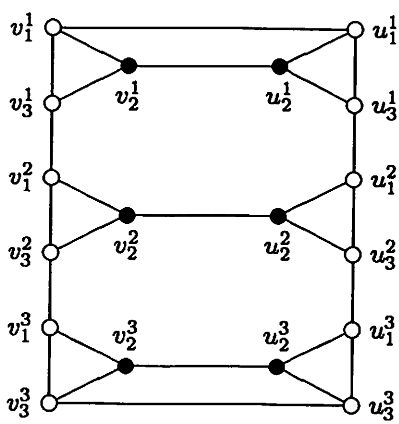

Figure 1: The graph G_3. (The darkened vertices form an open packing in G_3.)

For $\delta \ge 3$, let $F_1, F_2, \dots, F_m, H_1, H_2, \dots, H_m$ be $2m$ disjoint copies of the complete graph K_δ on δ vertices. For $i = 1, 2, \dots, m$, let $V(F_i) = \{v^i_1, v^i_2, \dots, v^i_m\}$ and $V(H_i) = \{u^i_1, u^i_2, \dots, u^i_m\}$. The graph G_δ is obtained from the disjoint union $(\cup_{i=1}^m F_i) \cup (\cup_{i=1}^m H_i)$ of the F_i's and H_i's by adding the edges $\{v^i_j u^i_j \mid 1 \le i \le m, 2 \le j \le \delta - 1\} \cup \{v^i_\delta v^{i+1}_1 \mid i = 1, 2, \dots, m - 1\} \cup \{u^i_\delta u^{i+1}_1 \mid i = 1, 2, \dots, m-1\} \cup \{v^1_1 u^1_1, v^m_\delta u^m_\delta\}$. Then G_δ is a δ-regular graph of order $n = 2m\delta$. (The graph G_3 with $m = 3$ is shown in Figure 1.) Furthermore, the set $S = \{v^i_2 \mid i = 1, 2, \dots, m\} \cup \{u^i_2 \mid i = 1, 2, \dots, m\}$ is an open packing of G_δ of cardinality $|S| = 2m = n/\delta$. Hence, by Corollary 1, G_δ is a graph of order n with minimum degree δ for which $\rho^o(G_\delta) = n/\delta$.

Next we present an upper bound on the open packing number of a connected graph.

Theorem 1 *Let $G = (V, E)$ be a connected graph of order $n \geq 3$. Then $\rho^o(G) \leq 2n/3$ and this bound is sharp.*

Proof: Let S be a maximum open packing of G. Let S_1 be set of all isolated vertices in the subgraph $\langle S \rangle$ induced by S, and let $S_2 = S - S_1$. Since each vertex of S is adjacent with at most one other vertex of S, $\langle S_2 \rangle$ consists of disjoint copies of K_2. So $|S_2|$ is even. Since G is a connected graph, we know that for any two adjacent vertices of S_2 at least one of them must have degree at least 2 and is therefore adjacent with at least one vertex in $V - S$. Every vertex of S_1 has degree at least 1 and is therefore adjacent with at least one vertex in $V - S$. However S is an open packing, so every vertex of $V - S$ is adjacent with at most one vertex of S. Thus $n - |S| = |V - S| \geq |S_1| + \frac{1}{2}|S_2| = \frac{1}{2}(|S| + |S_1|) \geq \frac{1}{2}|S|$. Hence $|S| \leq 2n/3$.

That the bound is sharp, may be seen as follows. Let H be any connected graph, and let G be the graph obtained from H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex disjoint. Then G is a connected graph of order $n = 3|V(H)|$. The set $V(G) - V(H)$ is an open packing of G of cardinality $2|V(H)| = 2n/3$. □

2.2 Paths and cycles

In this subsection, we determine the open packing number of paths and cycles.

Proposition 3 *For $n \geq 2$,*

$$\rho^o(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \,(mod 4) \\ \lfloor \frac{n+2}{2} \rfloor & \text{otherwise} \end{cases}$$

Proof: Let G be the path P_n on n vertices given by $v_1, v_2, \ldots, v_n$. Suppose firstly that $n = 4k$ for some integer $k \geq 1$. Then $V(G)$ can be partitioned into k subsets each of which induces a path P_4 on four vertices. However, any maximal open packing contains at most two vertices from any P_4, so $\rho^o(G) \leq 2k = n/2$. On the other hand, the set $S = \{v_i \mid i = 1 \text{ or } 2 \,(mod 4)\}$ is a maximal open packing of G of cardinality $n/2$, so $\rho^o(G) \geq n/2$. Thus, $\rho^o(G) = n/2$ if $n \equiv 0 \,(mod 4)$. If $n \not\equiv 0 \,(mod 4)$, then $S = \{v_i \mid i = 1 \text{ or } 2 \,(mod 4)\}$ is a maximal open packing of G of cardinality $\lfloor (n+2)/2 \rfloor$, so $\rho^o(G) \geq \lfloor (n+2)/2 \rfloor$. However, by Proposition 2, $\rho^o(G) \leq \max\{k \mid 1 + 1 + (k-2)2 \leq n\} = \max\{k \mid k \leq (n+2)/2\}$, so $\rho^o(G) \leq \lfloor (n+2)/2 \rfloor$. Hence $\rho^o(G) = \lfloor (n+2)/2 \rfloor$ if $n \not\equiv 0 \,(mod 4)$. □

Proposition 4 *For $n \geq 3$,*

$$\rho^o(C_n) = \begin{cases} \frac{n}{2} - 1 & \text{if } n \equiv 2\,(mod 4) \\ \lfloor \frac{n}{2} \rfloor & \text{otherwise} \end{cases}$$

Proof: Let G be the cycle C_n on n vertices given by $v_1, v_2, \ldots, v_n, v_1$. By Corollary 1, we know that $\rho^o(G) \leq \lfloor n/2 \rfloor$. If $n \not\equiv 2\,(mod 4)$, then $S = \{v_i \mid i = 0 \text{ or } 3\,(mod 4)\}$ is a maximal open packing of G of cardinality $\lfloor n/2 \rfloor$, so $\rho^o(G) \geq \lfloor n/2 \rfloor$. Consequently, $\rho^o(G) = \lfloor n/2 \rfloor$ if $n \not\equiv 2\,(mod 4)$. It remains for us to establish that $\rho^o(G) = n/2 - 1$ if $n \equiv 2\,(mod 4)$. Suppose, then, that $n \equiv 2\,(mod 4)$. We show that $\rho^o(G) \leq n/2 - 1$. If this is not the case, then there exist an open packing S of G of cardinality $n/2$. Thus $\cup_{v \in S} N(v) = V(G)$ and the sets $N(v)$ where $v \in S$ partition $V(G)$. In particular, each $v \in S$ is therefore adjacent with some other vertex of S. Thus, the subgraph $\langle S \rangle$ induced by S consists of disjoint copies of K_2, so $|S|$ is even. This produces a contradiction since $|S| = n/2$ and $n/2$ is odd in this case. We deduce, therefore, that for $n \equiv 2\,(mod 4)$, $\rho^o(G) \leq n/2 - 1$. On the other hand, if $n \equiv 2\,(mod 4)$, then $S = \{v_i \mid i = 0 \text{ or } 3\,(mod 4)\}$ is a maximal open packing of G of cardinality $n/2 - 1$, so $\rho^o(G) \geq n/2 - 1$. Consequently, $\rho^o(G) = n/2 - 1$ if $n \equiv 2\,(mod 4)$. □

2.3 Minimality

In this subsection, we investigate how the removal of an edge can effect the open packing number of a graph.

Theorem 2 *For any edge e in a graph G,*

$$\rho^o(G) \leq \rho^o(G - e) \leq \rho^o(G) + 2,$$

and these bounds are sharp.

Proof: Any open packing in G is also an open packing in $G - e$, so $\rho^o(G) \leq \rho^o(G - e)$. Furthermore, if $n \equiv 0\,(mod 4)$, then, by Propositions 3 and 4 we know that $\rho^o(C_n) = n/2 = \rho^o(P_n)$. Hence there exist graphs G for which $\rho^o(G) = \rho^o(G - e)$ for every edge e of G.

To show that $\rho^o(G) + 2$ is an upper bound on $\rho^o(G - e)$, let S be an open packing of $G - e$ of maximum cardinality and let $e = uv$. We show that $\rho^o(G) \geq |S| - 2$. If $u, v \in S$, then $S - \{u, v\}$ is an open packing of G, so $\rho^o(G) \geq |S| - 2$. If $u \in S$ and $v \notin S$, then $S - \{u\}$ is an open packing of G, so $\rho^o(G) \geq |S| - 1$. If $u, v \notin S$, then S is an open packing of G, so $\rho^o(G) \geq |S|$. Hence $\rho^o(G) \geq |S| - 2 = \rho^o(G - e) - 2$. Furthermore, if $n \equiv 2\,(mod 4)$, then, by Propositions 3 and 4 we know that $\rho^o(P_n) = n/2+1 = \rho^o(C_n)+2$. Hence there exist graphs G for which $\rho^o(G-e) = \rho^o(G)+2$ for every edge e of G. □

3 The lower open packing number

3.1 Lower bounds on the open packing number

In this subsection, we investigate bounds on the lower open packing number. We begin with a lower bound on the lower open packing number of a graph in terms of its order and maximum degree.

Theorem 3 *Let $G = (V, E)$ be a graph of order n with maximum degree Δ. Then*

$$\rho_L^o(G) \geq \frac{n}{\Delta(\Delta-1)+1},$$

and this bound is sharp.

Proof: Let S be a maximal open packing of G. Let S_1 be set of all isolated vertices in the subgraph $\langle S\rangle$ induced by S, and let $S_2 = S - S_1$. Since each vertex of S is adjacent with at most one other vertex of S, $\langle S_2\rangle$ consists of disjoint copies of K_2. So $|S_2|$ is even. Since S is a maximal open packing, every vertex of $V-S$ is adjacent with at most one vertex of S and is within distance 2 from some vertex of S. Furthermore, no vertex at distance 2 from a vertex of S belongs to S, i.e., $N_2(S) \subseteq V-S$. Thus $V-S = (N(S)-S) \cup N_2(S)$. For each $v \in S$, let $N_v^1 = N(v) \cap (V-S)$ and $N_v^2 = N_2(v)$. Then $(N(S)-S) \cup N_2(S) = \cup_{v\in S} (N_v^1 \cup N_v^2)$.

We show next that for each $v \in S$, $|N_v^1| + |N_v^2| \leq \Delta(\Delta-1)$. If $v \in S_1$, then $N_v^1 \subseteq N(v)$ so $|N_v^1| \leq deg\, v \leq \Delta$. However, each vertex w in N_v^1 must be adjacent to some other vertex of $N(S)-S$, for otherwise $S \cup \{w\}$ would be an open packing of G contradicting the maximality of S. Hence each vertex w of N_v^1 is adjacent with at most $deg\, w - 2$ vertices of N_v^2. Thus $|N_v^2| \leq \sum_{w\in N_v^1}(deg\, w-2) \leq |N_v^1|\cdot(\Delta-2) \leq \Delta(\Delta-2)$. Thus for $v \in S_1$, $|N_v^1| + |N_v^2| \leq \Delta(\Delta-1)$. On the other hand, if $v \in S_2$, then $|N_v^1| \leq deg\, v - 1 \leq \Delta-1$. Hence $|N_v^2| \leq \sum_{w\in N_v^1}(deg\, w-1) \leq |N_v^1|\cdot(\Delta-1) \leq (\Delta-1)^2$. Thus, for $v \in S_2$, $|N_v^1| + |N_v^2| \leq \Delta(\Delta-1)$. Thus,

$$\begin{aligned} n - |S| &= |(N(S)-S) \cup N_2(S)| \\ &= |\cup_{v\in S} (N_v^1 \cup N_v^2)| \\ &\leq \textstyle\sum_{v\in S} (|N_v^1| + |N_v^2|) \\ &\leq |S|\cdot\Delta(\Delta-1). \end{aligned}$$

Thus, $|S| \geq n/(\Delta(\Delta-1)+1)$. To show that the bound is sharp, we first recall the definition of a circulant. A *circulant* is a graph $H = C_p\langle a_1, \ldots, a_r\rangle$ such that, if the vertices are labelled $v_1, v_2, \ldots, v_p$, then $v_i v_j \in E(H)$ if and only if $|i-j| \in \{a_1, \ldots, a_r\}$ using arithmetic modular p. (We may assume that $1 \leq a_1 < \cdots < a_r \leq \lfloor p/2 \rfloor$.) We note that $C_p\langle 1\rangle \cong C_p$. We now construct a $2r$-regular graph G_r of order n with maximum degree $\Delta = 2r$ satisfying $\rho_L^o(G) = n/(\Delta(\Delta-1)+1)$.

For integers $\ell \geq 1$ and $r \geq 2$, let H be the circulant $C_{4\ell r(r-1)}\langle 1, \ldots, r-1, 2\ell r(r-1)\rangle$ with vertices labelled $v_1, v_2, \ldots, v_{4\ell r(r-1)}$. Then H is a $(2r-1)$-regular graph. We now add $2r(\ell+1)$ new vertices $\{w_j^i \mid 1 \leq i \leq \ell, 1 \leq j \leq 2r\} \cup \{u_i \mid 1 \leq i \leq \ell\}$ together with the edges $\{u_i w_j^i \mid 1 \leq i \leq \ell, 1 \leq j \leq 2r\} \cup \{w_{2k+1}^i w_{2k+2}^i \mid 1 \leq i \leq \ell, 0 \leq k \leq r-1\} \cup \{w_j^i v_{2(r-1)(j-1)+4r(r-1)(i-1)+k} \mid 1 \leq i \leq \ell, 1 \leq j \leq 2r, 1 \leq k \leq 2(r-1)\}$. Let G_r denote the resulting graph. Then G_r is a $2r$-regular graph of order $n = \ell(4r^2 - 2r + 1)$ with maximum degree $\Delta = 2r$, and $S = \{u_1, \ldots, u_\ell\}$ is a maximal open packing of G_r of cardinality $\ell = n/(\Delta(\Delta-1)+1)$. □

Next we present an upper bound on the lower open packing number of a connected graph. Since $\rho_L^o(G) \leq \rho^o(G)$ for all graphs G, an immediate corollary of Theorem 1 now follows.

Corollary 2 *Let G be a connected graph of order $n \geq 3$. Then $\rho_L^o(G) \leq 2n/3$.*

That the upper bound in Corollary 2 is in a sense best possible, may be seen as follows. For $m \geq 2$ an integer, let T be the tree obtained from a star $K_{1,m}$ by subdividing each edge twice. Let $T_1, T_2, \ldots, T_m$ be m disjoint copies of T, and let v_i denote the central vertex of T_i for $i = 1, 2, \ldots, m$. Finally, let G_m be the tree obtained from the disjoint union $\cup_{i=1}^m T_i$ of $T_1, T_2, \ldots, T_m$ by adding a new vertex v and the edges vv_i for $i = 1, 2, \ldots, m$. The graph G_m is shown in Figure 2. We show that $\rho_L^o(G_m) = 2m^2 - m + 1$. Let S be a maximal open packing of G_m. Then S contains at most one of $v_1, v_2, \ldots, v_m$. If $v_i \notin S$, then $|S \cap V(T_i)| = 2m$, while if $v_i \in S$, then $|S \cap V(T_i)| = m + 1$. Thus, $\rho_L^o(G_m) \geq 2m^2 - m + 1$. However, there exist maximal open packing sets of G_m of cardinality $2m^2 - m + 1$ as illustrated by the set of darkened vertices in Figure 2, so $\rho_L^o(G_m) \leq 2m^2 - m + 1$. Thus, $\rho_L^o(G_m) = 2m^2 - m + 1$. Hence G_m is a tree of order $n = 3m^2 + m + 1$ satisfying

$$\frac{\rho_L^o(G_m)}{n} = \frac{2m^2 - m + 1}{3m^2 + m + 1} = \frac{2 - 1/m + 1/m^2}{3 + 1/m + 1/m^2}.$$

Hence, $\rho_L^o(G_m)/n \to 2/3$ as $m \to \infty$.

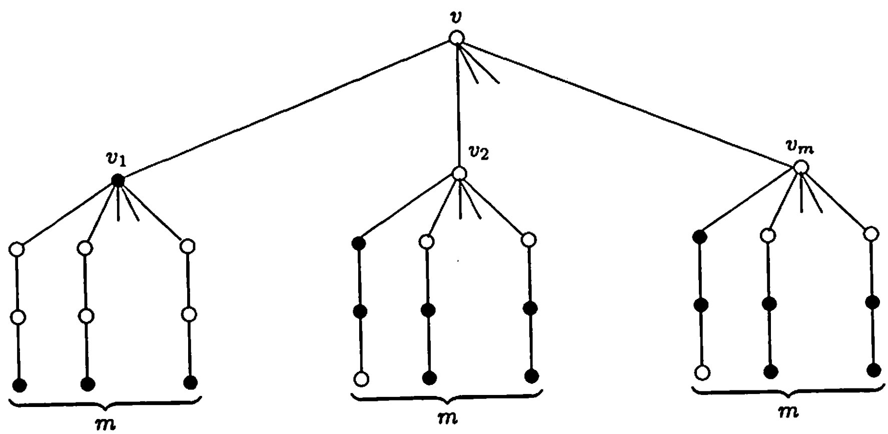

Figure 2: The graph G_m. (The darkened vertices form a maximal open packing in G_m.)

3.2 Paths and cycles

In this subsection, we determine the lower open packing number of paths and cycles.

Proposition 5 *For $n \geq 2$,*

$$\rho_L^o(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 2 \text{ or } 3 \pmod 6 \\ \lceil \frac{n}{3} \rceil & \text{otherwise} \end{cases}$$

Proof: If $n = 2$ or 3, then $\rho_L^o(P_n) = 2 = \lceil n/3 \rceil + 1$, so we may assume that $n \geq 4$. Let G be the path P_n on $n \geq 4$ vertices given by $v_1, v_2, \ldots, v_n$. By Theorem 3, we know that $\rho^o(G) \geq \lceil n/3 \rceil$. If $n \equiv 1$ or $4 \pmod 6$, then $S = \{v_i \mid i = 1 \pmod 3\}$ is a maximal open packing of G of cardinality $\lceil n/3 \rceil$. If $n \equiv 0 \pmod 6$, then $S = \{v_i \mid i = 2 \pmod 3\}$ is a maximal open packing of G of cardinality $n/3$. If $n \equiv 5 \pmod 6$, then $S = \{v_i \mid i = 3 \text{ or } 4 \pmod 6\}$ is a maximal open packing of G of cardinality $\lceil n/3 \rceil$. Hence if $n \not\equiv 2$ or $3 \pmod 6$, then $\rho^o(G) \leq \lceil n/3 \rceil$. Thus, $\rho^o(G) = \lceil n/3 \rceil$ if $n \not\equiv 2$ or $3 \pmod 6$.

Suppose that $n = 6k+2$ for some integer $k \geq 1$. Let $V_1 = \{v_1, v_2, v_3, v_4\}$, $V_{k+1} = \{v_{6k-1}, \ldots, v_{6k+2}\}$, and $V_i = \{v_{6i-7}, \ldots, v_{6i-2}\}$ for $2 \leq i \leq k$. Thus $V(G)$ can be partitioned into $k+1$ subsets $V_1, V_2, \ldots, V_{k+1}$ such that each of V_1 and V_{k+1} induce a path P_4 on four vertices and V_i induces a path P_6 on six vertices for $i = 2, \ldots, k$. Any maximal open packing of G

contains at least two vertices from each of the sets V_i $(1 \le i \le k+1)$, so $\rho^o(G) \ge 2(k+1)$. On the other hand, the set $S = \{v_i \,|\, i = 3$ or $4 \,(mod\, 6)\} \cup \{v_{n-1}, v_n\}$ is a maximal open packing of G of cardinality $2(k+1)$, so $\rho^o(G) \le 2(k+1)$. Thus, $\rho^o(G) = 2(k+1) = \lceil n/3 \rceil + 1$. Hence $\rho^o(G) = \lceil n/3 \rceil + 1$ if $n \equiv 2 \,(mod\, 6)$.

If $n = 6k+3$ for some integer $k \ge 1$, then $V(G)$ can be partitioned into $k+1$ subsets $V_1, V_2, \ldots, V_{k+1}$ such that V_1 induces a P_4 and $v_1 \in V_1$, V_{k+1} induces a P_5 and $v_n \in V_{k+1}$, and V_i induces a P_6 for $i = 2, \ldots, k$. Any maximal open packing of G contains at least two vertices from each of the sets V_i $(1 \le i \le k+1)$, so $\rho^o(G) \ge 2(k+1)$. On the other hand, the set $S = \{v_i \,|\, i = 1 \,(mod\, 3)\} \cup \{v_{n-1}\}$ is a maximal open packing of G of cardinality $2(k+1)$, so $\rho^o(G) \le 2(k+1)$. Thus, $\rho^o(G) = 2(k+1) = \lceil n/3 \rceil + 1$. Hence $\rho^o(G) = \lceil n/3 \rceil + 1$ if $n \equiv 3 \,(mod\, 6)$. □

Proposition 6 *For $n \ge 3$,*

$$\rho_L^o(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 2 \,(mod\, 6) \\ \lceil \frac{n}{3} \rceil & \text{otherwise} \end{cases}$$

Proof: Let G be the cycle C_n on n vertices given by $v_1, v_2, \ldots, v_n, v_1$. By Theorem 3, we know that $\rho^o(G) \ge \lceil n/3 \rceil$. If $n \equiv 0 \,(mod\, 3)$, then $S = \{v_i \,|\, i = 2 \,(mod\, 3)\}$ is a maximal open packing of G of cardinality $n/3$. If $n \equiv 1 \,(mod\, 3)$, then $S = \{v_i \,|\, i = 1 \,(mod\, 3)\}$ is a maximal open packing of G of cardinality $\lceil n/3 \rceil$. If $n \equiv 5 \,(mod\, 6)$, then $S = \{v_i \,|\, i = 3$ or $4 \,(mod\, 6)\}$ is a maximal open packing of G of cardinality $\lceil n/3 \rceil$. Hence if $n \not\equiv 2 \,(mod\, 6)$, then $\rho^o(G) \le \lceil n/3 \rceil$. Thus, $\rho^o(G) = \lceil n/3 \rceil$ if $n \not\equiv 2 \,(mod\, 6)$. It remains for us to establish that $\rho^o(G) = \lceil n/3 \rceil + 1$ if $n \equiv 2 \,(mod\, 6)$.

Suppose that $n = 6k+2$ for some integer $k \ge 1$. By Theorem 3, we know that $\rho^o(G) \ge \lceil n/3 \rceil = 2k+1$. Thus any maximal open packing of G (which can not contain two vertices at distance 2) must contain at least two adjacent vertices. Let S be a maximal open packing of G of minimum cardinality. We may assume that $\{v_1, v_2\} \subset S$ (and so $v_3 \notin S$ and $v_4 \notin S$). We now partition $V(G)$ into k subsets $V_1, V_2, \ldots, V_k$ where V_1 induces the path $v_1, v_2, \ldots, v_8$ on eight vertices and V_i induces a path P_6 on six vertices for $i = 2, \ldots, k$. Then S must contain one of v_5 and v_7 and one of v_6 and v_8, so S contains at least four vertices of V_1. Moreover, S contains at least two vertices from each of the sets V_i $(2 \le i \le k)$. Thus, $\rho^o(G) = |S| \ge 2(k+1)$. On the other hand, the set $S = \{v_i \,|\, i = 3$ or $4 \,(mod\, 6)\} \cup \{v_{n-1}, v_n\}$ is a maximal open packing of G of cardinality $2(k+1)$, so $\rho^o(G) \le 2(k+1)$. Thus, $\rho^o(G) = 2(k+1)$. Hence $\rho^o(G) = \lceil n/3 \rceil + 1$ if $n \equiv 2 \,(mod\, 6)$. □

3.3 Minimality

In this subsection, we investigate how the removal of an edge can effect the lower open packing number of a graph. We show firstly that deleting an edge from a graph can increase its lower open packing number by at most 2.

Theorem 4 *For any edge e in a graph G,*

$$\rho_L^o(G-e) \leq \rho_L^o(G)+2,$$

and this bound is sharp.

Proof: Let S be a maximal open packing of G of minimum cardinality and let $e = uv$. We show that $\rho_L^o(G-e) \leq |S|+2$. If $u, v \in S$, then the only possible vertices in $G-e$ that can be added to S to produce a maximal open packing are those vertices of $G-e$ in $N(u) \cup N(v)$. However, at most one vertex from any open neighborhood belongs to an open packing of $G-e$. Thus if S is not a maximal open packing of $G-e$, then it can be extended to a maximal open packing of $G-e$ by adding at most two vertices. If $u \in S$ and $v \notin S$, then the only possible vertices in $G-e$ that can be added to S to produce a maximal open packing are those vertices of $G-e$ in $N[v]$. However, at most two vertices from any closed neighborhood belongs to an open packing of $G-e$. Once again, S can be extended, if necessary, to a maximal open packing of $G-e$ by adding at most two vertices. Finally, if $u, v \notin S$, then S is also a maximal open packing of $G-e$. In all three cases we have $\rho_L^o(G-e) \leq |S|+2 = \rho_L^o(G)+2$. That the bound is sharp, may be seen by considering the tree G obtained from the disjoint union of two stars by joining the two vertices of maximum degree with an edge e. Then $\rho_L^o(G-e) = 4 = \rho_L^o(G)+2$. □

Although deleting an edge can increase the lower open packing number of a graph by at most 2, it can decrease its lower open packing number by an arbitrarily large amount.

Theorem 5 *For every positive integer n, there exists a graph G and an edge e of G satisfying $\rho_L^o(G-e) = \rho_L^o(G) - n$.*

Proof: Let T be the tree obtained from a star $K_{1,n}$ by subdividing each edge exactly once. Let T_1 and T_2 be two disjoint copies of T, and let v_1 and v_2 be vertices of maximum degree n in T_1 and T_2, respectively. Finally, let G be the graph obtained from $T_1 \cup T_2$ by adding a new vertex v and the edges $e = vv_1$, vv_2 and v_1v_2. Then $\{v_1, v_2\}$ is a maximal open packing of $G-e$, so $\rho_L^o(G-e) = 2$. We show next that $\rho_L^o(G) = n+2$. Let S be any maximal open packing of G. Then S contains at most one of v, v_1 and v_2. If $v \in S$, then S consists of v and the $2n$ leaves so $|S| = 2n+1$.

If $v_1 \in S$, then S contains the n leaves of T_2 and one vertex of T_1 that is adjacent with v_1, so $|S| = n+2$. Similarly, if $v_2 \in S$, then $|S| = n+2$. Finally, if S contains none of the vertices v, v_1 or v_2, then S contains all $2n$ leaves, one vertex of T_1 that is adjacent with v_1, and one vertex of T_2 that is adjacent with v_2, so $|S| = 2n+2$. Thus, $\rho_L^o(G) = n+2$. Hence $\rho_L^o(G-e) = \rho_L^o(G) - n$. □

4 Bounds relating ρ^o and ρ_L^o

In this section, we present an upper bound on the difference between the open packing number and the lower open packing number of a tree. If T is a rooted tree with root r and v is a vertex of T, then the *level number* of v, which we denote by $l(v)$, is the length of the unique r-v path in T. If a vertex v of T is adjacent to u and $l(u) > l(v)$, then u is called a *child* of v, and v is the *parent* of u. A vertex w is a *descendant* of v if the level numbers of the vertices on the v-w path are monotonically increasing. The subtree of T induced by v and all its descendants is called the *maximal subtree* of T rooted at v. We will refer to an end-vertex of T as a *leaf*.

Theorem 6 *If T is a tree of order $n \geq 2$, then*

$$\rho^o(T) - \rho_L^o(T) \leq \frac{n-2}{2},$$

and this bound is sharp.

Proof: We proceed by induction on the order $n \geq 2$ of a tree. If T is a tree of order $n \leq 5$ that is not a path on five vertices, then $\rho^o(T) = \rho_L^o(T) = 2$ and the result is immediate. If T is a path on $n = 5$ vertices, then $\rho^o(T) = 3 = \rho_L^o(T) + 1$, so $\rho^o(T) - \rho_L^o(T) = 1 \leq (n-2)/2$. Hence the result is true for all trees of order $n \leq 5$. So, assume that for all trees T' of order $n' \geq 2$ where $n' < n$ and $n \geq 6$, that $\rho^o(T') - \rho_L^o(T') \leq (n'-2)/2$. Let T be a rooted tree of order n. We show that $\rho^o(T) - \rho_L^o(T) \leq (n-2)/2$. Let w be a leaf of T at furthest distance from the root (so w is a vertex of T with maximum level number), and let v be the parent of w.

If T contains a vertex adjacent with at least two leaves, then removing one of these leaves produces a tree T' of order $n' = n-1$ satisfying $\rho^o(T') = \rho^o(T)$ and $\rho_L^o(T') = \rho_L^o(T)$. Thus, applying the inductive hypothesis, we have $\rho^o(T) - \rho_L^o(T) = \rho^o(T') - \rho_L^o(T') \leq (n'-2)/2 < (n-2)/2$. Hence we may assume that every vertex of T is adjacent with at most one leaf. In particular, v has degree 2. Let u be the parent of v in T.

If u has degree 2, then let x be the parent of u and consider the nontrivial tree $T' = T - \{u, v, w\}$ of order $n' = n - 3$. Since every maximal open packing of T contains two of the vertices u, v, w, x, we may assume without loss of generality, that there is a maximum open packing S of T containing v and w. Hence $S - \{v, w\}$ is a maximal open packing of T', so $\rho^o(T') \geq \rho^o(T) - 2$; equivalently, $\rho^o(T) \leq \rho^o(T') + 2$. On the other hand, every maximal open packing of T contains at least one of the vertices u, v, w, so $\rho^o_L(T) \geq \rho^o_L(T') + 1$. Thus, applying the inductive hypothesis, we have $\rho^o(T) - \rho^o_L(T) \leq \rho^o(T') - \rho^o_L(T') + 1 \leq (n' - 2)/2 + 1 = n'/2 < (n-2)/2$. Hence we may assume that u has degree $k + 1 \geq 3$.

If u is adjacent with a leaf x, then the tree $T' = T - x$ of order $n' = n - 1$ satisfies $\rho^o(T') = \rho^o(T)$ and $\rho^o_L(T') = \rho^o_L(T)$. Thus, applying the inductive hypothesis, we have $\rho^o(T) - \rho^o_L(T) = \rho^o(T') - \rho^o_L(T') \leq (n' - 2)/2 < (n - 2)/2$. Hence we may assume that every child of u has degree 2. Thus the maximal subtree of T rooted at u is isomorphic to $K_{1,k}$ with each edge subdivided once. Let $v_1, \ldots, v_k$ be the children of u, and let w_i be the leaf adjacent with v_i, $1 \leq i \leq k$.

We now consider the nontrivial tree $T' = T - \{v_1, w_1\}$ of order $n' = n - 2$. Every maximal open packing of T contains at most one child of u, so we may assume without loss of generality, that there is a maximum open packing S of T that does not contain v_1. If $u \in S$, then S contains none of the leaves $w_1, w_2, \ldots, w_k$. But then $(S - \{u\}) \cup \{w_1, w_2, \ldots, w_k\}$ would be an open packing of T of cardinality exceeding that of S, producing a contradiction. Thus $u \notin S$. Consequently, $\{w_1, w_2, \ldots, w_k\} \subset S$. Hence $S - \{w_1\}$ is a maximal open packing of T', so $\rho^o(T') \geq \rho^o(T) - 1$; equivalently, $\rho^o(T) \leq \rho^o(T') + 1$. On the other hand, let S be a maximal open packing of T of minimum cardinality. Once again, we may assume that S does not contain v_1. Then $S \cap V(T')$ is a maximal open packing of T' of cardinality at most $|S|$. Thus, $\rho^o_L(T) \geq \rho^o_L(T')$. Therefore, applying the inductive hypothesis, we have $\rho^o(T) - \rho^o_L(T) \leq \rho^o(T') - \rho^o_L(T') + 1 \leq (n' - 2)/2 + 1 = n'/2 = (n - 2)/2$. This completes the inductive proof.

That the bound is sharp as may be seen as follows. For an integer $k \geq 2$, let T_1 and T_2 be two disjoint copies of a star $K_{1,k}$ with each edge subdivided exactly once. For $i = 1, 2$, let v_i denote the central vertex of T_i. Finally, let T be the tree obtained from $T_1 \cup T_2$ by adding the edge v_1v_2. Then T is a tree of order $n = 4k + 2$. Furthermore, the set $\{v_1, v_2\}$ is a maximal open packing in T, so $\rho^o_L(T) = 2$. On the other hand, the set containing the $2k$ leaves of T, one vertex of T_1 that is adjacent with v_1 and one vertex of T_2 that is adjacent with v_2 is a maximum open packing of T, so $\rho^o(T) = 2k+2$. Thus, $\rho^o(T) - \rho^o_L(T) = 2k = (n - 2)/2$. □

5 Complexity results

In this section we show that the decision problem

OPEN PACKING (OPK)

INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

QUESTION: Does G have an open packing of cardinality k?

is NP-complete, even when restricted to bipartite and chordal graphs, by describing polynomial transformations from the following well-known NP-complete problem:

EXACT COVER BY 3-SETS (X3C)

INSTANCE: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.

QUESTION: Does C contain an exact cover for X, that is, a subcollection $C' \subseteq C$ such that every element of X occurs in exactly one member of C'.

Theorem 7 OPEN PACKING *is NP-complete, even for bipartite graphs.*

Proof: It is obvious that **OPK** is a member of NP since we can, in polynomial time, guess at set S of vertices and verify that S has cardinality at least m and is an open packing. We next show how a polynomial time algorithm for **X3C** could be used to solve **OPK** in polynomial time. Let $X = \{x_1, \ldots, x_{3q}\}$ and $C = \{C_1, \ldots, C_m\}$ be an arbitrary instance of **X3C**. We will construct a bipartite graph G such that this instance of **X3C** will have an exact three cover if and only if G has an open packing of cardinality k, where $k = m + 7q$.

The graph G is constructed as follows. Corresponding to each variable $x_i \in X$, we associate the graph H_i which consists of the path x_i, y_i, w_i, z_i on four vertices. Corresponding to each set C_j, we associate the graph F_j which consists of the path c_j, d_j on two vertices. The construction of the bipartite graph G is completed by adding the edges $\{x_i c_j \mid x_i \in C_j\}$. It is easy to see that the construction of the graph G can be accomplished in polynomial time. Let $W = \{w_1, w_2, \ldots, w_{3q}\}$, $Z = \{z_1, z_2, \ldots, z_{3q}\}$, $C = \{c_1, \ldots, c_m\}$ and $D = \{d_1, \ldots, d_m\}$. We show that C has an exact 3-cover if and only if G has an open packing of cardinality $k = m + 7q$.

Suppose C' is an exact 3-cover for X. Then $|C'| = q$. Let $S = \{c_j \mid C_j \in C'\} \cup D \cup W \cup Z$. Then S is an open packing of cardinality $k = m + 7q$. Suppose, conversely, that S is an open packing of cardinality $k = m + 7q$. Let $S' = S \cap C$. Since each vertex of S' is adjacent with three vertices of X, and since no two vertices of S have a common neighbor, there are $3|S'|$ vertices of X that are adjacent with vertices of S'. However there are precisely $3q$ vertices of X, so $|S'| \leq q$. Furthermore, at most two vertices of H_i are in the open packing S for every $i = 1, 2, \ldots, 3q$, so S contains at least $m + q$ vertices from $C \cup D$. Thus $D \subset S$ and $|S'| = q$. Consequently, $C' = \{C_j \mid c_j \in S\}$ is an exact 3-cover for X. □

Theorem 8 OPEN PACKING *is NP-complete, even for chordal graphs.*

Proof: It is clear that **OPK** is in NP. To show that **OPK** is an NP-complete problem, we will establish a polynomial transformation from **X3C**. Let $X = \{x_1, \ldots, x_{3q}\}$ and $C = \{C_1, \ldots, C_m\}$ be an arbitrary instance of **X3C**. We will construct a chordal graph H such that this instance of **X3C** will have an exact three cover if and only if H has an open packing of cardinality $k = m + 7q$.

Let H be obtained from the graph G constructed in the proof of Theorem 7 by adding an edge between every two vertices of X so that the x_i's induce a clique; that is, $\langle\{x_1, \ldots, x_{3q}\}\rangle \cong K_{3q}$. It is easy to see that the construction of the graph H can be accomplished in polynomial time. Proceeding now as in the proof of Theorem 7, we can show that C has an exact 3-cover if and only if H has an open packing of cardinality $k = m + 7q$. □

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