PATH-PAIRABLE GRAPHS

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ABSTRACT

A graph of even order is called path-pairable, if for any pairing of its vertices, there exist edge disjoint paths connecting the paired vertices. Extremal problems for path-pairable graphs with restrictions on the maximum degree will be considered. In particular, let $f(n,k)$ denote the minimum number of edges in a path-pairable graph of order n and maximum degree k. Exact values of $f(n,k)$ are determined for $k = n-1$, $n-2$ and $n-3$.

I. INTRODUCTION

This paper is the last one in a series of papers devoted to graph theoretic concepts emerging from a practical networking problem of L. Csaba. The initial problem and its graph theoretical model is discussed in [1]. A notion related to that model was the concept of *k-path-pairable graphs*, where any k pairs of *distinct* vertices of the

[1]Research is partially supported by NAS Exchange grant, ONR Grant No. N00014-91-J-1085 and NSA Grant No. MDA 904-90-H-4034

[2]Research supported by OTKA grant No. 2570 of the Hungarian Academy of Sciences

graph can be connected by k edge disjoint paths. (This notion is similar to the concept of weakly k-linked graphs introduced in [8] and also considered in [7], where the distinctness of the pairs is not required in the definition.) It was shown in [3] that there exists k-pairable 3-regular graphs for each positive integer k (which shows the drastic difference between k-path pairable and weakly k-linked, because a graph must be k-connected to be weakly k-linked). The cases $k = 2$ and $k = 3$ were treated in [4]. New types of graph factorization problems are also related to these network models, and these are discussed in [1] and [5].

In this paper we focus our attention on the concept of path-pairable graphs. A graph of even order is called *path-pairable*, (p.p.) if for any pairing of its vertices, there exist edge disjoint paths connecting the paired vertices. Notice that it is rather complicated to verify the p.p. property even for small graphs, such as the Cube and the Petersen graph, both of which are path-pairable. The latter takes about a three page case analysis (not given in this paper). The p.p. graphs are simplified models of communication networks which can simultaneously establish links among any configuration of pairs of its nodes. We are interested in extremal problems for p.p. graphs of restricted maximum degree, which is a usual assumption when dealing with a communications network.

It is obvious that with unrestricted maximum degree, the star has the least number of edges among the p.p. graphs of a given order. The first part of the paper is devoted to the problem of finding the minimum number of edges among p.p. graphs of order n and satisfying one of the assumptions below:

(i) maximum degree at most $n - 2$,

(ii) maximum degree $n - 2$ or $n - 3$, and

(iii) maximum degree $n - 4$.

For the two problems in (ii) we have exact answers, but for (i) and (iii) we can give only solutions that are asymptotical correct. (In all of the cases, the extremal numbers exceed $3n/2 - c \cdot \log_2 n$.)

The second part presents further problems and results about p.p. graphs.

2. VERY LARGE MAXIMUM DEGREE

A useful tool in studying path-pairable graphs is the cut condition property. A graph G satisfies the *cut condition*, if for any partition of the vertices of G into sets A and B with $|A| \leq |B|$, the number of edges in the edge cut between A and B is at least $|A|$. If G is a p.p. graph, then G must satisfy the cut condition, since if each of the vertices of A is paired with a vertex of B, then at least $|A|$ edge disjoint paths must cross the cut between A and B. The fact that a p.p. graph satisfies the cut condition will be used frequently in the next result, which gives a lower bound on the number of edges in a p.p. graph of order n that has no vertex of degree $n-1$.

Theorem 1. *If G is path-pairable graph of order n and $\Delta(G) \leq n-2$, then $|E(G)| \geq 3n/2 - log_2\, n - c$ for some constant c.*

Proof. The proof will be by induction on n. Using the fact that a p.p. graph must be connected, $|E(G)| \geq n-1 \geq 3n/2 - log_2\, n - c$ for $n \leq 83$, when $c = \frac{83}{2} + 1 - log_2\, 83$. Thus, the theorem is true for $n \leq 83$, and $n \geq 84$ is assumed for the rest of the proof.

If the minimum degree of G is three, then Theorem 1 is obviously true. Assume that $d(x) = 2$ for some $x \in V(G)$. There exists a $y \in V(G)$, $y \neq x$, such that $d(y) \leq 2$, for otherwise, G would have more edges than required. Removing x and y from G leaves a path-pairable graph G'. To see this, consider any pairing of G' and extend it to a pairing of G by pairing x and y. The xy path in the corresponding system of paths of G leaves one free edge at x and at most one free edge at y. These edges cannot be used in other paths in the path-matching. Therefore, the other paths use only edges of G'. Now the theorem follows by induction since

$$|E(G)| \geq |E(G')| + 3 \quad \text{and} \quad |V(G)| = |V(G')| + 2.$$

We may assume for $x \in V(G)$ that $d(x) \neq 2$, and therefore G has "many" vertices of degree one. The set of vertices adjacent to vertices of degree one will be denoted by $y_1, y_2, \cdots, y_r$. Assume that y_i is adjacent to $t_i - 1$ vertices of degree one, and this set of vertices together with y_i is denoted by A_i. Thus $|A_i| = t_i \geq 2$. The fact that G is p.p. and $\Delta(G) \leq n-2$ implies that $t_i \leq n/2$. Also, from the p.p. property, $d(y_i) \geq 2t_i - 1$. The vertex y_i is called *critical* if $d(y_i) \leq 2t_i$. Let $I \subset \{1, 2, \cdots, r\}$ denote the set of indices i for which y_i is critical. Set $|I| = p$. We will consider two cases that depend on the sizes of the sets A_i for each $i \in I$. In both cases the following simple lemma is needed.

Lemma 1. *If $a_1, a_2, \cdots, a_k$ are positive integers, and*

$$\sum_{i \in I_1} a_i \neq \sum_{i \in I_2} a_i$$

for $I_1 \neq I_2$ and $I_1 \cup I_2 \subset \{1, 2, \cdots, k\}$, then

$$k \leq log_2 \left(\sum_{i=1}^{k} a_i + 1 \right).$$

Proof. The numbers $\sum_{i \in I} a_i$ are all different as I runs through all non-empty subsets of $\{1, 2, \cdots, k\}$. Thus, $2^k - 1 \leq \sum_{i=1}^{k} a_i$, and the lemma follows. ∎

Case 1: *There exist two non empty disjoint sets $I_1, I_2 \subset I$ such that $\sum_{i \in I_1} t_i = \sum_{i \in I_2} t_i = a$.*

In this case select I_1 and I_2 so that $I_1 \cup I_2$ is as small as possible with this property. Let $s = |I_1 \cup I_2|$. Since $I_1 \cap I_2 = \phi$, there exists an $i \in I_1 \cup I_2$ such that $t_i \geq 2a/s$. The choice of I_1 and I_2 and Lemma 1 implies

$$s - 1 \leq log_2(2a - \frac{2a}{s} + 1) \leq log_2\, 2a = log_2\, a + 1,$$

and thus

$$s \leq log_2\, a + 2. \tag{1}$$

Now the proof can be finished by induction on the order of the graph G. Removing $\cup_{i \in I_1 \cup I_2} A_i$ from G leaves a pairable graph G'. To see this, consider a pairing of G' and extend it to a pairing of G by pairing the vertices of $A^* = \cup_{i \in I_1} A_i$ with the vertices of $A' = \cup_{i \in I_2} A_i$. After realizing this pairing in G by edge disjoint paths, note that the paths between A^* and A' leave at most one free edge at each vertex of $A^* \cup A'$, due to the critical property. Since these edges cannot be used by paths between pairs of G', the pairs of G' are connected with paths entirely in G'.

To estimate the number of edges incident to $A = \cup_{i \in I_1 \cup I_2} A_i$, two cases are considered. If $|A| \leq n/2$, then from the cut condition at least $|A|$ edges of G are between A and $V(G')$. Since A spans at least $|A| - s$ edges incident to vertices of degree 1, at least $2|A| - s = 4a - s$ edges are incident to A. Therefore, by induction

$$|E(G)| \geq 2|A| - s + |E(G')| \geq 4a - s + \frac{3(n-2a)}{2} - log_2(n-2a) - c$$
$$= \frac{2a - 2s}{2} + \frac{3n}{2} - log_2(n-2a) - c \geq \frac{3n}{2} - log_2 n - c,$$

because by the definition of A, $2a \geq 2s$.

If $|A| > n/2$, then we use the fact that the number of edges incident to A is at least $4a - s - \binom{s}{2}$. This is true since $d(y_i) \geq 2t_i - 1$ for any $i \in \{1, 2, \cdots, r\}$, and when adding $d(y_i)$ for $i \in I_1 \cup I_2$ at most $\binom{s}{2}$ edges are counted twice. To see that the induction works, we have to show

$$4a - s - \binom{s}{2} + \frac{3(n-2a)}{2} - log_2(n-2a) - c \geq \frac{3n}{2} - log_2 n - c,$$

and it is enough to see that

$$2a - s - s^2 \geq 0. \tag{2}$$

Since $2a \geq 2^{s-1}$ from (1), then (2) follows if $2^{s-1} - s - s^2 \geq 0$ i.e. for $s \geq 7$. If $s \leq 6$, then we use the fact that $|A| = 2a > n/2$, and (2) follows if $n/2 - 42 \geq 0$ (i.e. $n \geq 84$), and this was our initial assumption. Thus, in Case 1 induction gives Theorem 1.

Case 2: *For any two distinct sets $I_1, I_2 \subset I$,* $\sum_{i \in I_1} t_i \neq \sum_{i \in I_2} t_i$.

In this case

$$log_2(n+1) \geq |I| = p \tag{3}$$

follows from Lemma 1. Let I_1 denote the set of indices for which y_i is not critical, i.e. $I_1 = \{1, 2, \cdots, r\} - I$. Now, the sum of the degrees of G can be estimated as follows:

(4)

$$2|E(G)| = \sum_{x \in V(G)} d(x) = \sum_{d(x)=1} d(x) + \sum_{i \in I} d(y_i) + \sum_{i \in I_1} d(y_i) + \sum_{x \in V'} d(x),$$

where V' are the vertices not incident to a vertex of degree one. If $t = \sum_{i=1}^{r}(t_i - 1)$, then the first term in (4) is t. The second term is at least $\sum_{i \in I} d(y_i) \geq \sum_{i \in I}(2t_i - 1)$, since $d(y_i) \geq 2t_i - 1$ for any $i \in \{1, 2, \cdots, r\}$. For the third term, $\sum_{i \in I_1} d(y_i) \geq \sum_{i \in I}(2t_i + 1)$, since y_i is not critical for $i \in I_1$. The fourth term gives at least $3(n - t - r)$ since the vertices in V' have degree at least three. Thus,

$$\begin{aligned} 2|E(G)| &\geq t + \sum_{i \in I}(2t_i - 1) + \sum_{i \in I_1}(2t_i + 1) + 3(n - t - r) \\ &= t + \sum_{i \in I}(2t_i - 2 + 1) + \sum_{i \in I_1}(2t_i - 2 + 3) + 3(n - t - r). \end{aligned}$$

Therefore,

$$\begin{aligned} 2|E(G)| &\geq 3t + |I| + 3|I_1| + 3(n - t - r) = p + 3|I_1| + 3n - 3r \\ &= p + 3|I_1| + 3n - 3(p + |I_1|) = 3n - 2p. \end{aligned}$$

We only need to show that

$$3n - 2p \geq 2\left(\frac{3n}{2} - log_2\ n - c\right) = 3n - 2 \cdot log_2\ n - 2c,$$

which is equivalent to $log_2\ n + c \geq p$. This is insured by (3) and the fact that c is large. This completes the proof of Case 2 and of the Theorem 1. ∎

It seems likely that the error term $\log n$ can be eliminated from Theorem 1. Perhaps even a stronger statement is true which we put forward as a conjecture. Assume G is a graph of order $2n$. An even cut of G is a partition of the vertices of G into two equal parts. We say that G satisfies the *even cut condition* if each even cut of G has at least n edges. Obviously, each p.p. graphs satisfies the even cut conditon. The obvious example of a C_4 shows that the even cut condition is not equivalent to p.p.

Conjecture: *If G is a graph of (even) order n with the even cut condition and $\Delta(G) < n-1$ then for n sufficiently large, $|E(G)| \geq 3n/2 - O(1)$.*

It is not trivial that $|E(G)|$ is significally more than n if $|\Delta(G)| < n-1$ and G has the even cut condition. However, Z. Füredi (private communication) has a nice proof showing that with these conditions $|E(G)| \geq \frac{9}{8}n$.

For even n, $n \geq 4$, let $f(n,k)$ denote the minimum number of edges in a p.p. graph of maximum degree k. If there are no p.p. graphs of order n and maximum degree k, set $f(n,k) = \infty$.

Proposition 1. *If $k \leq n-2$ then $f(n,k) \geq n + \frac{k-3}{2}$.*

Proof. Let G be p.p., and let $x \in V(G)$ with $d(x) = k = \Delta(G)$. Since $\Delta(G) \leq n-2$, there is a $y \in V(G)$ that is non-adjacent to x. Then, since G is p.p. (the cut condition), a connected component of $V(G) - x$ must cover $V(G) - x - \Gamma(x)$ (here $\Gamma(x)$ denotes the set of neighbors of x), and this component C containing y must have at least $\frac{n}{2}+1$ vertices. Assume that $|C \cap \Gamma(x)| = t$. Then $\Delta - t + 1 \leq \frac{n}{2}$, so from the cut condition $t \geq \Delta - t + 1$ (*i.e.* $\frac{\Delta+1}{2} \leq t$). Then G must have at least $|C| - 1 + \Delta = t + n - \Delta - 2 + \Delta = t + n - 2 \geq n + \frac{\Delta-3}{2}$ edges. This completes the proof of Proposition 1. ∎

Theorem 2.

$$\begin{aligned} f(n, n-2) &= \frac{3n}{2} - 2 \\ f(n, n-3) &= \frac{3n}{2} - 3 \end{aligned}$$

Proof. Proposition 1 implies the required lower bound. The extremal graphs are shown in Figure 1 (in fact, they are unique). It is easy to check that both graphs are p.p.

The next theorem shows that there is a "jump" in $f(n,k)$ at $k = n-4$.

Theorem 3. $f(n, n-4) \geq \frac{3n}{2} + \frac{\sqrt{n}}{4} - 4$.

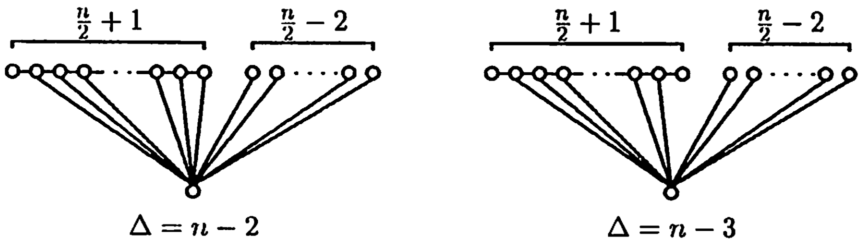

Figure 1: **Extremal Path Pairable (p.p.) Graphs**

Proof. Let G be path pairable, and let $x \in V(G)$ with $d(x) = \Delta(G) = n-4$. Then, just as in the proof of Propoosition 1, since G is p.p. (and thus the cut condition must be satisfied), a connected component C of $V(G)-x$ must cover $V(G)-x-\Gamma(x)$, and this component C must have more than $\frac{n}{2}$ vertices. Assume that C has $n/2+k$ vertices for some positive integer k.

We first claim that C cannot have as many as $2k+1$ vertices of degree 1. If so, then there would be at least $2k-2$ vertices of degree 1 in C that are adjacent to the vertex x, since only 3 vertices of G are not adjacent to x. Consider any pairing of vertices of G where the $2k-2$ vertices of degree 1 in $C \cap \Gamma(x)$ are paired with each other, and each of some $n/2-k$ remaining vertices of C are paired with the $n/2-k$ vertices not in C.

Consider a system of edge disjoint paths that realizes this pairing. With no loss of generality, you can assume the paths between the $2k-2$ vertices of degree 1 in C contain the vertex x. All of the remaining paths in this system contain the vertex x with one possible exception. Therefore, there are $n/2-1$ paths that, with at most one exception, contain 2 edges incident to x and the possible exceptional path contains at least 1 edge incident to x. This implies that $d(x) = n-3$, a contradiction that completes the proof of this claim.

The component C also does not contain a suspended path (a path whose inner vertices are of degree two in C) with $2k$ vertices. Assume that such a path existed, say with consecutive vertices $a_1, a_2, \cdots, a_k$,

$b_1, b_2, \cdots, b_k$. Consider a pairing of vertices of G such that a_i and b_i are paired for each i, and that each of the remaining $n/2 - k$ vertices of C are paired with the $n/2 - k$ vertices not in C. Consider a path system that realizes this pairing. With the pairing of vertices on the suspended path, at most 1 path for these pairs in the path system will not use the vertex x. Therefore, just as in the proof of the previous claim, there must be at least $n - 3$ edges incident to x. This gives a contradiction that completes the proof of this claim.

We will next show that G must have $\frac{3n}{2} + \frac{\sqrt{n}}{4} - 4$ edges by counting the number of edges in C. We will use the fact that in C there are restrictions on the number of vertices of degree 1 and on the length of suspended paths.

Let n_j be the number of vertices of degree j in C. Then, we have the following:

$$\begin{aligned} \sum_{j=1}^{t} j \cdot n_j &= 2|E(C)| \\ \sum_{j=1}^{t} n_j &= n/2 + k \end{aligned}$$

In addition, we know that $n_1 \leq 2k$. Shrink each suspended path in C to an edge to get a graph H that has no vertices of degree 2. The graph H has n_j vertices of degree j for each $j \neq 2$. Thus, since no suspended path of C can have $2k - 2$ vertices of degree 2, we have the number of vertices of degree 2 in C is no more than $2k - 2$ times the number of edges in H. Thus,

$$n_2 \leq \frac{1}{2}(2k-2)\left(\sum_{j \neq 2} j \cdot n_j\right).$$

This results in the following series of inequalities.

$$\begin{aligned} n_2 &\leq (k-1)\left(\sum_{j=1}^{t} j \cdot n_j\right) - 2(k-1)n_2 \\ n_2(2k-1) &\leq (k-1)\left(\sum_{j=1}^{t} j \cdot n_j\right) \end{aligned}$$

$$n_2 \leq \frac{k-1}{2k-1}(\sum_{j=1}^{t} j \cdot n_j) = \frac{2k-2}{2k-1}|E(C)|$$

We have

$$\begin{aligned} 2|E(C)| &= \sum_{j=1}^{t}(j-3)\cdot n_j + 3\left(\sum_{j=1}^{t} n_j\right) = \sum_{j=1}^{t}(j-3)n_j + 3(\frac{n}{2}+k) \\ &= \frac{3n}{2} + 3k - 2n_1 - n_2 + \sum_{j>3}^{t}(j-3)n_j \end{aligned}$$

This results in the following inequalities involving $|E(C)|$.

$$\begin{aligned} 2|E(C)| &\geq \frac{3n}{2} + 3k - 2n_1 - n_2 \\ 2|E(C)| &\geq \frac{3n}{2} + 3k - 2(2k) - \frac{2k-2}{2k-1}|E(C)| \\ |E(C)|\left(2 + \frac{2k-2}{2k-1}\right) &\geq \frac{3n}{2} - k \\ |E(C)| &\geq \frac{3n}{2}\left(\frac{2k-1}{6k-4}\right) - \frac{k(2k-1)}{6k-4} \end{aligned}$$

This gives the following inequalities involving $|E(G)|$.

$$|E(G)| \geq \frac{3n}{2}\left(\frac{2k-1}{6k-4}\right) - \frac{k(2k+1)}{6k-4} + n - 4$$

(5) $$|E(G)| \geq \frac{3n}{2} + \frac{n}{12k-8} - \frac{2k^2+25k-16}{6k-4}.$$

In addition, we also know that

(6) $$|E(G)| \geq \frac{3n}{2} + k - 5,$$

since $|C| = n/2 + k$.

If $k \geq \frac{\sqrt{n}}{4} + 1$, then (6) implies that $|E(G)| \geq \frac{3n}{2} + \frac{\sqrt{n}}{4} - 4$, and if $k \leq \frac{\sqrt{n}}{4}$, then (5), along with some straightforward arithmetic, implies the same lower bound for $|E(G)|$. This completes the proof of Theorem 3. ∎

3. PROBLEMS ON PATH-PAIRABLE GRAPHS

It is easy to see that the maximum degree of a p.p. graph tends to infinity as the order of the graph tends to infinity. More precisely we have the following.

Theorem 4. *If G is a path pairable graph of (even) order n and $\Delta(G) = k$, then*

$$n \leq 2k^k.$$

Proof. For $k \geq 2$ and $t \geq 2$, the number of vertices of distance $\leq t$ from a vertex v is at most $1 + k(k-1)^{t-1} \leq k^t$. Thus each vertex v of G has at least $n - k^t$ vertices at a distance greater than t from v.

Form a new graph H_t that has the same vertices as G, but two vertices are adjacent if their distance is $> t$. Let

$$t = \left\lfloor \log_k \frac{n}{2} \right\rfloor,$$

so $t + 1 \geq \log_k n/2$. Therefore, each vertex in H_t, has degree at least $n/2$, and so H_t is Hamiltonian by a theorem of Dirac [2]. Thus, there is a perfect matching in H_t, which implies the vertices of G can be paired so that the distance between each pair of vertices is at least $t + 1 \geq \log_k n/2$. The number of edges needed to realize the pairing is at least $(n/2)(t + 1)$, and so we have the following bounds on the number of edges in G:

$$\left(\frac{n}{2}\right) \log_k \frac{n}{2} \leq |E(G)| \leq \frac{kn}{2}.$$

It follows immediately from the previous equation that $n \leq 2k^k$, which completes the proof of Theorem 4. ∎

Problem: *Give a good estimate of $f(k)$, the maximum order of a p.p. graph with maximum degree k.*

It seems hopeless to determine $f(k)$ exactly. Even to determine $f(3)$ causes a lot of difficulties. Theorem 4 says that $f(3) \leq 54$. This is far from the truth, as the following result indicates.

Theorem 5. *$f(3) = 12$, and the unique 12-vertex p.p. graph is shown in Figure 2.*

Proof (outline). A refinement of the proof of Theorem 4 for the special case of $k = 3$ shows (with a very tedious case analysis) that $f(3) \leq 14$. It should be mentioned that the graphs K_4, $K_{3,3}$, Q_3 (the Cube), P_{10} (Petersen's graph), and the graph G_{12} in Figure 2 are 3-regular p.p. graphs of orders 4, 6, 8, 10, and 12 respectively.

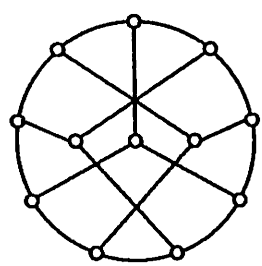

Figure 2: **A Path Pairable Graph of Order 12**

To show that no 3-regular p.p. graphs exist with 14 vertices is not easy. The most difficult case is to eliminate the Heawood graph, the unique 3−regular graph of girth 6 (see [6]). Interestingly, to show that G_{12} on Figure 2 is p.p. also requires long case analysis. (Even for smaller order, like for Petersen's graph we have only a complicated proof of the p.p. property). ∎

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