# Orientable Open Domination of Graphs

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## ABSTRACT

An open dominating set for a digraph $D$ is a set $S$ of vertices of $D$ such that every vertex of $D$ is adjacent from some vertex of $S$. The cardinality of a minimum open dominating set for $D$ is the open domination number $\rho_1(D)$ of $D$. The lower orientable open domination number $\mathrm{dom}_1(G)$ of a graph $G$ is the minimum open domination number among all orientations of $G$. Similarly, the upper orientable open domination number $\mathrm{DOM}_1(G)$ of $G$ is the maximum such open domination number.

For a connected graph $G$, it is shown that $\mathrm{dom}_1(G)$ and $\mathrm{DOM}_1(G)$ exist if and only if $G$ is not a tree. A discussion of the upper orientable open domination number of complete graphs is given. It is shown that for each integer $c$ with $\mathrm{dom}_1(\mathrm{K}_n) \le c \le \mathrm{DOM}_1(\mathrm{K}_n)$, there exists an orientation $D$ of $\mathrm{K}_n$ such that $\rho_1(D) = c$.

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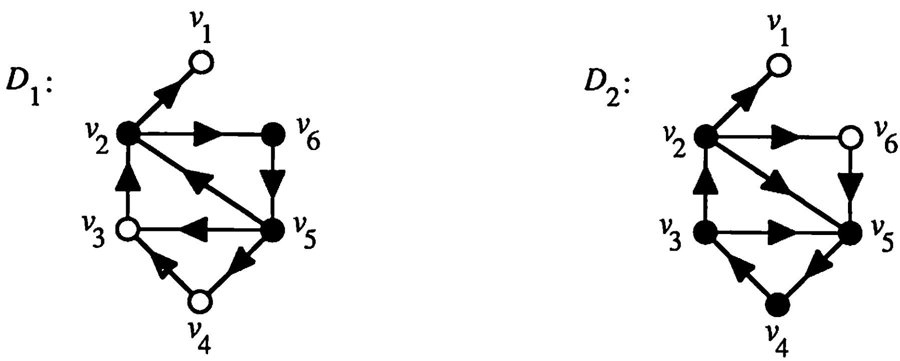
## 1. Orientable Open Domination

A vertex $v$ in a graph $G$ is said to *1-step dominate* or *openly dominate* each of its neighbors. A set $S$ of vertices of $G$ is an *open dominating set* of $G$ if every vertex of $G$ is openly dominated by some vertex of $S$. The minimum cardinality among the open dominating sets of $G$ is called the *open domination number* of $G$ and is denoted by $\rho_1(G)$.

Analogous definitions can be made for digraphs. In particular, for a digraph $D$, a vertex $v$ *openly dominates* all vertices $w$ with $(v, w) \in E(D)$. The *open domination number* $\rho_1(D)$ of $D$ is the minimum cardinality among the open dominating sets of $D$. Although for a graph $G$ without isolated vertices, the open domination number $\rho_1(G)$ always exists, such is not the case for digraphs. However, for a digraph $D$, a necessary and sufficient condition for $\rho_1(D)$ to exist is that $\text{id } v \geq 1$ for every vertex $v$ of $D$. This condition is satisfied for the digraphs $D_1$ and $D_2$ of Figure 1. In $D_1$, the vertex $v_1$ is uniquely openly dominated by $v_2$, the vertex $v_4$ is uniquely openly dominated by $v_5$, and $v_5$ is uniquely openly dominated by $v_6$. Hence if $S$ is an open dominating set of $D_1$, then $\{v_2, v_5, v_6\} \subseteq S$. Since $S_1 = \{v_2, v_5, v_6\}$ is itself an open dominating set of $D$, it follows that $S_1$ is a (in fact, the unique) minimum open dominating set of $D_1$. Thus $\rho_1(D_1) = 3$. Similarly, $S_2 = \{v_2, v_3, v_4, v_5\}$ is the unique minimum open dominating set of $D_2$ and so $\rho_1(D_2) = 4$. Since $D_1$ and $D_2$ are orientations of the same graph and $\rho_1(D_1) \neq \rho_1(D_2)$, this suggests some definitions.




$S_1 = \{v_2, v_5, v_6\}$ $\qquad$ $S_2 = \{v_2, v_3, v_4, v_5\}$

Figure 1

For a graph $G$, we say that $D$ is a *valid orientation* of $G$ if every vertex of $D$ has positive indegree. Let $D_1, D_2, \ldots, D_k$ be the distinct valid orientations of a graph $G$. We define the *lower* and *upper orientable open domination numbers* of $G$, respectively, as

$$\mathrm{dom}_1(G) = \min\{\rho_1(D_i) \mid 1 \le i \le k\} \text{ and } \mathrm{DOM}_1(G) = \max\{\rho_1(D_i) \mid 1 \le i \le k\}$$

These concepts were defined and investigated for ordinary domination in digraphs in [1]. In order to present a necessary and sufficient condition for these parameters to be defined for a graph $G$, we recall a theorem of Robbins [3].

**Theorem** (Robbins) A graph $G$ has a strong orientation if and only if $G$ is 2-edge-connected (connected and has no bridges).

**Theorem 1** Let $G$ be a connected graph. Then $\mathrm{dom}_1(G)$ and $\mathrm{DOM}_1(G)$ exist if and only if $G$ is not a tree.

**Proof** First assume that $\mathrm{dom}_1(G)$ and $\mathrm{DOM}_1(G)$ are defined and suppose, to the contrary, that $G$ is a tree of order $n$. Let $D$ be any valid orientation of $G$. Since $G$ is a tree, it follows that the sum of the indegrees of the vertices of $D$ is $\sum_{v \in V(D)} \mathrm{id}\, v = |E(D)| = |E(G)| = n - 1$. However, since every vertex has indegree at least 1, it follows that $\sum_{v \in V(D)} \mathrm{id}\, v \geq n$, producing a contradiction.

Next, suppose that $G$ is not a tree. We show that there exists a valid orientation $D$ of $G$. Since $G$ is not a tree, we know that $G$ contains a cyclic block. Let $B_1, B_2, \ldots, B_k$ denote the cyclic blocks of $G$. By Robbins' Theorem, each block $B_i$, $1 \leq i \leq k$, has a strong orientation. For the desired orientation $D$, begin by producing a strong orientation of each block $B_i$, $1 \leq i \leq k$. Then $\mathrm{id}\, v \geq 1$ for every vertex $v$ of the blocks $B_1, B_1, \ldots, B_k$. Next, for any shortest path $P$ between two cyclic blocks $B_i$ and $B_j$ $(i \neq j)$, orient the edges of $P$ from one block to the other (say, from $B_i$ to $B_j$). Then for each vertex $v$ of $P$, we have $\mathrm{id}\, v \geq 1$. Finally, let $V$ be the set $\bigcup_{i=1}^{k} V(B_i)$, along with the vertices of all shortest paths between two cyclic blocks, and let $u$ be any vertex of $G$ not belonging to $v$. Then there exists a shortest path from $u$ to some vertex, say $v$, of $V$. Direct the edges of the path from $v$ to $u$. The resulting orientation $D$ has minimum indegree at least 1. Therefore $\rho_1(D)$ exists, implying that $\mathrm{dom}_1(G)$ and $\mathrm{DOM}_1(G)$ exist. ❑

Some simple bounds for these parameters are presented next.

**Theorem 2** For every connected graph $G$ of order $n \geq 3$ that is not a tree,

$$3 \leq \mathrm{dom}_1(G) \leq \mathrm{DOM}_1(G) \leq n.$$

**Proof** Clearly, $\mathrm{DOM}_1(G) \leq n$. Let $D$ be a valid orientation of $G$ and let $S$ be a minimum open dominating set of $D$. For $u \in S$, there exists some vertex $v \in S$ that openly dominates $u$. So $(v, u) \in E(D)$. Also the vertex $v$ must be openly dominated by some vertex of $S$. Further, this vertex cannot be $u$ since $(u, v) \notin E(D)$. Hence there exists a vertex $w \in S$ with $w \neq u, v$ that openly dominates $v$. Therefore $\{u, v, w\} \subseteq S$, implying $\rho_1(D) = |S| \geq 3$. We now know $\rho_1(D) \geq 3$ for any valid orientation $D$ of $G$. Consequently, $\mathrm{dom}_1(G) \geq 3$. ❑

The following result shows that the upper orientable domination number of a graph attains the upper bound only in one special case. The proof of this result uses the following theorem of Hall [2].

**Theorem** (Hall) A collection $S_1, S_2, \ldots, S_n$, $n \geq 1$, of finite nonempty sets has a system of distinct representatives if and only if the union of any $k$ of these sets contains at least $k$ elements, for each $k$ such that $1 \leq k \leq n$.

**Theorem 3** For a connected graph $G$ of order $n \geq 3$, $\mathrm{DOM}_1(G) = n$ if and only if $G = C_n$.

**Proof** First assume that $G = C_n$. Then, we orient the edges so that the corresponding digraph $D$ is a directed cycle. It follows that every vertex is openly dominated only by the preceding vertex along the cycle. Therefore every vertex must belong to the open dominating set, implying that $\rho_1(D) = n$. Consequently, $\mathrm{DOM}_1(G) = n$.

For the converse, assume that $\mathrm{DOM}_1(G) = n$. Let $D$ be an orientation of $G$ such that $\rho_1(D) = \mathrm{DOM}_1(G) = n$. Assume that $V(D) = \{v_1, v_2, \ldots, v_n\}$. Define a digraph $D'$ by $V(D') = \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n'\}$ and $(u, v) \in E(D')$ if and only if

(1) $u = v_i$ for some $i \in \{1, 2, \ldots, n\}$,

(2) $v = v_j'$ for some $j \in \{1, 2, \ldots, n\}$, and

(3) $(v_i, v_j) \in E(D)$.

Let $S_j = \{v_i \mid (v_i, v_j') \in E(D')\}$ for $j = 1, 2, \ldots, n$. Thus $S_j$ is the set of vertices that openly dominate the vertex $v_j$ in $D$. We claim that $S_1, S_2, \ldots, S_n$ has a system of distinct representatives. Now let $J \subseteq \{1, 2, \ldots, n\}$ such that $|J| = k$. Let $S = \bigcup_{j \in J} S_j$ and suppose, to the contrary, that $|S| < |J|$. Let $W = \{v_j' \mid j \in J\}$ and let $W' = \{v_1', v_2', \ldots, v_n'\} - W$. In $D$ every vertex $v_j$, where $j \in J$, is openly dominated by some vertex of $S$. Now for each $v_\ell' \in W'$, we know since $\mathrm{id}_D\, v_\ell \geq 1$, that there exists $(v_i, v_\ell) \in E(D)$ and hence $(v_i, v_\ell') \in E(D')$. Let $v_{i_\ell}$ be such a vertex for each $v_\ell' \in W'$ and let $S' = \{v_{i_\ell} \mid v_\ell' \in W'\}$. Then $|S'| \leq |W'| = n - k$. Now, in $D$, every vertex $v_j$, where $j \notin J$, is openly dominated by some vertex of $S'$. Consequently, the set $S \cup S'$ is an open dominating set of $D$. However $|S \cup S'| \leq |S| + |S'| < k + n - k = n$. This implies that $\rho_1(D) < n$, producing a contradiction. Hence $|S| \geq |J|$, implying that $S_1, S_2, \ldots, S_n$ has a system of distinct representatives. It follows that $D'$ has an independent set of arcs. This independent set of arcs corresponds to a disjoint union of directed cycles, say $D_1, D_2, \ldots, D_m$, in $D$.

Suppose, to the contrary, that $\langle V(D_i) \rangle$ is not a directed cycle for some $i = 1, 2, \ldots, m$. Then there exist vertices $v_j$ and $v_k$ in $V(D_i)$ such that $(v_j, v_k) \in E(D)$ but $v_k$ does not follow $v_j$ on the directed cycle $D_i$. Let $v_\ell$ be the vertex preceding $v_k$ on the directed cycle $D_i$. Then $V(D_i) - \{v_\ell\}$ is an open dominating set of $\langle V(D_i) \rangle$. But this would imply that $\rho_1(D) < n$. Hence $\langle V(D_i) \rangle$ is a directed cycle for each $i = 1, 2, \ldots, m$.

Finally we show that $m = 1$. If $m \geq 2$, then since the underlying graph $G$ is connected, there exist vertices $v_i$ and $v_j$ such that $(v_i, v_j) \in E(D)$ where $v_i \in V(D_i)$ and $v_j \in V(D_j)$, $i \neq j$. Let $v_k$ be the vertex preceding $v_j$ on the directed cycle $D_j$. Then $V(D_i) \cup V(D_j) - \{v_k\}$ is an open dominating set of $\langle V(D_i) \cup V(D_j) \rangle$. Again, this would imply that $\rho_1(D) < n$. Hence $m = 1$ and $D = \langle V(D_1) \rangle$, a directed cycle of length $n$. That is, $G = C_n$. ❑

We next study a class of graphs, the difference of whose lower and upper orientable domination numbers is arbitrarily large. In particular, for $n \geq 4$, let $G_n = K_1 + P_{n-1}$ (see Figure 2).

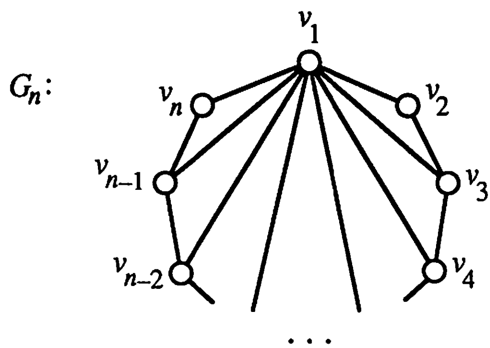
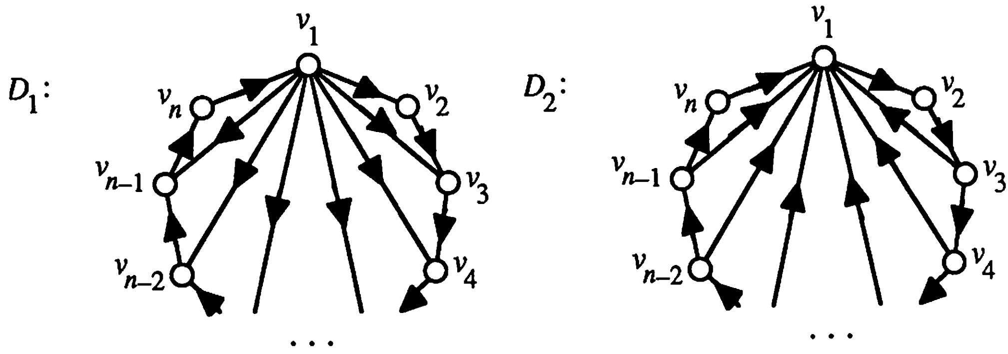



Figure 2

It can be shown that $\mathrm{dom}_1(G_n) = 3$ and $\mathrm{DOM}_1(G_n) = n - 1$ with the aid of the orientations $D_1$ and $D_2$ of $G_n$ shown in Figure 3.

For $D_1$, observe that $\mathrm{id}\ v_2 = 1$ and thus $v_1$ must belong to every open dominating set of $D_1$. Similarly, since $\mathrm{id}\ v_n = 1$ and $\mathrm{id}\ v_1 = 1$, it follows that $v_{n-1}$ and $v_n$ must also belong to each open dominating set of $D_1$. Further, $S_1 = \{v_1, v_{n-1}, v_n\}$ is an open dominating set of $D_1$. Thus $\rho_1(D_1) = |S_1| = 3$. Also $3 \leq \mathrm{dom}_1(G_n) \leq \rho_1(D_1) = 3$, which implies that $\mathrm{dom}_1(G_n) = 3$.

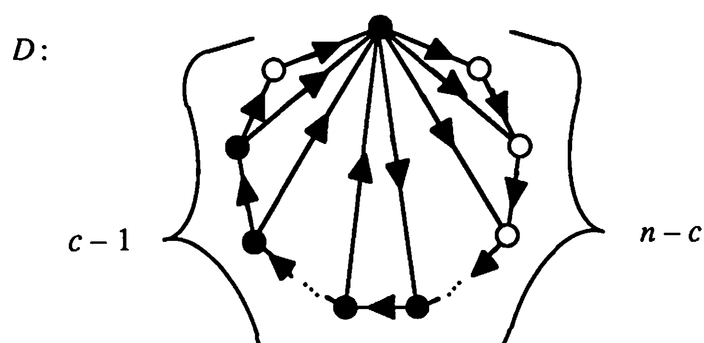
In a similar manner, observe that $\mathrm{id}_{D_2}\ v_i = 1$ for $i = 2, 3, \ldots, n$. Thus $v_1, v_2, \ldots, v_{n-1}$ must belong to every open dominating set. Since $S_2 = \{v_1, v_2, \ldots, v_{n-1}\}$ is an open dominating set, it follows that $\mathrm{DOM}_1(G_n) \geq \rho_1(D_2) = |S_2| = n - 1$. Further, by Theorem 3, $\mathrm{DOM}_1(G_n) \neq n$. Thus $\mathrm{DOM}_1(G_n) = n - 1$.




$S_1 = \{v_1, v_{n-1}, v_n\}$ $\qquad$ $S_2 = \{v_1, v_2, \ldots, v_{n-1}\}$

Figure 3

This class of graphs also has the property that for each integer $c$, $3 \le c \le n-1$, there exists an orientation of $G_n$, resulting in a digraph $D$, such that $\rho_1(D) = c$. (See Figure 4.)




$\rho_1(D) = c$

Figure 4

## 2. Upper Orientable Open Domination Numbers of Complete Graphs

In this section we investigate the growth of the function $\mathrm{DOM}_1(K_n)$. First, we show that $\mathrm{DOM}_1(K_n)$ is a nondecreasing function.

**Lemma 4** For all $n \geq 3$, $\mathrm{DOM}_1(K_n) \leq \mathrm{DOM}_1(K_{n+1})$.

**Proof** Choose a valid orientation $D_n$ of $K_n$ such that $\rho_1(D_n) = \mathrm{DOM}_1(K_n)$. Next define an orientation $D_{n+1}$ of $K_{n+1}$ by adding a vertex $w$ to $D_n$ and adding arcs $(u, w)$ for every $u \in V(D_n)$. Then $D_{n+1}$ is a valid orientation, and

$$\mathrm{DOM}_1(K_{n+1}) \geq \rho_1(D_{n+1}) = \rho_1(D_n) = \mathrm{DOM}_1(K_n). \quad \square$$

**Corollary 5** For $3 \leq m \leq n$, $\mathrm{DOM}_1(K_m) \leq \mathrm{DOM}_1(K_n)$.

In the next two lemmas, we investigate how slowly the function $\mathrm{DOM}_1(K_n)$ grows. Of course, every orientation of $K_n$ is a tournament of order $n$.

**Lemma 6** If $\mathrm{DOM}_1(K_{n+1}) > \mathrm{DOM}_1(K_n)$, then

(a) each tournament $T_{n+1}$ with $\rho_1(T_{n+1}) = \mathrm{DOM}_1(K_{n+1})$ is strong, and

(b) $\mathrm{DOM}_1(K_{n+1}) = \mathrm{DOM}_1(K_n) + 1$.

**Proof of (a)** Let $T_{n+1}$ be any valid tournament of order $n + 1$ such that $\rho_1(T_{n+1}) = \mathrm{DOM}_1(K_{n+1})$. Suppose, to the contrary, that $T_{n+1}$ is not strong. Then $V(T_{n+1})$ can be partitioned into sets $S_1, S_2, \ldots, S_k$ $(k \geq 2)$ such that (1) each $\langle S_i \rangle$ is a strong subdigraph and is maximal with respect to the property of being strong, that is, the subdigraphs $\langle S_1 \rangle, \langle S_2 \rangle, \ldots, \langle S_k \rangle$ are the strong components of $T_{n+1}$, and (2) for every vertex $u$ of $S_1$ and for every vertex $v$

of $S_i$, $i \geq 2$, the arc $(u, v)$ belongs to $T_{n+1}$. It is clear that $S_1$ is an open dominating set of $T_{n+1}$. Further, a minimum open dominating set of $\langle S_1 \rangle$ is also a minimum open dominating set of $T_{n+1}$. So $\mathrm{DOM}_1(K_{n+1}) = \rho_1(T_{n+1}) = \rho_1(\langle S_1 \rangle) \leq \mathrm{DOM}_1(K_{|S_1|}) \leq \mathrm{DOM}_1(K_n) < \mathrm{DOM}_1(K_{n+1})$, producing a contradiction. Thus $T_{n+1}$ is strong.

**Proof of (b)** It suffices to show that $\mathrm{DOM}_1(K_{n+1}) \leq 1 + \mathrm{DOM}_1(K_n)$. Let $T_{n+1}$ be a valid tournament of order $n + 1$ such that $\rho_1(T_{n+1}) = \mathrm{DOM}_1(K_{n+1})$. Choose a vertex $v$ of $T_{n+1}$ with od $v \geq 1$. Let $S = V(T_{n+1}) - N^+(v)$, where $N^+(v)$ represents the out-neighborhood of $v$. Then $v$ openly dominates all vertices belonging to $N^+(v)$. Also, since id $v \geq 1$, $S \neq \varnothing$. Since od $v \geq 1$, $|S| \leq n$. Now it follows that

$$\mathrm{DOM}_1(K_{n+1}) = \rho_1(T_{n+1}) \leq 1 + \rho_1(\langle S \rangle) \leq 1 + \mathrm{DOM}_1(K_n). \quad \square$$

**Theorem 7** If $\mathrm{DOM}_1(K_n) = m$ and $\mathrm{DOM}_1(K_{n+1}) = m + 1$, then $\mathrm{DOM}_1(K_i) = m + 1$ for $n + 1 \leq i \leq 2n + 2$.

**Proof** It suffices to show that $\mathrm{DOM}_1(K_{2n+2}) \leq m + 1$. Let $T$ be a valid tournament of order $2n + 2$. Then $T$ has a vertex $v$ such that od $v \geq n + 1$. Let $S = V(T) - (N^+(v) \cup \{v\})$. Then $|S| \leq n$. As in the proof of Lemma 6(b), we have

$$\rho_1(T) \leq 1 + \rho_1(\langle S \rangle) \leq 1 + \mathrm{DOM}_1(K_n) = 1 + m.$$

Since this is true for any such tournament $T$, it follows that $\mathrm{DOM}_1(K_{2n+2}) \leq m + 1$. $\square$

From our previous results, we know that $\mathrm{DOM}_1(K_n)$ is a nondecreasing function and any increase in functional values is a step increment of 1. However, our results so far have not shown that $\mathrm{DOM}_1(K_n)$ increases at all. The next result shows that, in fact, the function increases without bound.

**Theorem 8** The function $\mathrm{DOM}_1(K_n)$ is unbounded as $n \to \infty$.

**Proof** We proceed by a counting argument. First, let $T_n$ denote the number of labeled tournaments of order $n$. Since $K_n$ has $\binom{n}{2}$ edges, each of which can be oriented in one of two directions, $T_n = 2^{\binom{n}{2}}$.

Next, let $T_n^v$ denote the number of valid labeled tournaments of order $n$. We claim that $T_n^v = T_n - nT_{n-1}$. Observe that for any labeled tournament $T$ that is not valid, there cannot exist two vertices $v$ and $w$ with $\mathrm{id}\, v = 0$ and $\mathrm{id}\, w = 0$ because either $(v, w) \in E(T)$ or $(w, v) \in E(T)$. Hence for any labeled tournament $T$ that is not valid, there exists a unique vertex $v$ such that $\mathrm{id}\, v = 0$. Since there are $n$ choices for the label of the vertex of indegree 0 and, by removing this vertex, we obtain a labeled tournament of order $n - 1$, it follows that the number of labeled tournaments that are not valid is $nT_{n-1}$. Thus the number of valid labeled tournaments is $T_n^v = T_n - nT_{n-1}$.

Next, let $T_n^k$ denote the number of labeled tournaments of order $n$ with open domination number equal to $k$, $3 \le k \le n$. Any such labeled tournament $T$ has a minimum open dominating set $S$ such that $\langle S \rangle$ is a valid subtournament of order $k$. There are $\binom{n}{k}$ ways to label the vertices of $S$. So the number of possible orientations of $\langle S \rangle$ is $T_k^v$. Let $T'$ be the subtournament $\langle V(T) - S \rangle$. The number of possible orientations of $T'$ is $T_{n-k}$. Since $S$ is an open dominating set of $T$, it follows that for every vertex $v$ of $T'$, there is at least one vertex of $S$ adjacent to $v$. That is, each of the $n - k$ vertices of $T'$ is adjacent from at least one of the $k$ vertices of $S$. Hence the number of possible orientations of edges between $S$ and $V(T')$ is

$$\left[\binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{k}\right]^{n-k}.$$

Therefore,

$$T_n^k \le \binom{n}{k} \cdot T_k^v \cdot T_{n-k} \cdot [\binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{k}]^{n-k}.$$

$$= \binom{n}{k} \cdot T_k^v \cdot T_{n-k} \cdot (2^k - 1)^{n-k}$$

$$= \binom{n}{k} \cdot T_k^v \cdot 2^{\frac{(n-k)(n+k-1)}{2}} \cdot (\frac{2^k - 1}{2^k})^{n-k}$$

$$= \binom{n}{k} \cdot [2^{\frac{k(k-1)}{2}} - k2^{\frac{(k-1)(k-2)}{2}}] \cdot 2^{\frac{(n-k)(n+k-1)}{2}} \cdot (\frac{2^k - 1}{2^k})^{n-k}$$

$$= \binom{n}{k} \cdot [2^{\frac{n(n-1)}{2}} - k2^{\frac{n(n-1)-2(k-1)}{2}}] \cdot (\frac{2^k - 1}{2^k})^{n-k}$$

$$= \binom{n}{k} \cdot 2^{\binom{n}{2}} \cdot [1 - \frac{k}{2^{k-1}}] \cdot (\frac{2^k - 1}{2^k})^{n-k}$$

Also observe that

$$T_n^v = 2^{\binom{n}{2}} - n2^{\binom{n-1}{2}} = 2^{\binom{n}{2}} \cdot [1 - \frac{n}{2^{n-1}}].$$

Now let $g(n) = T_n^v - \sum_{i=3}^{k} T_n^i$, which counts the number of valid tournaments of order $n$ with minimum open dominating set of cardinality exceeding $k$. Then we have

$$g(n) = T_n^v - \sum_{i=3}^{k} T_n^i$$

$$\ge 2^{\binom{n}{2}}[1 - \frac{n}{2^{n-1}}] - \sum_{i=3}^{k} \binom{n}{i} \cdot 2^{\binom{n}{2}} \cdot [1 - \frac{i}{2^{i-1}}][\frac{2^i - 1}{2^i}]^{n-i}$$

$$= 2^{\binom{n}{2}}([1 - \frac{n}{2^{n-1}}] - \sum_{i=3}^{k} \binom{n}{i} \cdot [1 - \frac{i}{2^{i-1}}][\frac{2^i - 1}{2^i}]^{n-i}).$$

Since $\lim_{n\to\infty} \sum_{i=3}^{k} \binom{n}{i} \cdot [1 - \frac{i}{2^{i-1}}][\frac{2^i - 1}{2^i}]^{n-i} = 0$, it follows that $\lim_{n\to\infty} g(n) = \infty$. Thus, for any fixed $k$, $3 \le k < n$, there exists a sufficiently large integer $n$ such that for some valid labeled tournament $T$ of order $n$, $\rho_1(T) > k$, implying that $\mathrm{DOM}_1(K_n) > k$. □

Using the method described above, the first value of $n$ for which $g(n) > 0$ is 77. Thus $\mathrm{DOM}_1(K_{77}) > 3$. However this is not the least integer $n$ for which $\mathrm{DOM}_1(K_n) > 3$. It can be verified that $\mathrm{DOM}_1(K_n) = 3$ for $n = 3, 4, \ldots, 10$, the most difficult of which is $n = 10$. We verify this.

**Theorem 9** $\mathrm{DOM}_1(K_{10}) = 3$.

**Proof** Let $D$ be an orientation of $K_{10}$ such that $\rho_1(D) = \mathrm{DOM}_1(K_{10})$. There exists a vertex, say $v_1$, with maximum outdegree. Necessarily, od $v_1 \ge 5$.

*Case* 1 od $v_1 = 5$. Let $N^+(v_1) = \{u_1, u_2, u_3, u_4, u_5\}$, and let $N^-(v_1) = \{v_2, v_3, v_4, v_5\}$. Assume, without loss of generality, that $v_2$ is adjacent to at least two of $\{v_3, v_4, v_5\}$. In particular, assume the arcs $(v_2, v_4)$ and $(v_2, v_5)$ belong to $D$.

*Subcase* 1.1 $(v_2, v_3) \in E(D)$. Then, since id $v_2 \ge 1$, it follows that $(u_i, v_2) \in E(D)$ for some $i \in \{1, 2, 3, 4, 5\}$. It follows that $S = \{u_i, v_2, v_1\}$ is an open dominating set. Thus $\rho_1(D) = 3$.

*Subcase* 1.2 $(v_3, v_2) \in E(D)$. Since the maximum outdegree of $D$ is 5, it follows that at least 3 of the vertices of $N^+(v_1)$ are adjacent to $v_2$. Suppose $u_1, u_2$, and $u_3$ are among the vertices adjacent to $v_2$. Assume that none of the sets $\{u_i, v_2, v_1\}$, $1 \le i \le 3$, is an open dominating set. Then there must exist some vertex not openly dominated by any of these sets. In particular, $v_3$ is the only such vertex. So $(v_3, u_i) \in E(D)$ for $i = 1, 2, 3$. Now since the maximum outdegree of $D$ is 5, it follows that $(v_4, v_3) \in E(D)$ and $(v_5, v_3) \in E(D)$. Without loss of generality, assume $(v_4, v_5) \in E(D)$. Since the maximum outdegree of $D$ is 5, it follows that at least 3 of the vertices of $N^+(v_1)$ are adjacent to $v_4$. Hence there is at least one vertex, say $u_i$, for some $i$, of $N^+(v_1)$ which is adjacent to both $v_2$ and $v_4$. Then $S = \{u_i, v_1, v_4\}$ is an open dominating set, implying $\rho_1(D) = 3$.

*Case* 2 od $v_1 = 6$. Let $N^+(v_1) = \{ u_1, u_2, u_3, u_4, u_5, u_6\}$, and let $N^-(v_1) = \{v_2, v_3, v_4\}$. Assume, without loss of generality, that $v_2$ is adjacent to at least one of $v_3$ and $v_4$.

*Subcase* 2.1 *$\langle v_2, v_3, v_4\rangle$ is a directed 3-cycle.* If $\{v_2, v_3, v_4\}$ is not an open dominating set, then there exists $u_i$ $(1 \le i \le 6)$ such that $(u_i, v_2)$, $(u_i, v_3)$, $(u_i, v_4) \in E(D)$. Then $S = \{u_i, v_1, v_2\}$ is an open dominating set and $\rho_1(D) = 3$.

*Subcase* 2.2 *$\langle v_2, v_3, v_4\rangle$ is not a directed 3-cycle.* Then one of the vertices, say $v_2$, is adjacent to the other two, $v_3$ and $v_4$. And since the maximum outdegree of $D$ is 6, it follows that at least 3 of $N^+(v_1)$ are adjacent to $v_2$. Suppose $u_1, u_2$, and $u_3$ are among the vertices adjacent to $v_2$. Then $S = \{u_1, v_1, v_2\}$ is an open dominating set, implying $\rho_1(D) = 3$.

*Case* 3 od $v_1 = 7$. Let $N^+(v_1) = \{u_i \mid 1 \le i \le 7\}$ and $N^-(v_1) = \{v_2, v_3\}$. Assume, without loss of generality, that $(v_2, v_3) \in E(D)$. Since the maximum outdegree of $D$ is 7, there exist at least two vertices of $N^+(v_1)$ adjacent to $v_2$. Suppose $u_i$ is one of these vertices for some $i$. Then $S = \{u_i, v_1, v_2\}$ is an open dominating set. So $\rho_1(D) = 3$.

*Case* 4 od $v_1 = 8$. Let $N^+(v_1) = \{u_i \mid 1 \le i \le 8\}$ and $N^-(v_1) = \{v_2\}$. Since the maximum outdegree of $D$ is 8, there exists at least one vertex, say $u_i$, of

$N^+(v_1)$ adjacent to $v_2$. Then $S = \{u_i, v_1, v_2\}$ is an open dominating set. Therefore $\rho_1(D) = 3$. ❑

Finally, to show that the least integer $n$ for which $\mathrm{DOM}_1(K_n) = 4$ is $n = 11$, it is required to verify that $\mathrm{DOM}_1(K_{11}) \neq 3$.

**Theorem 10** $\mathrm{DOM}_1(K_{11}) = 4$.

**Proof** We need only show the existence of an orientation of $K_{11}$ with open domination number 4. We describe such an orientation $D$ as follows. Let $V(D) = \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5, w\}$. The tournament $D$ contains two strong subtournaments of order 5, say $T_1 = \langle\{u_1, u_2, \ldots, u_5\}\rangle$ and $T_2 = \langle\{v_1, v_2, \ldots, v_5\}\rangle$. Also the vertices of $T_2$ are adjacent to $w$, and the vertices of $T_1$ are adjacent from $w$. See Figure 5.

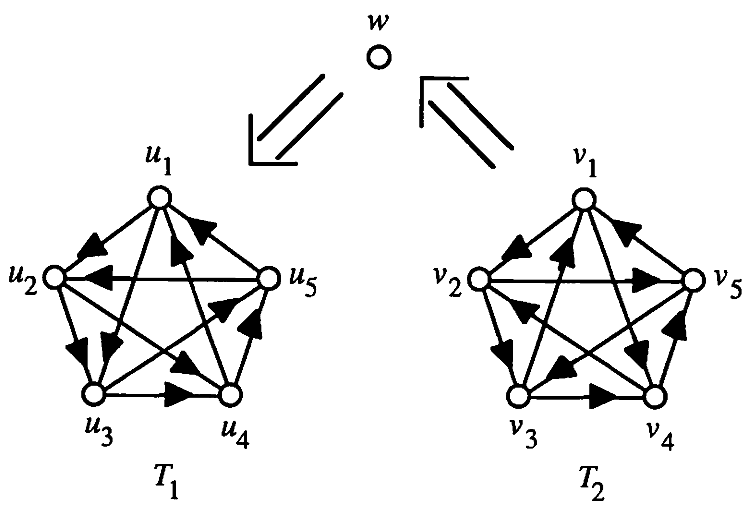



Figure 5

The orientations of edges between $V(T_1)$ and $V(T_2)$ remain to be described. We determine these orientations in the following way. There are precisely 5 directed 3-cycles in $T_2$. Let $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_3, v_4, v_5\}$, $S_3 = \{v_1, v_2, v_5\}$, $S_4 = \{v_2, v_3, v_4\}$, and $S_5 = \{v_1, v_4, v_5\}$ be the 5 sets of vertices which determine the directed 3-cycles of $T_2$. Now for each $i = 1, 2, 3, 4, 5$, let $u_i$ be adjacent to $V(S_i)$. All other arcs not mentioned thus far are directed from $V(T_2)$ to $V(T_1)$. The resulting orientation is the desired orientation $D$.

It remains to show that there is no minimum open dominating set of 3 vertices. Suppose, to the contrary, that $\rho_1(D) = 3$. Then there exists a directed 3-cycle, say $\langle S\rangle$, which openly dominates all vertices of $D$. Observe that $\langle S\rangle$ cannot be one of the triangles of $T_1$ since $w$ is adjacent to all vertices of $V(T_1)$. Also, by construction, each of the triangles of $T_2$ is openly dominated by some vertex of $V(T_1)$. So $\langle S\rangle$ cannot be one of the triangles of $T_2$. Hence we consider 3-cycles which contain vertices from both $T_1$ and $T_2$.

*Case* 1 $S = \{w, u_i, v_j\}$ *for some* $i$, *some* $j$. Then $\langle S\rangle$ is the cycle $w, u_i, v_j, w$. Now $u_i$ is directed toward a unique 3-cycle, say $C$, of $T_2$ where $v_j$ belongs to $C$. Of the two vertices of $V(T_2) - V(C)$, one is adjacent to $v_j$ and one is adjacent from $v_j$. Let $v_k$ be the vertex adjacent to $v_j$. Then $v_k$ openly dominates $S$. So $S$ cannot be an open dominating set of $D$.

*Case* 2 $S = \{u_i, u_j, v_k\}$ *for some* $i, j$ *and* $k$. There are 15 triangles of this type. However, each triangle is openly dominated by some vertex. (See Figure 6.) So $S$ cannot be an open dominating set of $D$.

| $S$ | openly dominated by vertex |
|---|---|
| $u_1, u_2, v_5$ | $u_5$ |
| $u_1, u_2, v_4$ | $u_5$ |
| $u_1, u_3, v_5$ | $v_4$ |
| $u_4, u_1, v_1$ | $v_5$ |
| $u_5, u_1, v_3$ | $u_4$ |
| $u_5, u_1, v_2$ | $u_4$ |
| $u_2, u_3, v_1$ | $u_1$ |
| $u_2, u_3, v_2$ | $u_1$ |
| $u_2, u_4, v_2$ | $v_1$ |
| $u_5, u_2, v_3$ | $v_2$ |
| $u_3, u_4, v_4$ | $u_2$ |
| $u_3, u_4, v_3$ | $u_2$ |
| $u_3, u_5, v_4$ | $v_3$ |
| $u_4, u_5, v_1$ | $u_3$ |
| $u_4, u_5, v_5$ | $u_3$ |

Figure 6

*Case* 3 $S = \{u_i, v_j, v_k\}$ *for some* $i, j$ *and* $k$. There are 15 triangles of this type. However, each triangle is openly dominated by some vertex. (See Figure 7.) Thus $S$ cannot be an open dominating set of $D$.

| $S$ | openly dominated by vertex |
|---|---|
| $v_1, v_2, u_5$ | $u_3$ |
| $v_3, v_1, u_2$ | $u_1$ |
| $v_3, v_1, u_4$ | $v_5$ |
| $v_1, v_4, u_3$ | $v_3$ |
| $v_1, v_4, u_1$ | $u_5$ |
| $v_5, v_1, u_2$ | $u_5$ |
| $v_2, v_3, u_3$ | $u_1$ |
| $v_4, v_2, u_2$ | $v_1$ |
| $v_4, v_2, u_5$ | $u_4$ |
| $v_2, v_5, u_1$ | $v_4$ |
| $v_2, v_5, u_4$ | $u_3$ |
| $v_3, v_4, u_1$ | $u_4$ |
| $v_5, v_3, u_3$ | $u_2$ |
| $v_5, v_3, u_5$ | $v_2$ |
| $v_4, v_5, u_4$ | $u_2$ |

Figure 7

We have now exhausted all possibilities. Thus there is no such open dominating set of 3 vertices, implying $\rho_1(D) \neq 3$. By Lemma 6, it follows that $\mathrm{DOM}_1(K_{11}) = 4$. ❑

## 3. An Intermediate Value Theorem for Orientable Open Domination in Complete Graphs

By the results of the preceding section, it follows that if $m$, $n$, and $c$ are positive integers such that $m \leq n$ and $\mathrm{DOM}_1(K_m) \leq c \leq \mathrm{DOM}_1(K_n)$, then there exists an integer $k$, with $m \leq k \leq n$, such that $\mathrm{DOM}_1(K_k) = c$. Hence we have a certain type of Intermediate Value Theorem. In this section, we consider the existence of another type of Intermediate Value Theorem. In particular, for a graph $G$, given an integer $c$ such that $\mathrm{dom}_1(G) \leq c \leq \mathrm{DOM}_1(G)$, does there exist an orientation $D$ of $G$ such that $\rho_1(D) = c$?

Although such a theorem has not yet been proven for an arbitrary graph $G$, the result does hold for complete graphs. We begin by establishing a few lemmas for graphs in general.

**Lemma 11** Let $G$ be a graph and let $v$ be any vertex of $G$. If $\mathrm{DOM}_1(G - v)$ is defined, then

$$\mathrm{DOM}_1(G) \geq \mathrm{DOM}_1(G - v) \geq \mathrm{DOM}_1(G) - 1.$$

**Proof** Assume $\mathrm{DOM}_1(G - v)$ is defined. Let $D - v$ be an orientation of $G - v$ such that $\rho_1(D - v) = \mathrm{DOM}_1(G - v)$. Define $D$ by directing all edges of $G$ incident with $v$ towards the vertex $v$. Notice that a minimum open dominating set of $D$ also openly dominates $D - v$. Hence $\rho_1(D) \geq \rho_1(D - v)$. Thus $\mathrm{DOM}_1(G) \geq \rho_1(D) \geq \rho_1(D - v) = \mathrm{DOM}_1(G - v)$.

Next let $D$ be any orientation of $G$ such that every vertex of $D$ has positive indegree. Then, for the given vertex $v$, there exists some arc, say $(u, v)$ in $D$. Now an open dominating set of $D$ can be formed from an open dominating set of $D - v$, possibly along with the vertex $u$. Thus $\rho_1(D) \leq \rho_1(D - v) + 1 \leq \mathrm{DOM}_1(G - v) + 1$. Since this is true for any valid orientation $D$ of $G$, it follows that $\mathrm{DOM}_1(G) \leq \mathrm{DOM}_1(G - v) + 1$, that is, $\mathrm{DOM}_1(G - v) \geq \mathrm{DOM}_1(G) - 1$. ❑

**Lemma 12** If $\mathrm{DOM}_1(G) > \mathrm{dom}_1(G)$, then there exists a vertex $v$ of $G$ such that

(a) $\mathrm{dom}_1(G - v)$ and $\mathrm{DOM}_1(G - v)$ are defined

(b) $\mathrm{dom}_1(G) \geq \mathrm{dom}_1(G - v) \geq \mathrm{dom}_1(G) - 1$

(c) either $\mathrm{DOM}_1(G - v) = \mathrm{DOM}_1(G)$ or $\mathrm{DOM}_1(G - v) = \mathrm{DOM}_1(G) - 1$.

**Proof** Clearly, if $G$ is a cycle, then $\mathrm{dom}_1(G) = \mathrm{DOM}_1(G)$. Thus, assuming $\mathrm{DOM}_1(G) > \mathrm{dom}_1(G)$, it follows that $G$ is not a cycle and $G$ is not a tree.

Now let $D$ be an orientation of $G$ such that every vertex has positive indegree. Let $S$ be a minimum open dominating set of $D$. Since $G$ is not a cycle, we know $|S| < n$. Hence there exists some vertex, say $v$, in $V(G) - S$. Now since $S$ is an open dominating set of $D$, it follows that $\langle S \rangle$ contains a cycle. Further every vertex of $V(G) - S$ is adjacent from some vertex of $S$. Consequently $G - v$ is connected and contains a cycle. In fact, every vertex of $D - v$ has positive indegree and $\rho_1(D - v)$ exists. So $\mathrm{dom}_1(G - v)$ and $\mathrm{DOM}_1(G - v)$ are defined, proving part (a).

Let $D$ be an orientation of $G$ such that $\rho_1(D) = \mathrm{dom}_1(G)$. Let $S$ be a minimum open dominating set of $D$. And let $v$ be a vertex of $V(G) - S$, as in the proof of part (a). Then $\mathrm{dom}_1(G) = \rho_1(D) \geq \rho_1(D - v) \geq \mathrm{dom}_1(G - v)$, proving the first inequality of part (b).

Next let $D - v$ be an orientation of $G - v$ such that $\mathrm{dom}_1(G - v) = \rho_1(D - v)$. Let $D$ be the orientation of $G$ formed by directing all edges incident with $v$ toward the vertex $v$. Then $\mathrm{dom}_1(G) \leq \rho_1(D) \leq \rho_1(D - v) + 1$. Hence $\mathrm{dom}_1(G) - 1 \leq \rho_1(D - v) = \mathrm{dom}_1(G - v)$, proving the second inequality in part (b).

The proof of part (c) follows directly from Lemma 11. ❑

**Lemma 13** Let $G$ be a connected graph such that $G$ is not a tree, and let $c$ be an integer such that

$$\mathrm{dom}_1(G) \leq c < \mathrm{DOM}_1(G).$$

Then there exists a sequence $v_1, v_2, \ldots, v_k$ of vertices of $G$ such that for $G_i = G - \{v_1, v_2, \ldots, v_i\}$, $1 \leq i \leq k$, we have

(a) $\text{dom}_1(G_i)$ and $\text{DOM}_1(G_i)$ are defined

(b) $\text{DOM}_1(G_k) = c$.

**Proof** The result is obvious if $c = \text{dom}_1(G)$. Thus we assume $\text{dom}_1(G) < c$. We proceed iteratively.

By Lemma 12, there exists a vertex, say $v_1$, such that $\text{dom}_1(G_1)$ and $\text{DOM}_1(G_1)$ are defined and either $\text{DOM}_1(G_1) = \text{DOM}_1(G)$ or $\text{DOM}_1(G_1) = \text{DOM}_1(G) - 1$. We consider the following two cases.

*Case* 1 $\text{DOM}_1(G_1) = \text{DOM}_1(G)$. Then, by Lemma 12, $\text{dom}_1(G_1) \le \text{dom}_1(G) < \text{DOM}_1(G) = \text{DOM}_1(G_1)$. That is, $\text{dom}_1(G_1) < \text{DOM}_1(G_1)$.

*Case* 2 $\text{DOM}_1(G_1) = \text{DOM}_1(G) - 1$ and $c < \text{DOM}_1(G_1)$. Then, by Lemma 12, $\text{dom}_1(G_1) \le \text{dom}_1(G) < c < \text{DOM}_1(G_1)$. That is, $\text{dom}_1(G_1) < \text{DOM}_1(G_1)$.

Observe that for each of these cases, the graph $G_1$ satisfies the hypothesis of Lemma 12. Thus there exists a vertex, say $v_2$, such that $\text{DOM}_1(G_2)$ is defined, and the process continues.

In general, as long as $\text{dom}_1(G_i) < \text{DOM}_1(G_i)$, there exists a vertex $v_{i+1}$ such that $\text{DOM}_1(G_{i+1})$ is defined. Further, the process continues as long as

(1) $\text{DOM}_1(G_{i+1}) = \text{DOM}_1(G_i)$ or

(2) $\text{DOM}_1(G_{i+1}) = \text{DOM}_1(G_i) - 1$ and $c < \text{DOM}_1(G_{i+1})$

We claim that the process terminates, that is, there exists $k$ such that $\text{DOM}_1(G_k) = \text{DOM}_1(G_{k-1}) - 1$ and $c = \text{DOM}_1(G_k)$. Let $k - 1$ be the largest integer such that $\text{dom}_1(G_{k-1}) \le c < \text{DOM}_1(G_{k-1})$. Then, it follows that there exists a vertex $v_k$ such that $\text{DOM}_1(G_k)$ is defined and $\text{DOM}_1(G_k) \le c$. But we know either $\text{DOM}_1(G_k) = \text{DOM}_1(G_{k-1})$ or $\text{DOM}_1(G_k) = \text{DOM}_1(G_{k-1}) - 1$. Since $\text{DOM}_1(G_{k-1}) > c$, we must have

$\mathrm{DOM}_1(G_k) = \mathrm{DOM}_1(G_{k-1}) - 1$. Hence $c \geq \mathrm{DOM}_1(G_k) = \mathrm{DOM}_1(G_{k-1}) - 1 > c - 1$, implying $\mathrm{DOM}_1(G_k) = c$. ❑

Finally, we have an Intermediate Value Theorem for the upper orientable open domination number of a complete graph.

**Theorem 14** Let $c$ be an integer such that $\mathrm{dom}_1(K_n) \leq c \leq \mathrm{DOM}_1(K_n)$. Then there exists an orientation $D$ of $K_n$ such that $\rho_1(D) = c$.

**Proof** Certainly if $c = \mathrm{dom}_1(K_n)$ or $c = \mathrm{DOM}_1(K_n)$, the result is clear. Thus we assume $\mathrm{dom}_1(K_n) < c < \mathrm{DOM}_1(K_n)$. By Lemma 13, there exists a set of vertices $W = \{v_1, v_2, \dots, v_k\}$, $k \geq 1$, such that $\mathrm{DOM}_1(K_n - W)$ is defined and $\mathrm{DOM}_1(K_n - W) = c$. Let $D'$ be an orientation of $K_n - W$ such that $\rho_1(D') = \mathrm{DOM}_1(K_n - W)$. Form an orientation $D$ of $K_n$ from $D'$ by letting $\langle D - W \rangle = D'$ and directing all edges incident with the vertices of $W$ toward $W$. Now let $S$ be a minimum open dominating set of $D'$, that is, $\rho_1(D') = |S|$. By construction, the arc $(u, v_i)$ belongs to $D$ for every $u \in S$ and for every $v_i \in W$. Thus every vertex of $W$ is openly dominated by a vertex of $S$. Hence, $S$ is an open dominating set of $D$. Further $S$ is, in fact, a minimum open dominating set, for otherwise we would have $\rho_1(D') < |S|$. Therefore

$$\rho_1(D) = |S| = \rho_1(D') = \mathrm{DOM}_1(K_n - W) = c. \quad ❑$$

Another consequence of the previous lemmas is the following.

**Corollary 15** Let $G$ be a connected graph which is not a tree. Let $c$ be any integer such that $\mathrm{dom}_1(G) \leq c \leq \mathrm{DOM}_1(G)$. Then $G$ contains an induced subgraph $H$ such that $\mathrm{DOM}_1(H) = c$.

We conclude with a conjecture.

**Conjecture** Let $G$ be a graph. If $c$ is an integer for which $\mathrm{dom}_1(G) \leq c \leq \mathrm{DOM}_1(G)$, then there exists an orientation $D$ of $G$ such that $\rho_1(D) = c$.

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