ON r–TYPE–CONSTRUCTIONS AND Δ–COLOUR–CRITICAL GRAPHS

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To the memory of M.L. Mehrabadi, mathematician and friend.

Abstract

In this paper we first generalize a classical result of B. Toft (1974) on r–type–constructions for graphs (rather than hypergraphs) and then we show how the result can be used to construct colour–critical graphs with a special focus on Δ–colour–critical graphs. This generalization covers most of known constructions which generate small critical graphs. We also obtain some upper bounds for the *minimum excess function* $\eta(k,p)$ when $4 \leq k \leq 6$; where

$$\eta(k,p) = \min_{G \in K(k,p)} \epsilon(G),$$

in which $\epsilon(G) = 2|E(G)| - |V(G)|(k-1)$, and $K(k,p)$ is the class of all k–colour–critical graphs on p vertices with $\Delta = k$. We use the techniques to construct an infinite family of Δ–colour–critical graphs for $\Delta = 5$ with a relatively small minimum excess function; and we prove that $\eta(6, 6n) \leq 6(n-1)$ $(n \geq 2)$ which shows that there exists an infinite family of Δ–colour–critical graphs for $\Delta = 6$.

1 Preliminaries And The Background

First, we go through some basic definitions. In this paper, $\mathrm{N} = \{1, 2, \ldots\}$ is the set of *natural numbers*. For any finite set X, $|X|$ is the *size* of X, i.e. the number of elements of X, and $P(X)$ is the *power set* of X, i.e. the set of all subsets of X. Also, for any statement $\mathcal{P}$, $\neg\mathcal{P}$ is the negation on $\mathcal{P}$.

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A *hypergraph* H is an ordered pair $(V(H), E(H))$ in which $V(H)$ is a finite nonvoid set of *vertices* and $E(H) \subset P(V(H))$ is such that each element of $E(H)$ contains at least two elements of $V(H)$. The elements of $E(H)$ are called *edges* of H; and an edge of H consisting of the subset A of $V(H)$ is denoted by (A). If $A = \{x, y\}$ then (A) is called a *simple edge* and we say that (A) *joins* x and y, or x is *connected* to y; and we write $x \leftrightarrow y$ or $A = xy$. An edge (A) with $|A| > 2$ is called a *hyperedge*. We usually use capital letters for hyperedges and small letters for simple edges of a hypergraph. Also, $H + (A)$ is the hypergraph obtained by adding the edge (A) to H; and $H - (A)$ is defined similarly. A hypergraph in which each edge consists of precisely n vertices is called *n-uniform*. A *graph*, G, is a 2-uniform hypergraph.
For two hypergraphs H_1 and H_2 we write $H_1 \leq H_2$ if $V(H_1) \subseteq V(H_2)$ and $E(H_1) \subseteq E(H_2)$ and we call H_1 a *subhypergraph* of H_2. Also, for a hypergraph H and $X \subseteq V(H)$, $H[X]$ denotes the (sub)hypergraph *induced* on X with $V(H[X]) = X$ and $E(H[X])$ consists of those edges of H whose vertices are completely contained in X.
A *k-colouring* of a hypergraph H is a map $\sigma : V(H) \longrightarrow \mathbf{N}$ such that the restriction of σ to any edge of H is non-constant and $|\sigma(V(H))| = k$. In this case, any $j \in \sigma(V(H))$ is called a *colour* and $\sigma^{-1}(j)$ which we denote by $[j]$ is called the *colour-class* of colour j. The least integer k for which H admits a k-colouring is called the *chromatic number* of H and is denoted by $\chi(H)$; and in this case, H is said to be a *k-chromatic* hypergraph. A hypergraph H is called *k-edge-colour-critical*, or *k-critical* for short, if $\chi(H) = k$ and for any edge (A) of H, $\chi(H - (A)) < \chi(H)$.
An *independent* set of vertices $I \subseteq V(G)$ in a graph G is a set of vertices such that none of them are connected to each other. Note that for a k-colouring of a graph G each colour-class is an independent set of vertices. A graph G is said to be *k-uniquely-vertex-colourable* if $k = \chi(G)$ and any k-colouring of G induces the same colour-classes on G. Such a graph is called a k-UCG for short.
K_n is the *complete graph* on n vertices for which $E(K_n)$ contains any subset of $V(K_n)$ whose size is 2. $cl(G)$, the *clique number* of a graph G, is the maximum number m such that $K_m \leq G$; and the *coclique number* of a graph G is defined to be $ccl(G) = \chi(G) - cl(G)$. Also, for any graph G, $N_S(v) = \{x \mid v \leftrightarrow x \text{ in } S\}$ is the *neighbourhood* of the vertex v in the subgraph $S \leq G$ and $deg_S(v) = |N_S(v)|$ is the *degree* of the vertex v in S. Note that, for simplification, we omit the subscript if $S = G$. Moreover, *splitting* a vertex x of a graph G into a set X of new vertices means replacing x by an independent set of new vertices X such that the neighbourhoods $\{N_G(z)\}_{z \in X}$ partition $N_G(x)$.
Consider a graph G and a collection of nonvoid subsets of $P(V(G))$ such as $\mathcal{F} = (W_1, \ldots, W_l)$ with $W_i \in P(V(G))$ for all $1 \leq i \leq l$. In this setting,

note that, it is possible that a subset of $P(V(G))$ appear more than once in the collection. Also, there are situations throughout this paper that such collections appear naturally as domains of some maps. Therefore, we make this a rule to consider a collection $\mathcal{F} = (W_1, \ldots, W_t)$ as a set of ordered pairs as (i, W_i) in which the first component is used as a counter. In this paper, this is called a *list* $\mathcal{F} \in \mathbf{N} \times P(V(G))$ of subsets of $P(V(G))$.
Considering the background of the subject of this paper, maybe, the following conjecture of Borodin and Kostochka [5, 17] can be distinguished as one of the most important motivations for the study of Δ-critical graphs.

Conjecture 1.[5] *In any graph G if $\Delta(G) \geq 9$ and $cl(G) < \Delta(G)$ then $\chi(G) < \Delta(G)$.*

Note that the conjecture essentially means

$$(\Delta \geq 9 \;\&\; \chi = \Delta) \quad \Rightarrow \quad ccl = 0.$$

Also, it is proved by B. Reed that the conjecture is correct for sufficiently large chromatic numbers [17, 24]. In [3] Beutelspacher and Hering prove the following.

Theorem 1.[3] *Let G be a graph with $\Delta(G) = \chi(G) = k$ and $ccl(G) > 0$. Then,*

a) G has at least $2k - 1$ vertices.

b) If G has exactly $2k - 1$ vertices, then $k \leq 8$.

The following interesting theorem is also stated without proof.

Theorem 2.[3] *Let G be a critical graph with $\Delta(G) = \chi(G) = k \geq 5$ and $ccl(G) > 0$. Then, for any vertex v (or any edge e) there exists a $(k-1)$-critical subgraph H containing v (or e) such that $\Delta(H) = \chi(H) = k - 1$ and $ccl(H) > 0$.*

Beutelspacher and Hering claim that there are only 13 k-critical graphs on $2k - 1$ vertices with $\Delta = \chi = k$; and they state that they have been able to find all of them using a computer search.
There are also some results about the minimal counterexample approach. In this regard, assume that G is a k-chromatic graph with $k = \Delta$ and $ccl(G) > 0$ which is vertex-minimal. Then if $k \geq 9$ Mahmoodian *et.al.* have proved that G is critical, any two $(k-1)$-cliques are disjoint in G and that $|V(G)| \geq 3k - 9$ [21].
On the other hand, when one considers the structure of Δ-critical graphs for small cases ($4 \leq \Delta \leq 8$), one encounters graphs with very interesting structures. It is also interesting to note that these structures are very much

related to small UCG's and small critical graphs where k–critical graphs with small *excess function*,

$$\epsilon(G) = \sum_{v \in V(G)} (deg(v) - (k-1)) = 2|E(G)| - |V(G)|(k-1),$$

are studied [2, 11, 16, 18, 19].
In the next section, first we focus on some classical results of B. Toft on (amalgam) r–type–construction [26] which we briefly recall in what follows. In this direction, we partially generalize the r–type–construction of Toft for graphs and we show how it can be used to construct Δ–critical graphs with small Δ. Also, in Section 3 we prove that there exists an infinite family of Δ–critical graphs with $\Delta = 6$ on $6n$ vertices for each $n > 1$; and we obtain some upper bounds for the *minimum excess function* of Δ–critical graphs when $\Delta \leq 6$.

Theorem 3.[26] *Let $k \geq 3$ be an integer and H_1 and H_2 denote two disjoint hypergraphs and let $X_i \subseteq V(H_i)$ for $i = 1, 2$, where $|X_1| = |X_2| \geq 2$, $X_1 \neq V(H_1)$ and $(X_2) \in E(H_2)$. Also, let H denote the hypergraph obtained from H_1 and $H_2 - (X_2)$ by identifying the two sets of vertices X_1 and X_2 and identifying nothing else. Let H_1 satisfy the following conditions.*

a) *H_1 is connected.*

b) $\chi(H_1) \leq k - 1$.

c) *In any $(k-1)$–colouring of H_1 the vertices of X_1 have at least two different colours.*

d) *For all $(k-1)$–colourings σ of $H_1[X_1]$ in which the vertices of X_1 have at least two different colours, there exists a $(k-1)$–colouring σ_1 of H_1 such that the restriction of σ_1 to X_1 is σ.*

e) *For all edges A_1 of H_1 there exists a $(k-1)$–colouring of $H_1 - (A_1)$ in which the vertices of X_1 all have the same colour.*

Then, H is k–critical if and only if H_2 is k–critical.

Toft considers the class $\mathcal{G}_k$ of graphs H_1 which satisfy conditions (a) to (e) of the preceding theorem where $X_1 \subseteq V(H_1)$, $X_1 \neq V(H_1)$ and $|X_1| \geq 2$. He defines a vertex x_1 of a k–critical graph G, $(k \geq 3)$ to be *universal* if any graph H_1 obtained from G by splitting x_1 into a set of vertices X_1 belongs to $\mathcal{G}_k$. We recall the following results from [26].

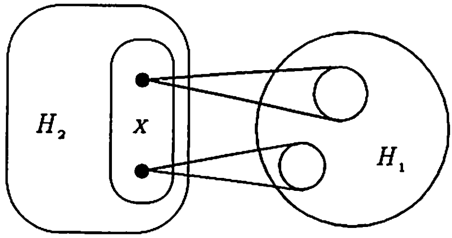

Figure 1: The graph H (Corollary 1)

Theorem 4.[26]

a) *A vertex x_1 of a k-critical graph G, $(k \geq 3)$ is universal if and only if the graph H_1 obtained from G by splitting x_1 into a set of vertices X_1, where $|X_1| = deg_G(x_1)$, satisfies the condition (d) of Theorem 3.*

b) *Let G be a k-critical graph $(k \geq 3)$ and let $x \in V(G)$. If $deg_G(x) \leq 2k-3$ then x is universal in G, except perhaps if $k = 4$ and $deg_G(x) = 5$.*

c) *Let G^* denote a $(k+1)$-chromatic graph obtained from a k-critical graph G, $(k \geq 3)$ by joining a new vertex y to all vertices of G. If x is a universal vertex of G, then x is also universal in G^*.*

d) *Let G_1 and G_2 be two disjoint k-critical graphs $(k \geq 3)$ and let G denote a k-critical graph obtained from G_1 and G_2 by the construction of Hajós, identifying the vertices x_1 and x_2 of G_1 and G_2, respectively, to the vertex x of G. If x_1 and x_2 are universal in G_1 and G_2, respectively, then x is universal in G.*

e) *If G^* is a 4-critical graph obtained from an odd cycle G by joining a new vertex y to all vertices of G, then y is universal in G^*.*

f) *Let G^* denote a $(k+1)$-chromatic graph $(k \geq 2)$ obtained from a k-critical graph G by joining a new vertex y to all vertices of G. If H is a graph obtained from G^* by splitting y into a set Y of new vertices and if the vertices of Y are coloured with precisely two different colours such that they do not all have the same colour, then this colouring of Y may be extended to a k-colouring of H.*

The following statement can be considered as a simplified version of Theorem 3, however, since we are going to use it explicitly in the sequel, we state and prove it to show our general view to this type of constructions.

Corollary 1. *Let $k \geq 3$ be an integer, H_1 be a $(k-1)$-critical graph, and H_2 be a $(k-1)$-chromatic graph with $X \subseteq V(H_2)$ such that,*

a) $|X| > 1$.

b) For any $(k-1)$-colouring σ of H_2, $|\sigma(X)| = 1$.

c) For any edge $e \in E(H_2)$ there is a $(k-1)$-colouring, σ, of $H_2 - e$ for which $|\sigma(X)| = 2$.

Now construct a graph H_1^* by joining a new vertex y to all vertices of H_1 and then splitting y into a set Y with $|Y| = |X|$. Let H be the graph formed from H_1^* and H_2 by identifying the two sets X and Y and identifying nothing else (Figure 1). Then H is a k-critical graph which is not a k-clique.

Proof. Trivially, H is not a clique. Now for any $(k-1)$-colouring of H_2 change the colour of one of the vertices in X to the kth colour; then by Theorem 4 (f), the k-colouring so obtained extends to a k-colouring of H. But any $(k-1)$-colouring of H would restrict to $(k-1)$-colourings of H_1^* and H_2 making incompatible demands on $X(=Y)$. Thus, $\chi(H) = k$.
On the other hand, condition (c) together with criticality of H_1, implies that H is k-critical. □

Note that as a classical example one can use $H-(X)$ as H_2 in Corollary 1 when H is a k-critical hypergraph with only one hyperedge (X). Also such constructions has been frequently used in the construction of critical graphs (for instance see [1]). Actually, we will see in the sequel that there is a close relationship between *k-chromatic graphs for which there is a fixed colour class in any k-colouring*, UCG's and *small critical graphs*. We leave our discussion on these relationships for the last section; however, we prefer to state a theorem which is actually a generalization of Lemma 4 in [27] and can be considered as the root of what will appear in the next section. In this regard we start with the following definition.

Definition 1. Consider a $(k-1)$-chromatic graph G. Then a list $\mathcal{F} = \{(i, W_i) \mid 1 \leq i \leq l\}$ of subsets of $P(V(G))$ is called a *transverse system*[3] for G if both of the following conditions are satisfied.

- For every $(k-1)$-colouring σ of G, if $(i, W) \in \mathcal{F}$ then W has nonempty intersection with all colour classes of σ.
- For every k-colouring $\sigma : V(G) \xrightarrow{\text{onto}} \{1, \ldots, k\}$ of G, there exists $(i, W) \in \mathcal{F}$ such that W has nonempty intersection with all colour classes of σ.

◇

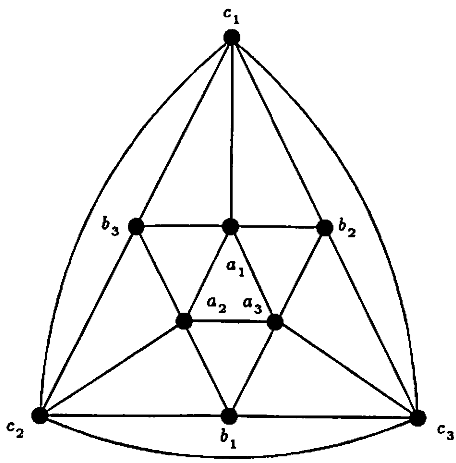

Figure 2: The graph D4.

Example 1. Consider the prism $\mathsf{P} = \mathsf{D4} - \{b_1, b_2, b_3\}$ (see Figure 2) and note that

$$\mathcal{F} = \{(1, \{a_1, a_2, c_1, c_2\}), (2, \{a_2, a_3, c_2, c_3\}), (3, \{a_3, a_1, c_3, c_1\})\}$$

is a transverse system for P. ◇

Theorem 5.[11] *Let H be a k-chromatic graph such that in every k-colouring of H there is a fixed colour-class V^* consisting of m specified vertices $v_1, \ldots, v_m$, $(m \geq 1)$; and consider $G = H - V^*$. Also define $\mathcal{F} = \{(i, N_G(v_i)) \mid v_i \in V^*\}$. Then,*

a) $\chi(G) = k - 1$ and $\mathcal{F}$ is a transverse system for G.

Moreover if $cl(H) \leq k - 1$ then,

b) $cl(G[W]) \leq k - 2$ for every $(i, W) \in \mathcal{F}$, and $cl(G) \leq k - 1$.

Conversely, let G be a $(k-1)$-colourable graph and let $\mathcal{F} \subseteq \mathrm{N} \times P(V(G))$ be a transverse system for G. Then the graph H obtained by adding to G new vertices v_i, for each $((i, W_i) \in \mathcal{F})$, and joining each v_i to all vertices in W_i is a k-chromatic graph such that in any one of its k-colourings the class $V^* = \{v_i \mid (i, W_i) \in \mathcal{F}\}$ is fixed. If in addition (b) is also fulfilled then $cl(H) \leq k - 1$.

[3]The author is very grateful to the anonymous referee who suggested this formulation and the name *transverse system* for this concept.

Proof. First note that for (a), if V^{*} is a fixed colour–class in H and $G = K_{k-1}$, then $N_G(v) = V(G)$ for all $v \in V^{*}$; or otherwise, v can take the colour of any vertex of G which is not connected to it. Also, if $G \neq K_{k-1}$ then the first condition of the transverse system is clearly satisfied. For the next condition, assume (by contradiction) that there exists a k–colouring σ of G such that for all $v \in V^{*}$, there exists a colour c_v which does not appear in $N_G(v)$. Then we can extend this k–colouring to H by colouring v with c_v and this is a contradiction. Moreover, if $cl(H) \leq k-1$ then (b) is clearly satisfied.
The converse also follows similarly. □

As an application of this theorem we can consider the transverse system of Example 1 and deduce that $\{b_1, b_2, b_3\}$ is a fixed colour–class in any 4–colouring of D4 (see Figure 2). Also, note that by joining any two vertices of $\{b_1, b_2, b_3\}$ we obtain a 5–chromatic graph with $ccl = 1$ and $\chi = \Delta = 5$. For more on this theorem and related subjects see [8, 11].

2 The Generalization

In this section we partially generalize Theorem 3 for graphs in two steps. First, we note that the theorem essentially means that H_1 and H_2 reject their $(k-1)$–colourings through X; considering the contradictory properties *taking the same colour* and *taking at least two different colours* in each $(k-1)$–colouring. This will make sure that the chromatic number of the new graph is greater than or equal to k. Also, the extension properties will guarantee edge–minimality and consequently we obtain a k–critical graph. We generalize the above concepts in the following definition.

Definition 2. Let G be a graph, $X \subseteq V(G)$, $k > 1, \kappa > 0$, and $|X| > \kappa$. Then, we call G to be of type $\mathcal{L}(X, k, \kappa)$ if

l1) G is k–colourable.

l2) For any k–colouring of G such as σ we have $0 < |\sigma(X)| \leq \kappa$.

l3) For any edge $e \in E(G)$ there is a k–colouring of $G - e$ such as σ^{*} such that $|\sigma^{*}(X)| > \kappa$.

l4) For any k–colouring σ of $G[X]$ for which $|\sigma(X)| \leq \kappa$ there is a k–colouring σ^{*} of G such that the restriction of σ^{*} to X is σ.

Also, we call G to be of type $\mathcal{M}(X, k, \kappa)$ if

m1) G is k–colourable.

m2) For any k–colouring of G such as σ we have $|\sigma(X)| > \kappa$.

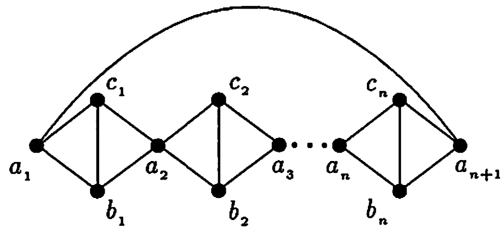

Figure 3: 4–critical graphs T_n^4 with $3n+1$ vertices $n > 1$.

m3) For any edge $e \in E(G)$ there is a k–colouring of $G - e$ such as σ^* such that $|\sigma^*(X)| \leq \kappa$.

m4) For any k–colouring σ of $G[X]$ for which $|\sigma(X)| > \kappa$ there is a k–colouring σ^* of G such that the restriction of σ^* to X is σ.

$\diamond$

Now, we can generalize Theorem 3 as follows.

Theorem 6. *Let $k > 2, \kappa > 0$ and let H_1 and H_2 be two graphs such that H_1 is of type $\mathcal{M}(X_1, k-1, \kappa)$, H_2 is of type $\mathcal{L}(X_2, k-1, \kappa)$ and $|X_1| = |X_2| > 1$. Then the graph H obtained by identifying the two sets of vertices X_1 and X_2 is k-critical.*

Proof. First, note that for any $(k-1)$–colouring σ of H_1 and H_2 we have $|\sigma(X_1)| > \kappa$ and $|\sigma(X_2)| \leq \kappa$. Hence $\chi(H) \geq k$.
Also, for any edge $e \in E(H_i)$, $(i = 1, 2)$ there is a $(k-1)$–colouring of $H - e$ by the hypothesis; and consequently H is k–critical. $\square$

Example 2. Note, that if G is a k–critical hypergraph with only one hyperedge (A), then $G - (A)$ is of type $\mathcal{L}(V(A), k-1, 1)$. Also, it is clear that any graph H_1 obtained from splitting the universal vertex x of a k–critical graph G into a new set of vertices X is of type $\mathcal{M}(X, k-1, 1)$. Hence, Theorem 3 can be considered as the special case of Theorem 6 for $\kappa = 1$.
The preceding paragraph shows that graphs of type $\mathcal{L}(X, k-1, 1)$ can be applied to construct new colour–critical graphs. Therefore, we note again

that if G is a graph of type $\mathcal{L}(\{u,v\},k-1,1)$, then one can consider n identical copies of G such as G_i's ($i=1,\ldots,n$) with the corresponding subsets $\{u_i,v_i\}$ such that each G_i is of type $\mathcal{L}(\{u_i,v_i\},k-1,1)$, and one can construct a new graph $\tilde{G}$ with identifying v_i with v_{i+1} for i odd and identifying u_i with u_{i+1} for i even. Now, it is easy to see that $\tilde{G}$ is of type $\mathcal{L}(\tilde{X}\cup\{u_1,z\},k-1,1)$ in which identified vertices are not distinguished, $\tilde{X}\subseteq\cup_{i=1}^{n}\{u_i,v_i\}$, $z=u_n$ for n even and $z=v_n$ for n odd.
As an example for this construction consider the graph $G=K_4-e$ in which the two ends u,v of the edge e are forced to take the same colour in any 3–colouring of G. Hence, G is of type $\mathcal{L}(\{u,v\},3,1)$. If we apply the above construction with $r=n$ and the partition $\{\{u\},\{v\}\}$ we obtain the graph $\mathsf{T}_n^4-a_1a_{n+1}$ of Figure 3 which is of type $\mathcal{L}(\{a_1,a_{n+1}\},3,1)$. Also, note that if we apply Theorem 6 with K_2 as a graph of type $\mathcal{M}(\{a_1,a_{n+1}\},3,1)$ we obtain the 4–critical graphs T_n^4.
It is not surprising to note that the above construction and Theorem 6 can be considered as generalizations of the well–known Hajós Construction. We note that it also covers a classical construction of G.A. Dirac for the family $\mathcal{D}_k$ and a construction of Kostochka and Stiebitz for the family $\mathcal{F}_k$ of edge–minimal k–critical graphs. [13, 14, 16, 18]. ◇

Example 3. In this example we wish to note that if G is a graph of type $\mathcal{L}(X,k,\kappa)$ then $G-e$ is not necessarily of type $\mathcal{L}(X,k,\kappa+1)$, even if l1, l2 and l3 are satisfied.
For this, consider the graph D4 depicted in Figure 2. It can be easily checked that the graph has only two 4–colourings in both of which $B=\{b_i \mid i=1,2,3\}$ is a fixed colour–class (see [7, 9, 10] and Theorem 5). Hence, it is easy to see that D4 is of type $\mathcal{L}(B,4,1)$. However, D4 $-\, b_1a_3$ is not of type $\mathcal{L}(B,4,2)$, although, l1, l2 and l3 are satisfied.
For this first note that there is no 4–colouring of D4 $-\, b_1a_3$ in which B takes three different colours. Moreover, consider the following colouring of B in D4 $-\, b_1a_3$.

$$\{b_1,b_2\}\subseteq[1],\quad b_3\in[2].$$

Then without loss of generality we may assume that $a_1\in[3]$; which forces c_1 to take the colour 4. This forces c_2 to take the colour 3, and consequently

$$c_3\in[2],\quad a_2\in[4].$$

This forces a_3 to take a new colour which shows that this colouring of B has no extension to a 4–colouring of D4 $-\, b_1a_3$; and this contradicts l4. ◇

Example 4. Consider the complete graph K_t, add m new vertices $X=\{x_i \mid i=1,\ldots,m\}$ and join each x_i to all vertices of K_t. Denote this new

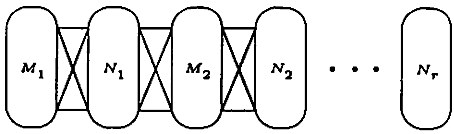

Figure 4: The graph $K^*_{m,n,r}$ (Example 4)

graph by K_t^m. Then it is easy to see that if $k > t$ and $m > k - t$, K_t^m is of type $\mathcal{L}(X, k, k-t)$.
Also, consider the graph $K^*_{m,n,r}$ depicted in Figure 4, in which there are r copies of K_{m+n} along with a partition of its vertices $\{A, B\}$ such that $|A| = m$ and $|B| = n$. Assume that M_i and N_i be the subgraphs of the ith copy of K_{m+n} which are induced on A and B respectively; and connect each vertex of N_i to each vertex of M_{i+1} $(i = 1, \ldots, r-1)$. Then if we define $X = V(M_1) \cup V(N_r)$ it is easy to see that $K^*_{m,n,r}$ is of type $\mathcal{M}(X, m+n, m+n-1)$.
Now, as an application of Theorem 6 consider graphs $K^*_{2,2,r}$ $(r > 1)$ and K_1^4 and apply the theorem with $X_1 = V(M_1) \cup V(N_r)$ and $X_2 = \{x_i \mid i = 1, \ldots, 4\}$. This gives rise to the graph of Figure 5 as a Δ–critical graph with $\Delta = 5$. Also, we should note that the same idea covers a construction of Kostochka and Stiebitz for the class $\mathcal{E}_k$ of edge–minimal k–critical graphs [18].
As one more application, note that if we split any vertex of a K_7 to 3 new vertices $X = \{x_i \mid i = 1, 2, 3\}$, then we obtain a graph of type $\mathcal{M}(X, 6, 1)$. Hence, if we identify these new vertices with the three vertices of K_5^3 we obtain a Δ–critical graph for $\Delta = 7$. This construction for different vertices of the K_7, recursively, gives rise to Δ–critical graphs with $\Delta = 7$ and $7n$ vertices for $n = 2, \ldots, 8$ (the symmetric graph with 56 vertices was independently constructed by R. Naserasr [22]). ◇

At this stage we wish to consider Theorem 6 once again. Intuitively, this theorem describes how the graphs H_1 and H_2 reject their $(k-1)$–colourings through X; however, it is quite conceivable to think about cases for which different $(k-1)$–colourings are rejected through different subsets of vertices. This is the main idea of the next part of this section which can be considered as our second step to the generalization of Theorem 3.
In this direction we, first, introduce a generalization of our previous definition for graphs of types $\mathcal{L}$ and $\mathcal{M}$. To do this, we need to define the concept of a *colouring property* with respect to a colouring σ and a list $\mathcal{F} \subseteq \mathbf{N} \times P(V(G))$, however, a precise definition of this concept needs

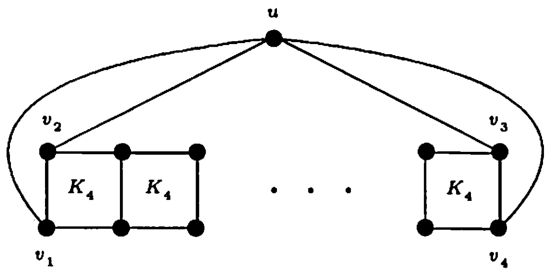

Figure 5: A class of 5–critical graphs with $2n + 1$ copies of K_4 $(n > 0)$.

some more preliminaries from logic. Therefore, we try to define the concept as precise as it is possible at this stage and we provide some concrete examples to clarify it. These examples are essentially the most important colouring properties which will be used in the sequel.

Definition 3. Let G be a graph. Then a *colouring property* on G is a (logical) statement $\mathcal{P}(\sigma, \mathcal{F})$ which contains a colouring σ and elements of a list $\mathcal{F} \subseteq \mathbf{N} \times P(V(G))$ as variables along with some constants called *parameters* and (possibly) some quantifiers. We show the truth values of this colouring property as $\mathcal{P}(\sigma, (i, W))$ for different elements of $\mathcal{F}$.
Two colouring properties are said to be of the same *type* if they have the same logical statements possibly with different parameters. ◇

Example 5. Let G be a graph, $\mathcal{F} \subseteq \mathbf{N} \times P(V(G))$ a list and consider $\phi : \mathcal{F} \longrightarrow \mathbf{N}$. Then $\mathcal{P}_\phi(\sigma, \mathcal{F})$ which is defined as

$$\forall\, (i, W) \in \mathcal{F} \quad |\sigma(W)| \leq \phi((i, W))$$

is a colouring property with parameter ϕ. We show this colouring property by $\mathcal{P}_\phi$ (note that we always trivially have $0 < |\sigma(W)|$ for all $(i, W) \in \mathcal{F}$). On the other hand, we may also strengthen this property to obtain the colouring property $\mathcal{P}^*_\phi$ which is defined as,

$$\forall\, (i, W_i), (j, W_j) \in \mathcal{F} \quad |\sigma(W_i)| \leq \phi((i, W_i)) \quad \& \quad \sigma(W_i) \neq \sigma(W_j) \quad (i \neq j).$$

Note that in some cases, this new property can also be described by upper and lower bounds on the number of colours. For instance, if ϕ is constant and equal to 1, the property is equivalent to conditions

$$\forall\, (i, W_i), (j, W_j) \in \mathcal{F} \quad |\sigma(W_i)| \leq 1 \quad \& \quad 2 \leq \sigma(W_i \cup W_j) \quad (i \neq j).$$

We will use these properties in forthcoming examples. Also, note that negating a colouring property produces a new colouring property. ◇

Definition 4. Let G be a graph. Consider a list $\mathcal{F} \subseteq \mathbf{N} \times P(V(G))$ and assume that $k > 1$. Also let $\mathcal{P}(\sigma, \mathcal{F})$ be a colouring property. Then, we call G to be of type $\mathcal{L}(\mathcal{F}, k, \mathcal{P})$ if

L1) G is k–colourable.

L2) For any k–colouring of G such as σ, $\mathcal{P}(\sigma, \mathcal{F})$ is true.

L3) For any edge $e \in E(G)$ there is a k–colouring of $G - e$ such as σ^* such that $\neg\mathcal{P}(\sigma^*, \mathcal{F})$ is true.

L4) If $X = \bigcup_{(i,W)\in\mathcal{F}} W$ then for any k–colouring σ of $G[X]$ for which $\mathcal{P}(\sigma, \mathcal{F})$ is true, there is a k–colouring σ^* of G such that the restriction of σ^* to X is σ.

Also, we call G to be of type $\mathcal{M}(\mathcal{F}, k, \mathcal{P})$ if and only if it is of type $\mathcal{L}(\mathcal{F}, k, \neg\mathcal{P})$; i.e.,

M1) G is k–colourable.

M2) For any k–colouring of G such as σ, $\neg\mathcal{P}(\sigma, \mathcal{F})$ is true.

M3) For any edge $e \in E(G)$ there is a k–colouring of $G - e$ such as σ^* such that $\mathcal{P}(\sigma^*, \mathcal{F})$ is true.

M4) If $X = \bigcup_{(i,W)\in\mathcal{F}} W$ then for any k–colouring σ of $G[X]$ for which $\neg\mathcal{P}(\sigma, \mathcal{F})$ is true, there is a k–colouring σ^* of G such that the restriction of σ^* to X is σ.

◇

Example 6. Consider the colouring property $\mathcal{P}_\phi(\sigma, (i, W))$ of Example 5 and first, note that if $\mathcal{F} = \{(1, X)\}$ and $\phi((1, X)) = \kappa$ then we have our previous local definition (Definition 2).
On the other hand, it is easy to see that a critical k–UCG along with $\mathcal{F}$ which consists of its k colour–classes is a graph of type $\mathcal{L}(\mathcal{F}, k, \mathcal{P}_\phi)$ in which ϕ is constant and equal to 1. As a special case, consider an m–UCG, U, with $m > 1$ colour–classes each of size n_i $(i = 1, \ldots, m)$ and then construct a new graph by joining a K_{t-1} to this graph in which each vertex of K_{t-1} is connected to all vertices of the m–UCG. Now, it is clear that this graph is a $(t + m - 1)$–UCG of type $\mathcal{L}(\mathcal{F}, t + m - 1, \mathcal{P}_\phi)$ in which, $\mathcal{F}$ consists of the colour–classes of the m–UCG and ϕ is constant and equal to 1. This graph will be denoted by $U^m_{t-1}(n_1, \ldots, n_m)$. ◇

Example 7. If we look more carefully at the structure of the graph $U^m_{t-1}(n_1, \ldots, n_m)$ of the previous example we find out that not only each colour-classes take a fixed colour but also each two colour-classes take different colours. In order to formulate this stronger property, consider the colouring property $\mathcal{P}^{\bullet}_{\phi}$ of Example 5. Hence, the graph $U^m_{t-1}(n_1, \ldots, n_m)$ is also of type $\mathcal{L}(\mathcal{F}, t+m-1, \mathcal{P}^{\bullet}_{\phi})$ where $\mathcal{F}$ consists of the colour-classes of the m-UCG and ϕ is constant and equal to 1.
Also, consider the graph K^m_t of Example 4 with m vertices $X = \{x_i \mid i = 1, \ldots, m\}$. If we split each x_i to n_i new vertices $\{y^i_1, \ldots, y^i_{n_i}\}$, we obtain a new graph $K^m_t(n_1, \ldots, n_m)$ which is of type $\mathcal{M}(\mathcal{F}, t+m-1, \mathcal{P}^{\bullet}_{\phi})$ in which,

$$\mathcal{F} = \{(i, \{y^i_l \mid l = 1, \ldots, n_i\}) \mid i = 1, \ldots, m\}, \quad \& \quad \phi = 1.$$

It is worth noting that if we add one edge between each two elements of $\mathcal{F}$ in $K^m_t(n_1, \ldots, n_m)$, then the new graph is also of type $\mathcal{M}(\mathcal{F}, t+m-1, \mathcal{P}_{\phi})$ with $\phi = 1$. ◇

Definition 5. Let H_1 and H_2 be two graphs of types $\mathcal{M}(\mathcal{F}_1, k-1, \mathcal{P}_1)$ and $\mathcal{L}(\mathcal{F}_2, k-1, \mathcal{P}_2)$ respectively; where $\mathcal{P}_1$ and $\mathcal{P}_2$ are of the same type but possibly with different parameters, and let

$$X_1 = \bigcup_{(j,W)\in\mathcal{F}_1} W \quad \text{and} \quad X_2 = \bigcup_{(j,W)\in\mathcal{F}_2} W.$$

Then we call H_1 and H_2 to be *repelling* if there exists a bijection $\iota^* : X_1 \longrightarrow X_2$ with $|X_1| = |X_2| > 1$ which induces a natural bijection $\iota : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ such that considering this correspondence $\mathcal{P}_1$ is equivalent to $\mathcal{P}_2$. ◇

Intuitively, the above definition means that we can identify the vertices of X_1 and X_2 through ι^* which locally identifies $\mathcal{F}_1$ and $\mathcal{F}_2$ such that each of their subsets W plays the role of X in Definition 2 for some $(k-1)$-colourings. For instance, note that if $\mathcal{P}_1 = \mathcal{P}_{\phi_1}$ and $\mathcal{P}_2 = \mathcal{P}_{\phi_2}$ then $\mathcal{P}_1$ is equivalent to $\mathcal{P}_2$ via ι means that $\phi_1 = \phi_2 \circ \iota$.
Now we can generalize Theorem 6 as follows.

Theorem 7. *Let $k > 2$ and let H_1 and H_2 be two repelling graphs of types $\mathcal{M}(\mathcal{F}_1, k-1, \mathcal{P}_1)$ and $\mathcal{L}(\mathcal{F}_2, k-1, \mathcal{P}_2)$ respectively, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are of the same type but possibly with different parameters. Then the graph obtained by identifying $\mathcal{F}_1$ and $\mathcal{F}_2$ is k-critical.*

Proof. First, note that for any $(k-1)$-colouring σ of H_1, by M2, $\neg\mathcal{P}_1(\sigma, \mathcal{F}_1)$ is true. And for any $(k-1)$-colouring σ of H_2, by L2, $\mathcal{P}_2(\sigma, \mathcal{F}_2)$ is true. But since these graphs are repelling, $\mathcal{P}_1$ and $\mathcal{P}_2$ behave identically on $\mathcal{F}_1$ and $\mathcal{F}_2$ through an identification $\iota : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$; and, therefore, these

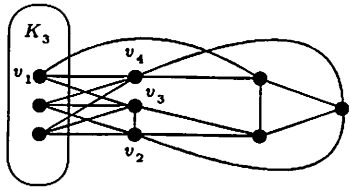

Figure 6: A graph of type $K(5,9)$ ($\mathcal{F} = \{(1,\{v_1,v_2\}),(2,\{v_3,v_4\})\}$).

properties are contradictory. This implies that the chromatic number of the new graph is greater than or equal to k.
On the other hand, assume that $e \in E(H_1)$. Then by M3 there is a $(k-1)$–colouring of $H_1 - e$ such as σ^* such that $\mathcal{P}_1(\sigma^*, \mathcal{F}_1)$ is true; and consequently, we may extend this $(k-1)$–colouring to H_2 by L4 and the identification ι. The same kind of reasoning, using L3 and M4, shows that for any edge $e \in E(H_2)$ we have a $(k-1)$–colouring of the new graph too; and this shows that the new graph is k–critical. □

Example 8. From our previous examples we know that a $(t+m-1)$–UCG along with m colour–classes as $\mathcal{F}$ is a graph which is both of types $\mathcal{L}(\mathcal{F}, t+m-1, \mathcal{P}_\phi)$ and $\mathcal{L}(\mathcal{F}, t+m-1, \mathcal{P}^*_\phi)$.
Also, $K^m_t(n_1, \ldots, n_m)$ is of type $\mathcal{M}(\mathcal{F}, t+m-1, \mathcal{P}^*_\phi)$ in which,

$$\mathcal{F} = \{(i, \{y^i_l \mid l = 1, \ldots, n_i\}) \mid i = 1, \ldots, m\}, \quad \& \quad \phi = 1.$$

Now if we choose the $(t+m-1)$–UCG such that the size of the m colour–classes are in one to one correspondence with numbers n_1 to n_m, then we can apply Theorem 7 to obtain a $(t+m)$–critical graph.
It should be noted that we could also, use property $\mathcal{P}_\phi$, however, in that case we should add the necessary edges between each two classes of $\mathcal{F}$ for $K^m_t(n_1, \ldots, n_m)$ in order to make sure that the graph is of type $\mathcal{M}(\mathcal{F}, t+m-1, \mathcal{P}_\phi)$. But this is possible since the other graph is a UCG and there is at least one edge between each two colour–classes of it. Therefore, we can add edges identically to obtain a well defined identification.
As an example of the above construction consider the graph $K^2_3(2,2)$ and $U^2_2(2,2)$ of Example 7 where $U = K_2$ and the UCG is a path on four vertices $\{v_1, v_2, v_3, v_4\}$. Then applying Theorem 7 we obtain the graph of Figure 6 which is a Δ–critical graph with $\Delta = 5$ ($\mathcal{F} = \{(1,\{v_1,v_2\}),(2,\{v_3,v_4\})\}$ [3]). ◇

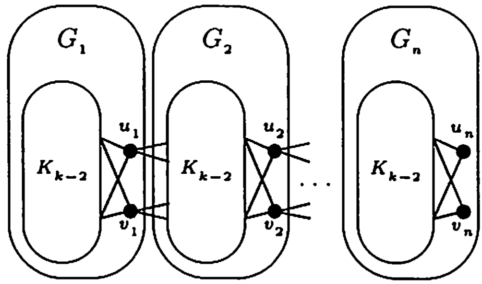

Figure 7: Graphs of type D^k_n (see Propositions 1 and 2).

The construction of Examples 7 and 8 using UCG's and most of the ideas of this paper reveals some links between critical graphs and UCG's. Also, we wish to note that almost all graphs of Beutelspacher and Hering can be constructed using Theorem 7. For more on this subject, other examples and related topics see [11] and the last section.

3 Some More Constructions

In this section we use a combination of ideas introduced so far to construct Δ–critical graphs. To begin, consider the general pattern depicted in Figure 7, in which G_i is constructed from joining two vertices u_i and v_i to all vertices of a $(k-2)$–clique. Also , u_i and v_i in G_i are connected to G_{i+1} such that their neighbourhoods in G_{i+1} form a partition of its $(k-2)$–clique. Any graph of this type is called to be of type D^k_n. Also, if we add a new vertex w to a graph of type D^k_n and join it to all vertices of the $(k-2)$–clique in G_1 then we obtain a new graph which will be called to be of type $D^k_n(w)$.

Proposition 1. *In any $(k-1)$–colouring of a graph G of type D^k_n all vertices of type u_i and v_i take the same colour. Also, if $X \subseteq V(G)$ is such that*

a) At least one of u_1 or v_1 is in X.

b) u_n and v_n are in X.

Then for any edge $e \in E(G)$ there is a $(k-1)$–colouring of $G-e$ such as σ such that $|\sigma(X)| = 2$.

Proof. Assume that we have a $(k-1)$–colouring of this graph $(n > 0)$. Then, the $(k-2)$–clique of G_1 takes all colours in $\{1, \ldots, k-2\}$ while u_1 and

v_1 in G_1 are forced to take the colour $k-1$. This forces the $(k-2)$-clique of G_2 to take its colours from $\{1,\ldots,k-2\}$; and since it is a $(k-2)$-UCG, u_2 and v_2 in G_2 are forced to take the colour $k-1$. This reasoning for each G_i $(i = 1,\ldots,n)$ proves the first part.
For the second part, first note that for $n = 1$ we are done by Example 4. Otherwise, let $1 \le j \le n$ be fixed. Then we consider three cases.

- $e \in E(K_{k-2}) \subseteq E(G_j)$.
  First, assume that $j \ne 1$, and as in the first part, assume that u_i and v_i have taken the colour $k-1$ and $K_{k-2} \le G_i$ has taken its colours from $\{1,\ldots,k-2\}$ for all $i < j$. Then for $K_{k-2} - e \le G_j$ we can use the colours $\{1,\ldots,k-3\}$.
  Now if $j = n$ we can use the colour $k-2$ for either of u_n or v_n which is in X. Otherwise, we let u_j take the colour $k-1$ and v_j take the colour $k-2$. But since $N_{G_{j+1}}(v_j) \ne \emptyset$ we can let one of the vertices in this neighbourhood take the colour $k-1$ and we can use the rest of colours in $\{1,\ldots,k-3\}$ for the $K_{k-2} \le G_{j+1}$. Hence, u_i and v_i can take the colour $k-2$ for $j+1 \le i \le n$.
  For the case $j = 1$ we can use the same technique but we let either of u_1 or v_1 which is in X take the colour $k-1$.

- $e \in E(G_j)$ and e is connected to u_j or v_j.
  In this case we can use exactly the same technique as in the previous case assuming that the other end of e in $K_{k-2} \le G_j$ has taken the colour $k-2$.

- $e \notin E(G_j)$ and e is connected to u_j or v_j.
  In this case, again, assume that u_i and v_i have taken the colour $k-1$ and $K_{k-2} \le G_i$ has taken its colours from $\{1,\ldots,k-2\}$ for all $i \le j$. Then we can assume that the vertex at the other end of e in $K_{k-2} \le G_{j+1}$ takes the colour $k-1$; and the rest of vertices in the $(k-2)$-clique take their colour from $\{1,\ldots,k-3\}$. Then, as before, all vertices u_i and v_i are forced to take the colour $k-2$ for $j < i \le n$.

□

Note that we can also prove the following counterpart of Proposition 1 using the same kind of reasoning.

Proposition 2. *In any $(k-1)$-colouring of a graph G of type $D_n^k(w)$ all vertices of type u_i and v_i and w take the same colour. Also, if $X \subseteq V(G)$ is such that*

a) w is in X.

b) u_n and v_n are in X.

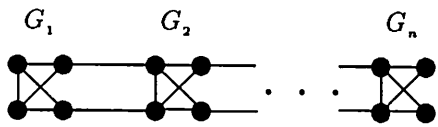

Figure 8: See Theorem 8.

Then for any edge $e \in E(G)$ there is a $(k-1)$-colouring of $G-e$ such as σ such that $|\sigma(X)| = 2$.

We define the *minimum excess function* as follows.

Definition 6. The *minimum excess function* is defined as

$$\eta(k,p) = \min_{G \in K(k,p)} \epsilon(G),$$

in which $K(k,p)$ is the class of all k-colour-critical graphs on p vertices with $\Delta = k$. Also, naturally, we define $\eta(k,p) = +\infty$ if $K(k,p) = \emptyset$. ◇

Theorem 8.

a)	$\eta(4, 3m+1) \leq m-1$	$(m \geq 2)$.
b)	$\eta(4, 4m) \leq 2m-2$	$(m \geq 2)$.
c)	$\eta(4, 4m+1) \leq 2m-1$	$(m \geq 3)$.
d)	$\eta(4, 4m+2) \leq 2m$	$(m \geq 2)$.
e)	$\eta(4, 4m+3) \leq 2m-1$	$(m \geq 1)$.
f)	$\eta(5, 5m) \leq 4m-4$	$(m \geq 2)$.
g)	$\eta(5, 5m+1) \leq 4m$	$(m \geq 7)$.
h)	$\eta(5, 5m+2) \leq 4m-2$	$(m \geq 5)$.
i)	$\eta(5, 5m+3) \leq 4m$	$(m \geq 3)$.
j)	$\eta(5, 5m+4) \leq 4m-2$	$(m \geq 1)$.
k)	$\eta(6, 6m) \leq 6(m-1)$	$(m \geq 2)$.

Proof. For (a) consider the graphs T_n^4 of Example 2 (Figure 3). For (b) and (d) apply Corollary 1 with H_2 a graph of type $D_n^4(w)$ and $H_1 = K_3$ or $H_1 = C_5$ respectively; and similarly, for (c) and (e) apply Corollary 1 with H_2 a graph of type D_n^4 and $H_1 = C_5$ or $H_1 = K_3$ respectively (Figure 8). For (f), (g) and (i) apply Corollary 1 with H_2 a graph of type $D_n^5(w)$ and $H_1 = K_4$, $H_1 = T_3^4$ or $H_1 = T_2^4$, respectively; and, similarly, for (h) and (j) apply Corollary 1 with H_2 a graph of type D_n^5 and $H_1 = T_2^4$ and $H_1 = K_4$, respectively (Figure 9).

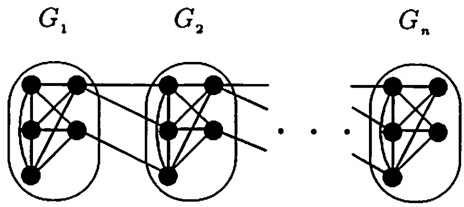

Figure 9: See Theorem 8.

Also, for (k) apply Corollary 1 with H_2 a graph of type $D^6_n(w)$, in which each u_i or v_i is connected to two vertices in G_{i+1}, and $H_1 = K_5$ (Figure 10). □

It is interesting to note that in our construction of D^k_n graphs for $k = 4, 5$, the number of vertices with degree less than k which take the same colour in any $(k-1)$–colouring, is a function of n. Hence, in these cases we can use any $(k-1)$–critical graph H with $\Delta(H) = k-1$ and Propositions 1 and 2 to construct critical graphs with $\chi = \Delta$, since by choosing n large enough we can have an arbitrary large number of these vertices with degree $k-1$. Also, note that instead of G_1 in our construction of D^k_n graphs, we could use any $(k-1)$–critical graph with the properties of H_2 in Corollary 1. For instance, as an interesting example of such a graph for $k = 5$ consider the graph of Figure 2 with $X = \{b_1, b_2, b_3\}$ [7, 9, 11].

4 Concluding Remarks

The existence of sparse k–chromatic graphs (with large girth) is a classic in graph theory [4, 12, 15, 20, 23, 28]; however, if we focus on *extremal* k–chromatic graphs which do not contain k–cliques, then the problem is something else. Actually, this problem can be considered from different angles.

First, note that the absence of k–cliques and extremality imply that not only the chromatic number of the graph is not obtained trivially through a direct forcing on a k–clique, but also it is a direct consequence of the whole (internal) structure of the graph. This guarantees that these graphs contain the most basic nontrivial k–chromatic structures.

On the other hand, the extremality condition can be applied in many different ways, from which *criticalness* and *uniqueness of the colouring* can be considered as the most basic ones. In other words, it is not surprising if we suspect that the structure of *minimal* k–uniquely–colourable graphs

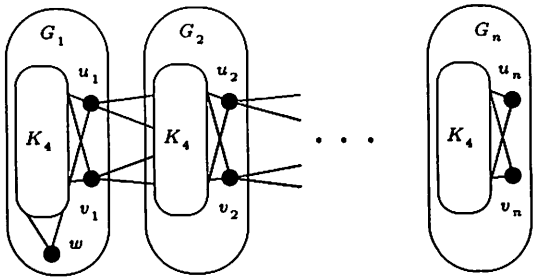

Figure 10: See Theorem 8.

should be similar to those of *minimal* $(k+1)$–critical graphs, when the *minimality* condition is applied either on the number of edges or on the number of vertices; since the structure of a minimal uniquely–colourable graph is so saturated that the chromatic number can be increased by just adding a small number of edges (properly). Actually, it is quite natural to think of graphs whose chromatic number is large because they contain some uniquely–colourable subgraphs or some subgraphs with a restricted number of colourings. This phenomenon is called a type 1 *forcing* in [9, 11, 7] while we used a simple form of it in the construction of D_n^k graphs in Section 3. As such a graph with a small (restricted) number of colourings, consider the graph of figure 2 which has only two different 4–colourings (see the paragraph proceeding Theorem 5) in which $\{b_1, b_2, b_3\}$ is a fixed colour-class. Then if we add the edge $e = b_1b_2$ we obtain a minimal 5–critical graph which does not contain a 5–clique. Note that this graph is the unique graph in $K_6(5,9)$ which appears in the list of Beutelspacher and Hering [3]. As one other aspect of this approach we also should note that it is quite probable that these special (chromatic) structures can be characterized if we add some other *extremality* conditions. For instance, one can mention a result of T. Gallai and B. Toft which states that a k–critical graph, which has a separating set of edges of size $k-1$, can be constructed using the classical r–type construction of Toft [26]. Therefore, generalizing this approach, toward characterization of k–chromatic structures, can be considered as one of the main sequels to the methods of construction appeared in Section 2.

The distribution of edges can also be considered as a constraint and in this case we face the same phenomena, because on one hand we want to keep the number of edges as small as possible and on the other hand we have to use all of the power of each edge in order to increase the chromatic number.

In other words, we do not expect a large *excess* and on the other hand we want to keep the chromatic number high (which intuitively needs a large number of edges). This explains why these conditions and $\chi = \Delta$ are more or less contradictory.
These interactions motivates the study of *minimal excess function* of k–critical and Δ–critical graphs [2, 16, 18] and the conjecture of Borodin and Kostochka (Conjecture 1 and [5]). As one of the main problems in this direction we have,

Problem 1. *Determine* $\eta(k,p)$ *for all* Δ*–critical graphs with* $4 \leq \Delta \leq 8$.

which seems to be quite hard to answer. However, we may formulate some simpler problems in this regard as follows.

Problem 2. *If* $0 < \alpha < 6$ *is fixed then, does there exist an infinite family of* 6*–critical graphs with* $\Delta = 6$ *and* $ccl > 0$ *such that the number of vertices of these graphs are always of the form* $6n + \alpha$*?*

Problem 3.

- *Is the number of* Δ*–critical graphs finite for any fixed* $7 \leq \Delta \leq 8$*?*
- *If the answer to the previous problem is "yes" then characterize all* Δ*–critical graphs when* $7 \leq \Delta \leq 8$.

Also, one may expect to formulate the same kind of problem for uniquely–colourable graphs and in this direction we face the following conjecture of S.J. Xu [25],

Conjecture 2.[25] *If* G *is a UCG and* $\Lambda(G) = 0$ *then* $ccl(G) = 0$*; where*

$$\Lambda(G) = |E(G)| - |V(G)|(k-1) + \binom{k}{2}.$$

(Note that for any k–uniquely–vertex–colourable graph G we have $\Lambda(G) \geq 0$ [6, 25, 27]). It is also interesting to note that both conjectures are quite hard to verify [7, 11, 24].
At the end, we wish to refer to some results of B. Toft concerning universal vertices of k–critical graphs recalled in Theorem 4; and to mention the structure of graphs of types $\mathcal{M}(\mathcal{F}, k, \mathcal{P})$ and $\mathcal{L}(\mathcal{F}, k, \mathcal{P})$. Needless to say, it is quite interesting to develop methods of construction for such graphs and trying to characterize conditions under which a k–critical graph can be constructed by Theorem 7 and a specific type of colouring property $\mathcal{P}$. This approach may help to characterize Δ–critical graphs for small Δ.

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