Unitary Designs with a Common Collection of Blocks

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Abstract

It is well known that one can construct a family of $\frac{q^2-q}{2}$ Miquelian inversive planes on the same pointset such that any two share exactly the blocks through a fixed point. Further, Ebert [10] has shown that this family can be augmented for even q by adding some Suzuki-Tits inversive planes. We wish to apply the method of Ebert combined with a technique from Dover [7] to obtain a family of unitals which have the same property.

1 Introduction

Since the 1970's, finding Steiner systems on a particular pointset which share a fixed set of blocks has been an outstanding problem in the theory of designs. Much work has been done on this sort of problem (see Rosa[12]), but has usually focused on designs with small block size. Of particular interest to us is the following problem: how many inversive planes ($3-(q^2+1, q+1, 1)$ designs) can one have on a set of points such that any two of the inversive planes have exactly the blocks through a fixed point in common? While there is much folklore about this problem, the main results have been precisely formulated and proved in Ebert[10].

In his presentation, Ebert gives a method for constructing such a family of inversive planes by "projecting" Buekenhout-Metz unitals. While this method duplicates some classical constructions concerning Miquelian inversive planes which share only the blocks through a fixed point, it is also shown how some Suzuki-Tits inversive planes can be added to this collection without disturbing our intersection property.

We wish to give an extension of Ebert's method which gives us a family of unitals ($2-(q^3+1, q+1, 1)$ designs) on a fixed pointset which have only the blocks through a fixed point in common. Further, we will combine this

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construction technique with another technique by the author [7] to enlarge our family significantly.

2 Some Preliminary Results

We would like to begin by giving the Bose-André model of $\mathcal{PG}(2,q^2)$ using the method of Bruck and Bose [4]. We then discuss Buekenhout's construction of unitals in this model (see Buekenhout [5] for more details).

Let $\Sigma = \mathcal{PG}(4,q)$ be modelled as a five-dimensional vector space over $\mathcal{GF}(q)$ with homogeneous coordinates. Let Σ^* be a hyperplane of Σ. Finally, let $\mathcal{S}$ be a regular spread of lines of Σ^*. Then we can model $\mathcal{PG}(2,q^2)$ by taking as points the points of $\Sigma \setminus \Sigma^*$ together with the lines of $\mathcal{S}$. The lines are the planes of Σ which meet Σ^* in a line of $\mathcal{S}$ together with $\mathcal{S}$ itself. Incidence is given by containment. This model is called the *Bose-André model* for $\mathcal{PG}(2,q^2)$, and is denoted $\pi(\Sigma,\Sigma^*,\mathcal{S})$.

An *ovoid* of $\mathcal{PG}(3,q)$ is a set of q^2+1 points such that no three are collinear. The classical example of an ovoid is the set of points on an elliptic quadric; in fact, these are the only ovoids when q is odd. However, if q is an odd power of 2, except 2 itself, there exists another type of ovoid, called the *Tits ovoid*, which is strongly related to the Suzuki simple groups.

We can now give Buekenhout's construction of unitals. Let $\mathcal{O}^*$ be the point cone over an ovoid which meets the hyperplane Σ^* in exactly a line ℓ of the spread $\mathcal{S}$. Then the q^3 affine points of $\mathcal{O}^*$ together with the line ℓ form the points of a unital whose blocks are the intersections of $\mathcal{O}^*$ with nontangent planes to $\mathcal{O}^*$ which meet Σ^* in a line of $\mathcal{S}$. Again incidence is given by containment. Such a unital is called a *parabolic Buekenhout unital*, or *B-unital* for short.

Let $\pi(\Sigma,\Sigma^*,\mathcal{S})$ be the Bose-André model of $\mathcal{PG}(2,q^2)$, and let $\mathcal{O}^*$ be an ovoidal cone with base $\mathcal{O}$ which meets Σ^* in only the line ℓ of $\mathcal{S}$. Finally, let U be the B-unital determined by $\mathcal{O}^*$, and let the point of U represented by ℓ be called P. Associated with this unital, we can define an incidence structure Ω as follows: the points of Ω are the blocks of U which pass through P. The blocks of U are the distinct sets of blocks of U through P which are met by some block of U not through P. The following result has been shown in Dover [7] and independently in Barwick and O'Keefe [2]:

Theorem 2.1 *Let $\Sigma = \mathcal{PG}(4,q)$ with special hyperplane Σ^*. Let $\mathcal{S}$ be any spread of Σ^*, and let $\mathcal{O}^*$ be any ovoidal cone with base ovoid $\mathcal{O}$ which meets Σ^* in a line ℓ of $\mathcal{S}$. Finally let U be the unital represented by $\mathcal{O}^*$. Then the design Ω obtained from U as above is isomorphic to the point residual of the inversive plane defined by $\mathcal{O} \setminus \Sigma^*$. Further, this isomorphism maps the points of Ω onto the points of $\mathcal{O}\setminus\Sigma^*$ as follows. A point of Ω is a block of U which passes through ℓ. This block is represented in Σ by a plane which contains ℓ and another line n of $\mathcal{O}^*$. This line n meets $\mathcal{O} \setminus \Sigma^*$ in a unique point, which is defined to be the image of the point of Ω with which we started.*

We will call this design the *local projection* of U at P. It is clear that this projection can be defined for any point P of any unital, but may not be of interest if P is not the special point of the unital represented by ℓ.

3 A First Construction

Let $\{U_1, \dots U_n\}$ be a family of unitals on the same pointset. We say this family has *property I* if there exists a point P such that each of the unitals U_i share the blocks through the point P but no two of the unitals share any other blocks. So our goal is to construct a family of unitals which have property I.

To this end, we first define a common pointset on which these unitals will lie. Let q be any prime power, and define a three-dimensional vector space over $\mathcal{GF}(q)$ via $\{(x, y) | x \in \mathcal{GF}(q^2), y \in \mathcal{GF}(q)\}$, i.e. $\mathcal{GF}(q^2) \times \mathcal{GF}(q)$. This three-dimensional vector space can be used to model a three-dimensional affine space $\mathcal{AG}(3, q)$. Finally, define the set $\mathcal{A}$ to be $\mathcal{AG}(3, q) \cup \{\infty\}$, where ∞ is a new point not in the affine space. Note that $\mathcal{A}$ has exactly $q^3 + 1$ elements.

Our first construction will focus on the Buekenhout-Metz unitals (B-unitals in $\mathcal{PG}(2, q^2)$ obtained from the cone over an elliptic quadric) which were studied in Baker and Ebert [1] and Ebert [8]. Using the notation in these two papers, we can actually work strictly in $\mathcal{PG}(2, q^2)$ without referring to the Bose-André model. Indeed, we can define the set U_{ab} via:

$$U_{ab} = \{(0, 1, 0)\} \cup \{(z, az^2 + bz^{q+1} + r, 1) | z \in \mathcal{GF}(q^2), r \in \mathcal{GF}(q)\} \qquad (1)$$

where $(b^q - b)^2 + 4a^{q+1}$ is a nonsquare if q is odd or $\frac{a^{q+1}}{(b^q+b)^2}$ is an element of trace 0 over $\mathcal{GF}(q)$ if q is even. It has been shown in these two papers that under these conditions U_{ab} is a B-unital, and that the local projection of U_{ab} at $(0, 1, 0)$ is the point residual of a Miquelian inversive plane.

We'd like our unitals to lie on the pointset $\mathcal{A}$. So letting $\mathcal{U}_{ab}$ be an arbitrary Buekenhout-Metz unital, define the map $\phi : U_{ab} \longrightarrow \mathcal{A}$ via $(0, 1, 0)\phi = \infty$ and $(z, az^2 + bz^{q+1} + r, 1)\phi = (z, r)$. It is easy to see that ϕ is a bijection from the points of U_{ab} onto $\mathcal{A}$.

We begin with a proposition which describes what the images of blocks through the point $(0, 1, 0)$ are in $\mathcal{A}$.

Proposition 3.1 *Let U_{ab} be a Buekenhout-Metz unital in $\mathcal{PG}(2, q^2)$, and let B be a block of U_{ab} through the point $(0, 1, 0)$. Then the image of B under ϕ is of the form:*

$$B\phi = \{\infty\} \cup \{(z, r) | r \in \mathcal{GF}(q)\}$$

for some $z \in \mathcal{GF}(q^2)$. In particular, $B\phi$ is an affine line together with the point ∞.

Proof: Let B be a block of U_{ab} through point $(0, 1, 0)$. Then for some fixed $z \in \mathcal{GF}(q^2)$, B is the set of points of U_{ab} which lie on the line with coordinates $[1, 0, -z]$. One can easily compute that this set of points is $\{(z, az^2 + bz^{q+1} + r, 1) | r \in \mathcal{GF}(q)\}$ together with $(0,1,0)$. Therefore, $B\phi = \{(z,r) | r \in \mathcal{GF}(q)\} \cup \{\infty\}$ as claimed. □

So, by taking the image under ϕ of all unitals of the form U_{ab}, we obtain a family of unitals on the pointset $\mathcal{A}$ which have all of the blocks through the point ∞ in common. However, these unitals may also have other blocks in common. To control this, we want to consider the local projection of U_{ab} at $(0,1,0)$ and its image under ϕ.

The key to obtaining our result is the following result from Ebert [10]; however, the terminology used there is slightly different than ours. For a fixed B-unital U_{ab}, they define the block ℓ_z as the set $\{(z, az^2 + b^{q+1} + r, 1) | r \in \mathcal{GF}(q)\}$ together with the point $(0,1,0)$ for every $z \in \mathcal{GF}(q^2)$. They then define a design Ω_{ab} on the elements of $\mathcal{GF}(q^2)$ whose blocks are the distinct sets of the form $\{z \in \mathcal{GF}(q^2) | \ell \cap \ell_z \neq \emptyset\}$ as ℓ varies over all the blocks of U_{ab} not containing $(0,1,0)$. It is clear that if one identifies the field elements z with the blocks ℓ_z that Ω_{ab} is simply the local projection of U_{ab} at $(0,1,0)$. With this notation, we can now state:

Theorem 3.2 *For any prime power q, there exist $\frac{q(q-1)}{2}$ distinct pairs (a_i, b_i) such that the designs $\Omega_{a_i b_i}$ are pairwise disjoint.*

Before giving a family of unitals which have property I, we need the following notation. Let U_{ab} be any Buekenhout-Metz unital. Then, $U_{ab}\phi$ is the design whose points are the points of $\mathcal{A}$, and whose blocks are the images of blocks of U_{ab} under ϕ. Since we are merely exchanging pointsets, it is clear that the resulting design $U_{ab}\phi$ is indeed a unital. Further, one can easily see that the local projection of $U_{ab}\phi$ at ∞ is exactly the image of the local projection of U_{ab} at $(0,1,0)$ under ϕ. Using Proposition 3.1 and identifying the point $\{(z,r) | r \in \mathcal{GF}(q)\}$ of this local projection with the field element $z \in \mathcal{GF}(q^2)$, we again see that this local projection is exactly the design Ω_{ab}.

We can now prove the following:

Theorem 3.3 *Let (a_i, b_i) be a set of $\frac{q(q-1)}{2}$ pairs such that the designs $\Omega_{a_i b_i}$ are pairwise disjoint. Then the $\frac{q(q-1)}{2}$ unitals $U_{a_i b_i}\phi$ have property I.*

Proof: By Proposition 3.1, we know that these unitals all share the blocks through ∞, so all that remains to show is that no two of these unitals have any other block in common. By way of contradiction, suppose there exist distinct unitals $U_{ab}\phi$ and $U_{\alpha\beta}\phi$ which share a block B not through ∞. Then the set of blocks through ∞ which meet B is a block of the local projections of both $U_{ab}\phi$ and $U_{\alpha\beta}\phi$ at ∞. This implies that the designs Ω_{ab} and $\Omega_{\alpha\beta}$ have a common block, which is a contradiction, since these designs are pairwise disjoint. Thus, no pair of these unitals have a block not through ∞ in common, and therefore this family of unitals has property I. □

4 An Extension

We would now like to give an extension of our family which involves the construction method in [7]. Let $\pi(\Sigma, \Sigma^*, \mathcal{S})$ be a Bose-André model of $\mathcal{PG}(2, q^2)$, and let $\mathcal{O}^*$ be any ovoidal cone which meets Σ^* in a line ℓ of $\mathcal{S}$.

It can easily be shown that the q^4 lines of Σ^* disjoint from ℓ can be partitioned into q^2 partial spreads of size q^2 each. In particular, we can append ℓ to each of these partial spreads to obtain q^2 spreads which pairwise meet in the line ℓ. Call these spreads $S_1, \ldots, S_{q^2}$. It can further be shown that these spreads may be assumed to be regular, so without loss of generality, we can assume $S_1 = \mathcal{S}$.

By Buekenhout's construction, for any spread S containing ℓ, we know that if we take our points to be the points of $\mathcal{O}^* \setminus \Sigma^*$ together with the line ℓ, and our blocks to be the intersections of $\mathcal{O}^*$ with nontangent planes to $\mathcal{O}^*$ containing a line of S, the resulting incidence structure is a unital. Performing this construction for each of our spreads S_i yields q^2 unitals on the pointset $\mathcal{O}^* \setminus \Sigma^* \cup \{\ell\}$. It is shown in [7] that these q^2 unitals share the blocks through ℓ and no other blocks, i.e. they form a family of unitals with property I.

We would like to prove the following result regarding local projections in such a family of unitals.

Proposition 4.1 *Let $\Sigma = \mathcal{PG}(4, q)$ with special hyperplane Σ^*. Let $\mathcal{O}^*$ be an ovoidal cone which meets Σ^* in only the line ℓ, and let S_1 and S_2 be any two spreads of Σ^* which contain ℓ. Finally, let U_1 and U_2 be the two unitals on $\mathcal{O}^* \setminus \Sigma^* \cup \{\ell\}$ determined by S_1 and S_2 respectively. Then, the local projections at ℓ of these two unitals are identical.*

Proof: Let Ψ be a hyperplane of Σ which meets $\mathcal{O}^*$ in an ovoid $\mathcal{O}$, and note that $\mathcal{O}$ has a unique point Q lying on Σ^*. Consider the local projection of U_1 at ℓ. By Theorem 2.1, this design is isomorphic to the point residual of the inversive plane defined by $\mathcal{O} \setminus \{Q\}$, with isomorphism ψ_1. Similarly, the local projection of U_2 at ℓ is isomorphic to this point residual with isomorphism ψ_2.

Consider how ψ_1 acts. Let B be a block of U_1 through ℓ. Then B is represented in Σ by a plane π which meets $\mathcal{O}^*$ in a pair of lines, one of which is ℓ. Let n be the other line of $\mathcal{O}^*$ contained in π. Then n meets $\mathcal{O} \setminus \{Q\}$ in a unique point, which is the image of B under ψ_1. ψ_2 acts in exactly the same way, since U_1 and U_2 share the blocks through ℓ. So considered as a mapping on points, ψ_1 and ψ_2 are identical. Thus $\psi_1\psi_2^{-1}$ is an isomorphism from the local projection of U_1 at ℓ to the local projection of U_2 at ℓ. But $\psi_1\psi_2^{-1}$ is the identity on the common pointset of the two local projections, which forces these local projections to have the same blocks. □

Let $\{U_{a_i b_i} | i \in \{1, \ldots \frac{q(q-1)}{2}\}\}$ be the family of unitals with property I obtained in Theorem 3.3. Let $U_i = U_{a_i b_i}$ for short. Since U_i is a Buekenhout-Metz unital, there exists a Bose-André model $\pi(\Sigma, \Sigma^*, \mathcal{S})$ for $\mathcal{PG}(2, q^2)$ such that U_i

is represented by an ovoidal cone and $(0,1,0)$ is represented by a line of $\mathcal{S}$. By the result in Dover [7], for each $i \in \{1, \dots \frac{q(q-1)}{2}\}$ there exists a family of unitals with property I, U_i^j, $j \in \{1, \dots q^2\}$ on the same pointset of U_i such that $U_i = U_i^1$. (Note that while the construction takes place in Σ, we can easily map these blocks over to be subsets of $\mathcal{PG}(2, q^2)$.)

We can now prove:

Theorem 4.2 *Let $U_i^j\phi$, $i \in \{1, \dots \frac{q(q-1)}{2}\}$, $j \in \{1, \dots, q^2\}$ be a family of unitals on the pointset $\mathcal{A}$. Then this family of unitals has property I.*

Proof: All of these unitals lie on the pointset $\mathcal{A}$. Also for each $i \in \{1, \dots \frac{q(q-1)}{2}\}$, the unitals U_i^j, $j \in \{1, \dots, q^2\}$ all share the blocks through $(0,1,0)$. Therefore when applying ϕ to each of these unitals, their images will share the blocks through ∞. Then since the unitals $U_i^1\phi$, $i \in \{1, \dots \frac{q(q-1)}{2}\}$ share the blocks through ∞, we have that all of our unitals share the blocks through ∞.

Now suppose two unitals $U_i^j\phi$ and $U_k^m\phi$ share a block not through ∞. If $i = k$, this immediately contradicts that the family U_i^x, $x \in \{1, \dots q^2\}$ has property I. So we can assume $i \neq k$.

Let B be this shared block. Then B meets $q+1$ of the blocks through ∞. This set of blocks is then a block of the local projections of both $U_i^j\phi$ and $U_k^m\phi$ at ∞. But by Proposition 4.1, $U_i^j\phi$ and $U_i\phi$ have the same local projection at ∞, and $U_k^m\phi$ and $U_k\phi$ have the same local projection at ∞. Thus the local projections of $U_i\phi$ and $U_k\phi$ at ∞ share a block, which is a contradiction as in the proof of Theorem 3.3. Therefore, no such shared block can exist, and our family must have property I. □

Corollary 4.3 *Let q be a prime power. Then there exists a family of $\frac{q^3(q-1)}{2}$ unitals of order q which have property I.*

5 Some Additions for q Even

In this section, we would like to explore the possibility of enlarging our family of unitals, at least for $q = 2^{2e-1}$ for some $e \geq 0$, by considering Buekenhout unitals of $\mathcal{PG}(2, q^2)$ which can be obtained from a cone over a Tits ovoid. We follow the presentation in Ebert[9].

As in the previous section, let $\Sigma = \mathcal{PG}(4, q)$ with a special hyperplane Σ^*, where $q = 2^{2e-1}$ for some $e \geq 0$. Let $\mathcal{S}$ be any regular spread of Σ^* which contains the line $\ell = \langle (0,0,1,0,0), (0,0,0,1,0) \rangle$. Define σ to be 2^e, and note that raising to the σ power is a field automorphism, and denote the expression $s^{\sigma+2} + t^\sigma + st$ as $\langle s, t \rangle$.

We define the ovoidal cone $\ell \cup \{(s, t, r, \langle s, t \rangle, 1) | s, t, r \in \mathcal{GF}(q)\}$. That this is the cone over a Tits ovoid with vertex $(0,0,1,0,0)$ follows from Dembowski [6]. Using Buekenhout's construction, we obtain a unital T from this ovoidal cone.

Consider $\mathcal{GF}(q^2)$ as a two-dimensional vector space over $\mathcal{GF}(q)$, and let β be a primitive element of $\mathcal{GF}(q^2)$. Then we can write every element of $\mathcal{GF}(q^2)$ as $a + b\beta$ for unique elements $a, b \in \mathcal{GF}(q)$. With this convention, we define a map ϕ_1 from the points of T to $\mathcal{A}$ via $\ell\phi_1 = \infty$ and $(s, t, r, \langle s, t\rangle, 1)\phi_1 = (s + t\beta, r)$. Since $\{1, \beta\}$ is a basis for $\mathcal{GF}(q^2)$ this map is an isomorphism from the points of T to $\mathcal{A}$.

It is easy to check that the blocks of T through ℓ are of the form $\{\ell\} \cup \{(a, b, r, \langle a, b\rangle, 1) : r \in \mathcal{GF}(q)\}$ as a and b vary over $\mathcal{GF}(q)$. So under ϕ_1, these blocks map to the sets $\{(z, r) : r \in \mathcal{GF}(q)\}$ as z varies over $\mathcal{GF}(q^2)$.

We would like to construct the local projection Ω of the unital T at ℓ. Using Lemma 2.1, we know that this design is isomorphic to the point residual of the Tits ovoid $\mathcal{O} = \{(s, t, 0, \langle s, t\rangle, 1) | s, t \in \mathcal{GF}(q)\}$. (Note that $(0, 0, 0, 1, 0)$ is the point removed.) Further, we know that this isomorphism maps the point $\{(x, y, r, \langle x, y\rangle, 1) : r \in \mathcal{GF}\} \cup \{\ell\}$ of Ω onto the point $(x, y, 0, \langle x, y\rangle, 1)$ of $\mathcal{O}$. From Dembowski, any block of the point residual of the Tits ovoid has the form $\{(x, y, 0, \langle x, y\rangle, 1) | \langle x, y\rangle = Ax + By + C\}$, where $A, B, C \in \mathcal{GF}(q)$ and $C \neq \langle B, A\rangle$.

As we did before, we can use this local projection to obtain the local projection of $T\phi_1$ at ∞. The points of the local projection of $T\phi_1$ at ∞ are the sets of the form $\{(s + t\beta, r) | r \in \mathcal{GF}(q)\} \cup \{\infty\}$ for all $s, t \in \mathcal{GF}(q)$. If we identify this block with $s + t\beta \in \mathcal{GF}(q^2)$ as before, we get that the blocks of this local projection are the sets of the form $\{x + y\beta | \langle x, y, \rangle = Ax + By + C\}$, where $A, B, C \in \mathcal{GF}(q)$ and $C \neq \langle B, A\rangle$. Call this design Ω_T.

We would now like to see how this unital $T\phi_1$ fits in with our family U_i^j, $i \in \{1, \dots \frac{q(q-1)}{2}\}$, $j \in \{1, \dots q^2\}$ of unitals with property I that was constructed in Theorem 4.2. We know that $T\phi_1$ is a unital with pointset $\mathcal{A}$ and that it shares the blocks through ∞ with the unitals $U_i^j\phi$. By way of contradiction, suppose $T\phi_1$ shares a block not through ∞ with a unital $U_i^j\phi$, and call such a block B. Then the set of $q + 1$ blocks through ∞ met by B is a block of the local projections of both $U_i^j\phi$ and $T\phi_1$. Identifying blocks through ∞ with elements of $\mathcal{GF}(q^2)$ as usual, this forces the designs $\Omega_{a_i b_i}$ and Ω_T to have a common block. This was shown to be impossible in Ebert [10]. So $T\phi_1$ can be added to our family without disturbing property I.

Further, using the method of the previous section, we can actually construct a family of q^2 unitals T^j, $j \in \{1, \dots, q^2\}$ such that the unitals $T^j\phi_1$ share the local projection Ω_T with $T\phi_1$. An argument identical to that of Theorem 4.2 shows that these unitals can also be added to our family.

To extend our family still further, we wish to construct unitals which have a different Suzuki-Tits point residual as local projection. We recall the following result from Ebert [10]:

Theorem 5.1 *Let I_0 be the point residual of a Suzuki-Tits inversive plane with points $\{(x, y) | x, y \in \mathcal{GF}(q)\}$ and blocks of the form $\{(x, y) | Ax + By + C = \langle x, y\rangle\}$ with $A, B, C \in \mathcal{GF}(q)$, $C \neq \langle B, A\rangle$. Then, the $q^2 - q$ point residuals with points*

$\{(x,y)|x,y \in \mathcal{GF}(q)\}$ and blocks $\{(fx+g, fy)|Ax+By+C = \langle x,y\rangle\}$, with $A,B,C \in \mathcal{GF}(q)$ and $C \neq \langle B,A\rangle$, as f and g vary over $\mathcal{GF}(q)$ with $f \neq 0$ share no blocks. Further, none of these designs share a block with any of the designs $\Omega_{a_i b_i}$.

With this result in mind, we define the mapping $M_{f,g} : \mathcal{A} \longrightarrow \mathcal{A}$ via $\infty M_{f,g} = \infty$ and $(x+y\beta, r)M_{f,g} = ((fx+g)+fy\beta, r)$, which is clearly a bijection. We can now define the unitals $T^j_{f,g}$ with $f,g \in \mathcal{GF}(q)$, $f \neq 0$ and $j \in \{1,\dots q^2\}$ via $T^j_{f,g} = (T^j\phi_1)M_{f,g}$.

Let us construct the local projection of $T^j_{f,g}$ at ∞. A block of this design will be the image of a block of the local projection of $T^j\phi_1$ under $M_{f,g}$. With our usual identification of blocks through ∞ with field elements, we can see that the blocks of the local projection of $T^j_{f,g}$ at ∞, which we call $\Omega_{T_{f,g}}$, have the form $\{fx+g+fy\beta|\langle x,y\rangle = Ax+By+C\}$, where $A,B,C \in \mathcal{GF}(q)$, $C \neq \langle B,A\rangle$. This allows us to prove:

Theorem 5.2 *Let $q = 2^{2e-1}$ with $e \geq 0$. Consider the family of unitals $U^j_i\phi$ constructed in Theorem 4.2, and adjoin the unitals $T^j_{f,g}$ as defined above. Then this family of $\frac{3q^3(q-1)}{2}$ unitals has property I.*

Proof: It is clear that all of our unitals are defined on the same pointset. We know that all of the unitals $U^j_i\phi$ share the blocks $\{(z,r) : r \in \mathcal{GF}(q)\} \cup \{\infty\}$ through ∞, and we know that each of the unitals $T^j\phi_1$ contains these blocks as well. Finally, $M_{f,g}$ permutes the blocks of $T^j\phi_1$ amongst themselves, so all of the unitals $T^j_{f,g}$ also contain these blocks.

It remains to show that these unitals share no other blocks. Suppose first that $U^j_i\phi$ and $T^k_{f,g}$ share a block B not through ∞. Then the set of blocks through ∞ met by B form a block of the local projections of both U^j_i and $T^k_{f,g}$ at ∞. This forces $\Omega_{a_i b_i}$ and $\Omega_{T_{f,g}}$ to share a block, which contradicts Theorem 5.1.

The proof that $T^k_{f,g}$ and $T^z_{x,y}$ share no block not through ∞ is similar, and thus omitted. □

6 Conclusion

We have used essentially two techniques here. The first is derived from Ebert's solution of the corresponding problem for inversive planes, while the second comes from the author's hyperspace method. In some sense, these two methods are complementary; the first gives us a large number of local projections from which to choose, while the second allows us to stack unitals with a fixed local projection. The local projection technique essentially bounds the number of Buekenhout unitals that can be stacked in this manner. In particular, we believe that the only way a Buekenhout unital can be appended to this family would be if we could add an inversive plane to the family of Ebert.

On the other hand, we do not conjecture that our results are the best possible. There are many types of unitals which do not arise from Buekenhout's construction, and it is quite possible that some of these could be added to our family while retaining property I. In particular, it may very well be possible to add Ree unitals for $q = 3^{2e+1}$, $e \geq 0$ to our family (see Lüneberg [11]).

Another potential application comes from the theory of translation planes. The set $\mathcal{A}$ we used to carry our unitals is intimately tied to the three-dimensional circle geometries of Bruck [3]; in particular, $\mathcal{A}$ can be considered as the pointset of this design, and various subsets are its blocks. Perhaps the blocks of some of our unitals are blocks of the circle geometry as well. If so, this could have interesting implications for the theory of three-dimensional translation planes, as these circle geometries are isomorphic to the planes and reguli of regular spreads of $\mathcal{PG}(5,q)$; perhaps some interesting sort of derivation can be obtained.

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