Remarks on a Generalization of Radon's Theorem

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ABSTRACT. Let S^n be the n-dimensional sphere and K be the simplicial complex consisting of all faces of some $(n+1)$-dimensional simplex. We present an explicit construction of a function $g : S^n \to |K|$ such that for every $x \in S^n$ the supports of $g(x)$ and $g(-x)$ are disjoint. This construction provides a new proof of the following result of Bajmóczy and Bárány [1] that is a generalization of a theorem of Radon [4]: If $f : |K| \to \mathbb{R}^n$ is a continuous map, then there are two disjoint faces Δ_1, Δ_2 of Δ such that $f(\Delta_1) \cap f(\Delta_2) \neq \emptyset$.

1 Introduction.

If $x_0, x_1, \ldots, x_k$ are points in $\mathbb{R}^m$ such that $\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ is a linearly independent set of k vectors in $\mathbb{R}^m$, then we say that these points are *affinely independent*. Let $0 \leq k \leq m$, and $x_0, x_1, \ldots, x_k$ be affinely independent points in $\mathbb{R}^m$. The *(open) k-simplex* $\Delta = (x_0, x_1, \ldots, x_k)$ is the following subset of $\mathbb{R}^m$:

$$\left\{ x = \sum_{i=0}^{k} \mu_i x_i : \sum_{i=0}^{k} \mu_i = 1,\ \mu_i > 0 \text{ for } i = 0, \ldots, k \right\}. \tag{1}$$

Since the points $x_0, x_1, \ldots, x_k$ are affinely independent, the reals μ_i, $0 \leq i \leq k$, are uniquely determined by x and $x_0, x_1, \ldots, x_k$. We shall call the sum

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in (1) the *barycentric representation* of x with respect to $(x_0, x_1, \dots, x_k)$. The points $x_0, \dots, x_k$ are the *vertices* of Δ, and k is the dimension of Δ. A simplex Δ_1 is a *face* (*proper face*) of a simplex Δ_2 if the vertex-set of Δ_1 is a subset (proper subset) of the vertex-set of Δ_2.

A (*geometric*) *simplicial complex* K is a finite set of disjoint simplices such that every face of every simplex of K is also a simplex of K. The *polyhedron* $|K|$ of the simplicial complex K is the union of all its simplices; the complex K is then also called a *simplicial decomposition* of $|K|$.

If $\{x_1, x_2, \dots, x_k\}$ is the set of vertices of the simplicial complex K and $x \in |K|$, then there are unique reals $\mu_1, \mu_2, \dots, \mu_k$ such that

$$x = \sum_{i=1}^{k} \mu_i x_i, \qquad (2)$$

where $\mu_i \geq 0$ for every $i = 1, 2, \dots, k$,

$$\sum_{i=1}^{k} \mu_i = 1,$$

and the set $\{x_i : \mu_i > 0\}$ is the vertex-set of a simplex Δ_x of K. We call the simplex Δ_x the *support* of x, and we say that the sum in (2) is the *barycentric representation* of x with respect to K, or just the barycentric representation of x if the complex is clear from the context.

The simplicial complex K' is a *subcomplex* of the simplicial complex K if the set of simplices of K' is a subset of the set of simplices of K, in particular the set of vertices of K' is a subset of the set of vertices of K.

The well-known theorem of Radon [4] says that, for any $A \subset \mathbb{R}^n$ satisfying $|A| \geq n + 2$, there are disjoint subsets B and C of A such that their convex hulls have nonempty intersection. Since, for any $A \subset \mathbb{R}^n$ satisfying $|A| = n + 2$ the convex hull of A is the image of the closure of an $(n+1)$-dimensional simplex under a linear map, Radon's theorem is an immediate corollary to the following theorem.

Theorem 1. *Let $\Delta \subset \mathbb{R}^{n+1}$ be an $(n+1)$-dimensional simplex and let K be the simplicial complex containing all faces of Δ. If $f : |K| \to \mathbb{R}^n$ is a linear map, then there are two disjoint faces Δ_1, Δ_2 of Δ such that $f(\Delta_1) \cap f(\Delta_2) \neq \emptyset$.*

Thus the following theorem of Bajmóczy and Bárány [1] can be thought of as a generalization of Radon's theorem.

Theorem 2. *Let Δ and K be as in Theorem 1. If $f : |K| \to \mathbb{R}^n$ is a continuous map, then there are two disjoint faces Δ_1, Δ_2 of Δ such that $f(\Delta_1) \cap f(\Delta_2) \neq \emptyset$.*

Bajmóczy and Bárány use the following antipodal theorem of Borsuk and Ulam [3] in their proof.

Theorem 3. *For any continuous map $h : S^n \to \mathbb{R}^n$, there exists $x \in S^n$ with $h(x) = h(-x)$.*

Theorem 2 follows immediately from Theorem 3 and the following theorem.

Theorem 4. *Let Δ and K be as in Theorem 1. There exists a continuous map $g : S^n \to |K|$ such that for every $x \in S^n$ the supports of $g(x)$ and $g(-x)$ are disjoint.*

In this brief note we are going to give a new very simple proof of Theorem 4. We present in it an explicit construction of the function g.

2 Proof of Theorem 4

Assume that $\Delta = (x_0, \dots, x_{n+1})$. Let K_1 be the simplicial complex with $\{x_0, \dots, x_{n+1}\}$ as its set of vertices and all proper faces of Δ as its simplices. Let K_2 be the barycentric subdivision of K_1. Let $\omega : |K_2| \to |K_2|$ be the free Z_2-action defined as follows. If $T \subset \{x_0, \dots, x_{n+1}\}$ is the vertex-set of a simplex σ of K_1 and c_σ is the barycentre of σ, then let

$$\omega(c_\sigma) = c_{\sigma'},$$

where σ' is the simplex of K_1 whose vertex-set T' is the complement of T, that is

$$T' = \{x_0, \dots, x_{n+1}\} \setminus T.$$

Thus we have defined ω on the vertices of K_2. Let us extend ω linearly to $|K_2|$, that is for any $x \in (c_{\sigma_1}, \dots, c_{\sigma_r}) \in K_2$ having the following barycentric representation

$$x = \sum_{i=1}^{r} \mu_i c_{\sigma_i},$$

let

$$\omega(x) = \sum_{i=1}^{r} \mu_i \omega(c_{\sigma_i}).$$

Clearly, ω is well defined and there is a homeomorphism $f : S^n \to |K_2|$ which is equivariant with respect to the antipodal map on S^n and ω on $|K_2|$, that is such that for every $x \in S^n$ the following equality holds:

$$f(-x) = \omega(f(x)).$$

Therefore, to prove our theorem, it is enough to show the existence of a continuous map $h : |K_2| \to |K|$ such that for every $x \in |K_2|$ the supports of $h(x)$ and $h(\omega(x))$ are disjoint.

Let K_3 be the barycentric subdivision of K_2. We shall define h on the vertices of K_3 first. Let d_A be the barycentre of the simplex $A = (c_{\sigma_1}, \dots, c_{\sigma_r})$ of K_2. Since A is a simplex of K_2, we can assume that σ_i is a proper face of σ_{i+1} for $i = 1, \dots, r-1$. Define

$$h(d_A) = c_{\sigma_1}.$$

Now let us extend h linearly to $|K_3| = |K_2|$, that is for $x \in (d_{A_1}, \dots, d_{A_s}) \in K_3$ with the barycentric representation

$$x = \sum_{i=1}^{s} \mu_i d_{A_i},$$

let

$$h(x) = \sum_{i=1}^{s} \mu_i h\left(d_{A_i}\right).$$

Now we shall show that for every $x \in |K_2|$ the supports of $h(x)$ and $h(\omega(x))$ in K are disjoint. Note first that if d_A is the barycentre of a simplex $A = (c_{\sigma_1}, \dots, c_{\sigma_r})$, then

$$\omega(d_A) = \omega\left(\frac{1}{r}\sum_{i=1}^{r} c_{\sigma_i}\right) = \frac{1}{r}\sum_{i=1}^{r} \omega(c_{\sigma_i}) = d_B$$

where

$$B = (\omega(c_{\sigma_1}), \dots, \omega(c_{\sigma_r})).$$

For $x \in |K_2|$, let

$$\{A_1, \dots, A_r\}$$

be the support of x in K_3 and

$$\{B_1, \dots, B_r\}$$

the support of $\omega(x)$ in K_3, where $B_i = \omega(A_i)$ for $i = 1, \dots, r$. Let

$$\{\sigma_{i,1}, \dots, \sigma_{i,s_i}\}$$

be the vertex-set of A_i for $i = 1, \dots, r$, where $\sigma_{i,j}$ is a proper face of $\sigma_{i,j+1}$ for $j = 1, \dots, s_i - 1$. Now let

$$\{\sigma'_{i,1}, \dots, \sigma'_{i,s_i}\}$$

be the vertex-set of B_i for $i = 1, \dots, r$. Since the vertex-set of $\sigma'_{i,j}$ is the complement of the vertex-set of $\sigma_{i,j}$, the simplex $\sigma'_{i,j+1}$ is a proper face of $\sigma'_{i,j}$, for all $i = 1, \dots, r$ and $j = 1, \dots, s_i - 1$.

Since $h(A_i) = \sigma_{i,1}$ for $i = 1, \dots, r$, the support of $h(x)$ in K_2 is the set

$$\{\sigma_{1,1}, \sigma_{2,1}, \dots, \sigma_{r,1}\},$$

and since $h(B_i) = \sigma'_{i,s_i}$ for $i = 1, \dots, r$, the support of $h(\omega(x))$ in K_2 is the set

$$\{\sigma'_{1,s_1}, \sigma'_{2,s_2}, \dots, \sigma'_{r,s_r}\}.$$

We can assume that A_i is a proper face of A_{i+1} for $i = 1, \dots, r$. Then $\sigma_{i+1,1}$ is a (not necessarily proper) face of $\sigma_{i,1}$ for $i = 1, \dots, r$, and thus the support of $h(x)$ in Δ^{n+1} is the vertex-set of $\sigma_{1,1}$. Since A_i is a proper face of A_{i+1}, the simplex B_i is a proper face of B_{i+1} for $i = 1, \dots, r$. Therefore, $\sigma'_{i+1,s_{i+1}}$ is a face of σ'_{i,s_i} for $i = 1, \dots, r$, and thus the support of $h(\omega(x))$ in K is the vertex-set of σ'_{1,s_1}. Now recall that the vertex-set of σ'_{1,s_1} is the complement of the vertex-set of σ_{1,s_1}. But the vertex-set of $\sigma_{1,1}$ is contained in the vertex-set of σ_{1,s_1} so the supports of $h(x)$ and $h(\omega(x))$ in K are disjoint, and the theorem is proved.

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