# Balanced Transitive Orientations

C. A. Rodger*
Department of Discrete and Statistical Sciences
120 Mathematics Annex
Auburn University
Auburn, AL 36849-5307 USA

**Abstract**

A transitive orientation of a partial triple system $(S,T)$ of index $2\lambda$ is a partial transitive triple system formed by replacing each triple $t \in T$ with a transitive triple defined on the same vertex set as $t$ in such a way that each ordered pair occurs in at most $\lambda$ of the resulting transitive triples. A transitive orientation $(S_1,T_1)$ of $(S,T)$ is said to be balanced if for all $\{u,v\} \subseteq S$, if $\{u,v\}$ occurs in $\ell$ triples in $T$ then $\lfloor \ell/2 \rfloor$ and $\lceil \ell/2 \rceil$ transitive triples in $T_1$ contain the arc $(u,v)$ and $(v,u)$ respectively. In this paper it is shown that every partial triple system has a balanced transitive orientation. This result is then used to prove the existence of certain transitive group divisible designs.

## 1 Introduction

A *(partial) triple system* of order $n$ and *index* $\lambda$ $(p)TS(n,\lambda)$ is an ordered pair $(S,T)$ where $S$ is a set of size $n$, the elements of which are called *symbols*, and $T$ is a collection of 3-element subsets of $S$, the elements of which are called *triples*, such that each pair of symbols in $S$ occurs in exactly (at most) $\lambda$ triples in $T$. The existence of $TS(n,\lambda)$s was settled by

*Research supported by NSF grant DMS-9531722

Hanani [6]. However many special partial $TS(n, \lambda)$s are also of particular interest. For example, a *group divisible design* $GDD(n, m; \lambda_1, \lambda_2)$ is an ordered triple $(V, G, B)$ where $V$ is a set of size $nm$, $G$ is a partition of $V$ into $m$ sets of size $n$, the elements of which are called *groups*, and $B$ is a collection of triples of $V$ such that each pair of symbols that occur in the same group (in different groups) occur together in exactly $\lambda_1$ (exactly $\lambda_2$) triples in $B$. Recently, the existence problem for $GDD(n, m; \lambda_1, \lambda_2)$s was solved by Fu, Rodger and Sarvate [4, 5].

**Theorem 1.1** **([4, 5])** *Let $n, m, \lambda_2 \geq 1$ and $\lambda_1 \geq 0$. There exists a $GDD(n, m; \lambda_1, \lambda_2)$ if and only if*

*(1) 2 divides $\lambda_1(n-1) + \lambda_2(m-1)n$,*

*(2) 3 divides $\lambda_1 mn(n-1) + \lambda_2 m(m-1)n^2$,*

*(3) if $m = 2$ then $\lambda_1 \geq \lambda_2 n/2(n-1)$, and*

*(4) if $n = 2$ then $\lambda_1 \leq (m-1)\lambda_2$.*

Also, Hoffman [8] has considered a $(d, v, \lambda_1, \lambda_2)$-triple system; that is, an ordered triple $(D, V, B)$ where $D \cap V = \emptyset$, $|D| = d$, $|V| = v$, and $B$ is a collection of triples of $D \cup V$ such that each pair $p \subseteq D \cup V$ occurs in exactly $0, \lambda_1$ or $\lambda_2$ triples in $B$ if $|D \cap p|$ is $0, 1$ or $2$ respectively.

**Theorem 1.2** **([8])** *There exists a $(d, v, \lambda_1, \lambda_2)$-triple system if and only if*

*(1) if $v \neq 0$ then 2 divides $\lambda_1 d$,*

*(2) if $d \neq 0$ then 2 divides $\lambda_2(d-1) - \lambda_1 v$,*

*(3) 3 divides $d(\lambda_2(d-1) - \lambda_1 v)$, and*

*(4) if $d \neq 0$ then $\lambda_1 v \leq \lambda_2(d-1)$, with equality holding if $d = 2$.*

It is well known that a (partial) triple system of order $n$ and index $\lambda$ can also be defined as an edge-disjoint decomposition of (a subgraph of) $\lambda K_n$ into copies of $K_3$. Then it is of no surprise to see that the following directed analogues of triple systems have also been well studied.

Let $\lambda D_n$ be the complete directed graph $(V, A)$ of index $\lambda$ (so $|V| = n$, and for all $u$, $v \in V$, $A$ contains $\lambda$ arcs joining $u$ to $v$, and $\lambda$ arcs joining $v$ to $u$). A *transitive triple* is a tournament on 3 vertices that is not a directed cycle. Denote the transitive triple $t$ with arc set $A(t) = \{(a,b),(a,c),(b,c)\}$ by $[a,b,c]$. A *(partial) transitive triple system* of *order* $n$ and *index* $\lambda$ $(p)TTS(n,\lambda)$ is an ordered pair $(S,T)$, where $S$ is a set of size $n$, and $T$ is a collection of transitive triples whose arcs form a partition of (a subset of) the arc set of $\lambda D_n$ with vertex set $S$.

There is an obvious connection between transitive triple systems and triple systems: if each transitive triple $[a,b,c]$ in a $TTS(n,\lambda)$ is replaced by a triple $\{a,b,c\}$ then the result is a $TS(n,2\lambda)$, known as the *underlying* $TS(n,2\lambda)$. It turns out that the reverse is also possible; namely, each triple $\{a,b,c\}$ in a $TS(n,2\lambda)$ can be replaced by a transitive triple with vertex set $\{a,b,c\}$ such that the result is a $TTS(n,\lambda)$. (This reverse statement is not true if decompositions of $\lambda D_n$ into directed 3-cycles are considered instead of into transitive triples.) This was first proved by Colbourn and Harms [3] (see [2] when $\lambda = 1$), but a simpler proof was recently found [9]. This new simpler algorithm also easily allows the study of directing the triples in a $pTS(n,2\lambda)$, and it is these results which we present here. In particular, we obtain analogues of Theorems 1.1 and 1.2 for transitive triple systems (see Theorems 3.1 and 3.2).

A *transitive orientation* of a partial $TS(n,2\lambda)(S,T)$ is a partial $TTS(n,\lambda)$ formed by replacing each triple $\{a,b,c\}$ in $T$ with some transitive triple with vertex set $\{a,b,c\}$. We say a transitive orientation $(S_1,T_1)$ of a partial $TS(n,2\lambda)$ $(S,T)$ is *balanced* if for all $\{u,v\} \subseteq S$, if $\{u,v\}$ occurs in $\ell$

triples in $T$ then $\lfloor \ell/2 \rfloor$ and $\lceil \ell/2 \rceil$ transitive triples in $T_1$ contain the arc $(u,v)$ and $(v,u)$ respectively.

For any undefined graph theoretic terminology see [1].

## 2 Balanced Transitive Orientations

A $k$-edge-coloring of a graph $G$ is said to be *equitable* if $|c_i(v) - c_j(v)| \leq 1$ for $1 \leq i < j \leq k$ and for all $v \in V(G)$, where $c_i(v)$ is the number of edges in $G$ incident with $v$ colored $i$. An easy proof of the following result can be found in [9].

**Theorem 2.1** ([10]) *For all $k \geq 1$, every bipartite graph has an equitable $k$-edge-coloring.*

The proof of the following theorem is essentially the transitive orientation algorithm in [9]. (This result may also be obtained by redefining the "next" function in the algorithm of Harms and Colbourn in [7].)

**Theorem 2.2** *Every partial $TS(n, 2\lambda)$ has a balanced transitive orientation.*

**Proof:** Let $(\mathbb{Z}_n, T)$ be a partial $TS(n, 2\lambda)$ with $T = \{t_1, \ldots, t_{|T|}\}$. For each triple $t_i = \{a, b, c\} \in T$, let $E(t_i) = \{\{a,b\}, \{a,c\}, \{b,c\}\}$ be the edges in $t_i$, and with $a < b < c$ let $p(t_i) = \{a, c\}$. Let $\ell(\{a,b\})$ be the number of triples in $T$ that contain $\{a,b\}$.

Form a simple bipartite graph $B$ with bipartition $X = \{t'_1, \ldots, t'_{|T|}\}$ and $Y = E(T) = \{\{a,b\} \mid \ell(\{a,b\}) \geq 1\}$ by joining $t'_i$ to $\{u,v\} \in E(T)$ if and only if $\{u,v\} \in E(t_i) \backslash p(t_i)$. Clearly $d_B(t'_i) = 2$ for $1 \leq i \leq |T|$, and $d_B(e) \leq \ell(e) \leq 2\lambda$ for all $e \in E(T)$.

Using Theorem 2.1, give $B$ an equitable 2-edge-coloring with colours 1 and 2. Then,

(i) for $1 \leq i \leq |T|$, $t'_i$ is incident with one edge of each color, and

(ii) for each $e \in E(T)$, $e$ is incident with at most $\lceil \ell(e)/2 \rceil \leq \lambda$ edges of each color.

To obtain a balanced transitive orientation of $(\mathbb{Z}_n, T)$, proceed as follows. For each edge joining $t'_i$ and $e = \{a, b\}$ with $a < b$ that is colored 1 or is colored 2 in $B$, replace the edge $\{a, b\}$ in $T$ with the arc $(a, b)$ or $(b, a)$ respectively. Then by (i), each triple $t_i = \{a, b, c\} \in T$ with $a < b < c$ now has the edges $\{a, b\}$ and $\{b, c\}$ replaced with arcs $(a, b)$ and $(c, b)$ or with arcs $(b, a)$ and $(b, c)$. So, regardless of how the edge $p(t_i)$ is oriented, $t_i$ will be replaced by a transitive triple. Also, by (ii), for each $u$, $v \in \mathbb{Z}_n$ there are at most $\lceil \ell(e)/2 \rceil$ arcs joining $u$ and $v$ in each direction. Therefore the arcs $p(t_i)$ for $1 \leq i \leq |T|$ can be oriented greedily, ensuring that $\lfloor \ell(e)/2 \rfloor$ or $\lceil \ell(e)/2 \rceil$ arcs join $u$ and $v$ in each direction. □

# 3 Some Consequences

With Theorem 2.2 in hand, many existence results for special partial triple systems can now immediately be used to prove existence results for corresponding partial transitive triple systems. We give two such examples here.

Define a *transitive group divisible design* $TGDD(n, m; \lambda_1, \lambda_2)$ to be an ordered triple $(V, G, B)$, where $V$ is a set of size $nm$, $G$ is a partition of $V$ into $m$ sets of size $n$, each element of $G$ being called a *group*, and $B$ is a collection of transitive triples defined on $V$ such that each ordered pair of symbols occurring in the same (different) groups occur together in exactly $\lambda_1$ (exactly $\lambda_2$) triples in $B$.

**Theorem 3.1** *Let $n, m, \lambda_2 \geq 1$ and $\lambda_1 \geq 0$. There exists a $TGDD(n, m; \lambda_1, \lambda_2)$ if and only if*

*(1)* 3 *divides* $\lambda_1 mn(n-1) + \lambda_2 m(m-1)n^2$

*(2) if* $m = 2$ *then* $\lambda_1 \geq \lambda_2 n/2(n-1)$, *and*

*(3) if* $n = 2$ *then* $\lambda_1 \leq (m-1)\lambda_2$.

**Proof:** By Theorem 2.2, there exists a $TGDD(n, m; \lambda_1, \lambda_2)$ if and only if there exists a
$GDD(n, m; 2\lambda_1, 2\lambda_2)$. The result therefore follows from Theorem 1.1. □

Define a $(d, v, \lambda_1, \lambda_2)$-transitive triple system to be an ordered triple $(D, V, B)$ where $D \cap V = \emptyset$, $|D| = d$, $|V| = v$, and $B$ is a collection of transitive triples such that each ordered pair of symbols in $D \cup V$ occurs in exactly $0, \lambda_1$ or $\lambda_2$ triples in $B$ if $|D \cap p|$ is $0, 1$ or $2$ respectively.

**Theorem 3.2** *There exists a* $(d, v, \lambda_1, \lambda_2)$*-transitive triple system if and only if*

*(1)* 3 *divides* $d(\lambda_2(d-1) - \lambda_1 v)$, *and*

*(2) if* $d \neq 0$ *then* $\lambda_1 v \leq \lambda_2(d-1)$, *with equality holding if* $d = 2$.

**Proof:** By Theorem 2.2 there exists a $(d, v, \lambda_1, \lambda_2)$-transitive triple system if and only if there exists a $(d, v, 2\lambda_1, 2\lambda_2)$-partial triple system, so the result follows from Theorem 1.2. □

# References


[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North Holland, 1976.

[2] C. J. Colbourn and M. J. Colbourn, *Every twofold triple system can be directed*, J. Combin. Theory Ser. (A), 34 (1983), 375-378.

[3] C. J. Colbourn and J. J. Harms, *Directing triple systems*, Ars Combin., 15 (1983), 261-266.

[4] H. L. Fu, C. A. Rodger and D. G. Sarvate, *The existence of group divisible designs with first and second associates, having block size* 3, submitted.

[5] H. L. Fu and C. A. Rodger, *Group divisible designs with two associate classes:* $n = 2$ *or* $m = 2$, submitted.

[6] H. Hanani, *The existence and construction of balanced incomplete block designs*, Ann. Math. Statist., 32 (1961), 361-386.

[7] J. J. Harms and C. J. Colbourn, *An optimal algorithm for directing triple systems using Eulerian circuits*, Ann. Discrete Math, 27 (1985), 433-438.

[8] D. G. Hoffman, *An extension of the Doyen-Wilson Theorem*, submitted.

[9] C. A. Rodger, *A solution for the embedding problem for partial transitive triple systems*, Bull. London Math Soc., to appear.

[10] D. de Werra, *Equitable colorations of graph*, Rev. Fran. Rech. Oper., 5 (1971), 3-8.