A Faster Algorithm for Least Deviant Path

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ABSTRACT. The least deviant path was defined by Klostermeyer [1] as the path between two vertices u and v that minimizes the difference between the largest and smallest weights on the path. This paper presents an $O(E \log E)$ time algorithm for this problem in undirected graphs, improving upon the previously given $O(E^{1.793})$ time algorithm. The same algorithm can also be used to solve the problem in $O(VE)$ time in directed graphs.

1 Introduction

The least deviant path in a graph was defined by Klostermeyer [1] to be a path between vertices u and v that minimizes the difference between the largest edge weight and the smallest edge weight on the path. For example, if there exists a path (regardless of length) between u and v that has all edge weights equal, that path has *deviation* zero and is optimal by definition. An algorithm was given in [1] that runs in time $O(E \log E + Em^{0.793})$ where m is the number of distinct edge weights in the graph. In the worst case, this running time is $O(E^{1.793})$. That algorithm applies to both directed and undirected graphs and is based on a type of two-dimensional binary search. In this paper we give an $O(E \log E)$ time algorithm for undirected graphs that uses a property shown by a simple lemma to prune the search space and the dynamic tree data structure of Sleator and Tarjan [2]. The same

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algorithm is also shown to solve the problem in $O(VE)$ time in directed graphs, using a different data structure.

2 The Algorithm

Let $G = (V, E)$ be the input graph; for now we focus on undirected graphs; in Section 3 we shall consider directed graphs. Let u be the source vertex and v the destination vertex. That is, we wish to find the least deviant path from u to v. The algorithm consists of three phases. In Phase 1, the edges are sorted in increasing order of weight, which requires $O(E \log E)$ time in general.

2.1 Queuing Phase

Phase 2 is called the queuing phase. In the queuing phase, edges are processed in increasing order of weight. Iteratively, all edges of a given weight are added to the graph until a path exists from u to v, at which point the last weight which was added is recorded in a queue, Q. The graph is then emptied, and the process continues with the next highest edge weight. Let m denote the number of distinct edge weights. We refer to the ith smallest distinct edge weight as $w[i]$. Formally:

Phase 2.
$Q :=$ empty queue
$G' = (V,$ empty edge set$)$
$best := \infty$
$lo := w[1]$
for $i := 1$ to m
        add all edges of weight $w[i]$ to G'
        if there is a path in G' from u to v then
                enqueue($w[i]$)
                $best := min(best, w[i] - lo)$
                let $G' := (V,$ empty edge set$)$
                $lo := w[i+1]$
end
if G' contains any edges then enqueue($w[m]$)

Let us call the edge weights contained in Q the *queued* weights, except possibly for $w[m]$ which is called the *leftover* edge weight if it was enqueued outside the "for" loop. $w[m]$ is called a queued weight if it was enqueued inside the "for" loop.

In order to determine whether or not a path exists between u and v at a given point in time, we use the dynamic tree data structure of Sleator and Tarjan [2]. This data structure, which is actually a forest of vertex disjoint trees, enables add edge, delete edge, and find_component operations to be done in $O(\log V)$ time each. We note that these structures are trees; prior to the adding of any individual edge, we test if the two end-vertices are in the same (connected) component of G' already, and if so, we do not add the new edge. That is, the vertices in a tree will be those vertices in a component of G'. We also note that some of the tree operations of [2] require that a certain vertex either be the root or a non-root vertex. However, this constraint is easily satisfied, as Sleator and Tarjan provide an $evert(x)$ operation, done in $O(\log V)$ time, that makes vertex x the root of its tree. Using these operations, it is easy to see that Phase 2 requires $O(E \log V)$ time, since each edge in E is added and deleted from G' exactly one time.

We now prove two lemmas.

Lemma 1. *Let the edge weights on a least deviant path from u to v have weights in the range $[a..b]$. Then there is at most one queued edge weight in the range $[a..b]$.*

Proof: Let y and z be two queued edge weights. Also denote edges of those weights by y and z. Suppose by way of contradiction that a least deviant path P contains both y and z weight edges. Without loss of generality, assume $y > z$. Then the deviation of P is at least $y - z$. By definition, in Phase 2, a path from u to v was formed in G' causing y to be enqueued – since the leftover edge weight (if it exists) is not a queued weight. Then when y was enqueued, the resulting path between u and v had deviation at most $y - t$ where t was the first edge weight, denoted by lo in the algorithm, to G' added since G' was previously emptied. But it must be that $t > z$, hence P cannot be a least deviant path. □

Lemma 2. *Let the edge weights on a least deviant path from u to v have weights in the range $[a..b]$. Then there is at least one queued edge weight or leftover edge weight in the range $[a..b]$.*

Proof: Suppose no queued/leftover weights fall in the range of the weights in a least deviant path P. There must be at least one queued weight in Q if a path exists in G between u and v. If there exists only one queued/leftover edge weight in Q, then that weight must be $w[m]$ – in which case it is clear that any least deviant path must contain $w[m]$. Now suppose there are at least two queued/leftover edge weights. Suppose P contains edge weights in the range $[c..d]$. Let y be the largest queued weight less than c and z the smallest queued/leftover weight greater than d. Let y' be the first edge weight added to G' after y was enqueued. Then no path was formed in G' until z was added. Since $d < c$, it must be that P is not a path from u to v. □

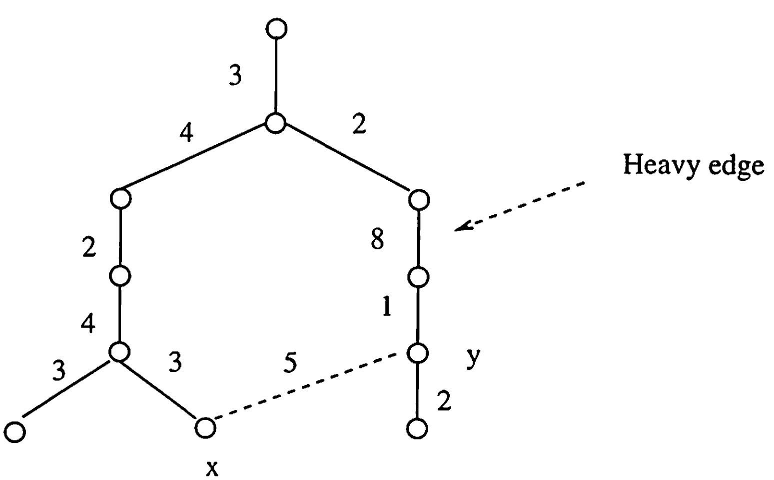
2.2 Path Testing Phase

Phase 3 of the algorithm uses the queued/leftover weights from Phase 2 to build and test paths. Lemmas 1 and 2 allow us to restrict our search for a least deviant path to paths that contain a queued/leftover weight. The idea of Phase 3 is to start with an empty graph and add all edges having weight equal to the current weight, w, where w is an edge weight from Q. We then add edge weights that are larger than w until a path from u to v is formed or until the deviation is larger than the best known deviation. We then remove these "large" weights one at a time (largest first) and add small edge weights one weight at a time – weights less than w (again, largest first), comparing the deviation each time with the best known deviation, provided a uv path exists. Formally:

```
for each w in Q do
        let G' = (V, empty edge set)
/* denote the index of w's weight in the sorted order by w[i] */
        dev := 0
        hi := i
        while (u and v not in same component of G') and (dev ≤ best) do
                add all edges of weight w[hi] to G'
                if u and v in same component of G then
                 best := min(best, w[hi] − w)
                else hi := hi+1
        end while
/* denote the index of the next lowest weight in Q by Q[prev] and assume
 Q[prev] = 1 for the first weight in Q */
        lo := i − 1
        while (w − w[lo] ≤ best) and (lo > Q[prev]) and (hi ≥ i) do      (1)
/* Delete the largest edges in G' until u and v in different components of
 G' */
                while (u and v in same component of G') do                (2)
                        remove edges from G' of weight hi
                        hi := hi − 1
                end while
/* Now add small edges */
                add all edges of weight lo to G'
                if u and v in same component of G then
                        best := min(best, w[hi] − w[lo])
                else lo := lo − 1
        end while
        remove w from Q
end for
```

Another way of thinking about Phase 3 that we have a narrow range surrounding each weight w in Q and look for a least deviant path within that range. The narrowness of the range is bounded by the deviation of the best path found so far and the weights of the next highest and lowest weights in Q.

We need to specify some additional details about when edges are added and deleted from the dynamic tree structure. The key to the correctness of the algorithm is ensuring that the acyclic dynamic tree structure correctly models the components of G'. An obstacle occurs when we add an edge (x, y) to G' if x and y are already in the same component of G' (and hence the same tree). Recall that w denotes the weight of the current edge from Q. There are two scenarios. First, if the weight $(x, y) > w$, we disregard the edge – it is not added to the dynamic tree structure. Edge (x, y) is then "marked" so that subsequently, when (x, y) is deleted from G', no tree operation is performed. On the other hand, if weight $(x, y) < w$, we do the following. In the tree containing x and y, find the largest weight edge that is an "ancestor" of x or y. That is, the ancestor edge lies on the path from x (y) to the nearest common ancestor vertex of x and y. Call this edge the *heavy* edge. See Figure 1 for an example. This can be done in $O(\log V)$ time by a simple modification of the *findcost* operation of [2]. Delete the heavy edge from the tree and add edge (x, y) – this is called *replacing* an edge in the tree.



Tree when (x, y) is added to graph
Figure 1. Replacing a heavy edge

We now need to prove a lemma.

Lemma 3. *Vertices x and y are in the same tree if and only if x and y are in the same component of G', for all vertex pairs, x, y.*

Proof: "$\rightarrow$" This direction is simple, since we only "link" disjoint trees when an edge is added to G' whose endvertices are in different trees. Furthermore, if two vertices become "disconnected" in the graph (lie in different components), the corresponding edges in the tree must have been deleted as well, thereby partitioning the tree.

"$\leftarrow$" In Phase 3, edges are first added in increasing order of weight; if an edge induces a cycle in a tree, it is not added to the tree. Once a path from u to v is found (or the edge weights become too large), edges are deleted in decreasing order of weight as new edges (those with weight less than w) are added in decreasing order of weight. Suppose x and y are in the same component of G', but are not in the same dynamic tree. Assume without loss of generality that x and y are the first such pair of vertices this happened to in the course of the algorithm. Therefore we can assume that x and y were in the same tree, T, at some point, but were later placed in different trees when an edge(s) was deleted. This partitioning occurs only if an edge in the path between them in T was deleted. Let (a, b) be this edge. Thus in G', there exists a path(s) between x and y not using (a, b). But from our rules for replacing edges, it must be that at least one edge on this path(s) is of weight greater than or equal to (a, b), which implies x and y are not in the same component of G'. □

The correctness of the algorithm follows from the three lemmas, since we are essentially testing each possible deviation that contains exactly one edge weight from Q (provided the deviation of the path is not too large) and since the data structure accurately models G'.

It is easy to see that each edge in G is added/deleted from G' only a constant number of times, since each edge can be considered during at most three iterations of the "for each w in Q" loop. In fact, an edge can only be considered in three iterations if there is a leftover edge weight; otherwise each edge is considered in at most two iterations. Each edge add/delete requires at most two find_component operations and two findcost operations, plus at most one add and at most one delete tree operation. Hence Phase 3 runs in $O(E \log V)$ time, which means the entire algorithm runs in $O(E \log E)$ time.

3 Directed Graphs

We show how to make the algorithm presented in Section 2 run in $O(VE)$ time in directed graphs, an improvement over the $O(E^{1.793})$ algorithm from [1] for digraphs that are not "sparse." As above, let u be the source vertex and v the destination. We use the following simple data structure. As edges are added in Phase 3 of the algorithm (remember, we start with queued weight edges and keep adding and deleting edges of certain weights, checking for a uv path), we maintain a (directed) tree T with u as the root.

T contains all vertices reachable from u using edges that have been added during the phase (and not deleted). In addition, the path from u to x in T will be a path in G. For practical considerations, it is necessary for each vertex in T to have a pointer to its parent. Whenever v is added to the tree we proceed to the next step in the algorithm: systematically deleting edges until v is no longer in the tree. Each vertex x in T also has a *value* field that records that largest edge weight on the path from u to x in the tree.

Some details are now given. When considering an edge (a, b) in Phase 3 of the algorithm, we add it to T if (1) a is in T and b is not in T; or (2) if b is in T and $value(b) > \max(value(a), weight(a, b))$. In the latter case, we disconnect b from its parent in T and connect it as a child of vertex a with an updated value field. If vertex a is not in T (which can be determined in $O(1)$ time, by maintaining an extra pointer or bit), we buffer edge (a, b) as follows: an array $FROM[1..V]$ has pointers to linked lists maintained for each vertex y in G; the list for vertex y contains edges of the form (y, z) and the weight of those edges. Of course, the linked lists only contain edges that have been "added" so far (and which are not in T). After edge (a, b) is added to T, we do the following. Traverse the $FROM[b]$ linked list adding to T each edge whose weight is within the current range under consideration by the algorithm, using the rules for adding edges to T given above. Edges whose weights exceed the current range may be removed permanently from the linked list. We then do the same for each of b's neighbors that were just added to the tree.

When edge (a, b) is deleted by the algorithm we remove b from T and buffer all edges in the subtree rooted at b; this can be done in $O(V)$ time. By buffering, we mean insert each "orphan" edge (x, y) to the end of the appropriate linked list, $FROM[x]$, which can be done in $O(1)$ time for each edge buffered. In this way it is easy to detect when v is deleted from the tree. Thus over the life of the algorithm, we spend $O(VE)$ time doing deletes. In addition, we spend $O(E+V)$ time in total doing edge additions if we only count the first time any edge is added. But an edge may be added, deleted and added again many times in the course of considering a particular queued weight. However, each edge (a, b) can only be added $O(V)$ times over the life of the algorithm, since vertex a has degree at most $V - 1$. Therefore the total cost of adding all edges will be $O(VE)$ over the life of the algorithm.

Given the simplicity of this data structure, we speculate that it is possible to find a faster algorithm for the case of directed graphs, perhaps one that runs in $O(E^{1.5})$ time.

CYCLES IN 2-FACTORIZATIONS

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Abstract. For odd v, we determine (apart from eight unresolved cases) the total number of cycles that may occur in a 2-factorization of K_v.

1. Introduction

Recent papers by Billington and Bryant [1] and by Dejter, Franek, Mendelsohn and Rosa [4] have addressed questions about counting cycles in decompositions of the complete graph K_v. The former paper establishes the number of cycles achievable while the latter determines, for $v \equiv 1$ or 3 (mod 6), the number of triangles achievable in those decompositions which form a 2-factorization of K_v. In this paper we determine, for odd values of v, the number of cycles achievable in a 2-factorization of K_v (apart from eight unresolved cases). All our results come from direct constructions.

A *2-factor* of a graph G is a subgraph of G which contains all the vertices of G and is regular of degree 2. A *2-factorization* of G is a partition of the edges of G into 2-factors. We use the notation $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ to denote a 2-factorization $\mathcal{F}$ with 2-factors $F_1, F_2, \ldots, F_k$. Clearly, for G to possess a 2-factorization it must be regular of even degree.

If v is an odd integer then the complete graph K_v can be decomposed (2-factored) into $(v-1)/2$ Hamiltonian cycles. When $v \equiv 3 \pmod 6$ there exists a Kirkman triple system of order v, KTS(v), and such a system provides an alternative 2-factorization of K_v in which each 2-factor is itself composed entirely of triangles (i.e. cycles of length 3). In the Hamiltonian case the total number of cycles in the 2-factorization is $(v-1)/2$ while in the Kirkman triple system case the total is $v(v-1)/6$.

It is easy to see that the total number of cycles in a 2-factorization of K_v (v odd) must lie in the interval $I(v) = [(v-1)/2, M_v]$, where $M_v = \frac{v-1}{2}\lfloor \frac{v}{3} \rfloor$. In other words, if $C(v) = \{c$: there exists a 2-factorization of K_v with exactly c cycles$\}$ then $C(v) \subseteq I(v)$. We refer to $C(v)$ as the *cycle spectrum* of v. The purpose of this paper is to prove that for $v \geq 41$, $C(v) = I(v)$. (Throughout this paper an interval $[a, b]$ will be regarded as a set of integers, so that $I(v)$ only contains integers.) We will also show that for v in the interval $[23, 39]$ there is at most one element of $I(v)$ which does not lie in $C(v)$ and that for $v \in [11, 21]$, $C(v) = I(v)$. We believe that, in fact, $C(v) = I(v)$ for all $v \geq 11$.

Throughout this paper v will denote an odd integer. We deal with $v \geq 41$ using two basic constructions, one of which is design-theoretical and establishes that the upper part of $I(v)$ lies in $C(v)$. The other construction is graph-theoretical and establishes that the lower part of $I(v)$ lies in $C(v)$. For $v \geq 41$ the upper and lower parts overlap. For $v \leq 39$ we use the two basic constructions together with a variety of ad-hoc methods.

We now give definitions of the designs used in our basic construction. If v, λ are positive integers and K is a subset of positive integers, then a (v, K, λ)-PBD (*Pairwise Balanced Design*) is a pair $(V, \mathcal{B})$ where V is a v-element set and $\mathcal{B}$ is a collection of subsets of V, called blocks, with the following properties:

a) If $B \in \mathcal{B}$ then $|B| \in K$.

b) Each 2-element subset of V is contained in precisely λ blocks.

In the case when $K = \{k\}, k < v$, the design is called a (v, k, λ)-BIBD (*Balanced Incomplete Block Design*). We will only be concerned with the case $\lambda = 1$ in this paper. A KTS(v) is a (v,3,1)-BIBD with the additional property that the blocks may be partitioned into $(v-1)/2$ *parallel classes* in such a way that within each class the $v/3$ blocks are mutually disjoint (i.e. are parallel). Such a design exists if and only if $v \equiv 3 \pmod 6$.

We will also make use of *Group Divisible Designs* (GDDs). Suppose $k, s, a_1, a_2, \ldots, a_s, g_1, g_2, \ldots, g_s$ are positive integers with $g_1, g_2, \ldots, g_s$ all distinct, and, to avoid trivial cases, if $s = 1$ we require $a_1 \neq 1$. A k-GDD of type ${g_1}^{a_1}{g_2}^{a_2}\ldots{g_s}^{a_s}$ is a triple $(V, \mathcal{G}, \mathcal{B})$ where V is a set of cardinality $v = g_1a_1 + g_2a_2 + \cdots + g_sa_s$, $\mathcal{G}$ is a partition of V into a_1 sets of cardinality g_1, a_2 sets of cardinality $g_2, \ldots, a_s$ sets of cardinality g_s (the groups) and $\mathcal{B}$ is a collection of subsets of V (the blocks) with the following properties:

a) If $B \in \mathcal{B}$ then $|B| = k$.

b) Every 2-element subset of V is contained in precisely one group or one block, but not both.

The reader is referred to [2] for further information about PBDs, BIBDs and GDDs.

2. The design-theoretic construction

A 2-factorization $\mathcal{F}$ of K_v is said to be a *2^*-factorization* if there exists a vertex x of K_v which lies in a triangle in each $F \in \mathcal{F}$. Put $C^*(v) = \{c$: there exists a 2^*-factorization of K_v with exactly c cycles$\}$. We refer to $C^*(v)$ as the *-cycle spectrum of v. Clearly, $C^*(v) \subseteq C(v)$. Also, for $v \geq 7$, a 2^*-factorization of K_v must contain at least 2-cycles per 2-factor (one of which is a triangle). Hence $C^*(v) \subseteq [v-1, M_v]$.

Our construction (the "PBD-construction") comes from [4]; it is itself a modification of Wilson's construction for resolvable designs [8].

Construction 2.1

Suppose $(U, \mathcal{B})$ is a $(u, L, 1)$-PBD, and for each $k \in L$ there exists a 2^*-factori- zation of K_{2k+1}. Then there exists a 2^*-factorization of K_{2u+1} on $U \times \{1,2\} \cup \{\infty\}$. Furthermore, if $c_B \in C^*(2|B|+1)$ for a block $B \in \mathcal{B}$ then $\Sigma_{B\in\mathcal{B}}(c_B - |B|) + u \in C^*(2u+1)$.

The proof is given essentially in [4]. We make two observations which allow us to make slight modifications to the construction. Firstly, if $\mathcal{B}$ contains a single 2-element block $\{x, y\}$ then the construction cannot be applied as described because there is no 2^*-factorization of K_5. However, we may omit this block from the construction and subsequently replace the triangles $(\infty x_1 x_2)$ and $(\infty y_1 y_2)$ which are formed in the construction from the other blocks containing x and y, by the pentagons $(\infty x_1 x_2 y_1 y_2)$ and $(\infty x_2 y_2 x_1 y_1)$ which form a 2-factorization of K_5. We still obtain a 2-factorization of K_{2u+1}, although not a 2^*-factorization. Moreover, if $c_B \in C^*(2|B|+1)$ for a block $B \in \mathcal{B}$ then $\Sigma(c_B - |B|) \in C(2u+1)$, where the summation now extends over all blocks $B \in \mathcal{B}$ apart from the block $\{x, y\}$.

Our second observation is that if $B_1, B_2, \ldots, B_l$ are disjoint blocks of cardinality $k_1, k_2, \ldots, k_l$, respectively, then we may replace 2^*-factorizations of K_{2k_i+1} on $B_i \times \{1,2\} \cup \{\infty\}$ by any 2-factorization on the same set and still obtain a 2-factorization of K_{2u+1}. Also, if $c_B \in C^*(2|B|+1)$ for $B \notin \{B_1, B_2, \ldots, B_l\}$ and $c_B \in C(2|B|+1)$ for $B \in \{B_1, B_2, \ldots, B_l\}$, then we have $\Sigma_{B\in\mathcal{B}}(c_B - |B|) + u \in C(2u+1)$.

Both modifications may be made simultaneously provided that the single 2-element block is disjoint from $B_1, B_2, \ldots, B_l$. We will apply Construction 2.1 (and the modified versions) to various PBDs. We need the following results which are established in Section 4 below.

$C^*(7) = \{6\}, C^*(9) = \{8, 9, 10, 12\},$
$\{10, 14, 15\} \subseteq C^*(11), \{12, 23, 24\} \subseteq C^*(13)$
$\{14, 34, 35\} \subseteq C^*(15), \{26, 116, 117\} \subseteq C^*(27).$

We will make extensive use of $(u, \{4, 13\}, 1)$-PBDs which contain precisely one 13-block. These exist for $u \equiv 1$ or 4 (mod 12) provided $u \geq 40$ [cf. [2], §1.13, p.187]. As is customary, we emphasize the fact that there is a single 13-block by appending an asterisk * , and thus, we use the notation $(u, \{4, 13^*\}, 1)$-PBD, and a similar notational device elsewhere. We also use $(u, \{4, 7^*\})$-PBDs containing a single 7-block; these exist for $u \equiv$ 7 or 10 (mod 12) provided $u \geq 22$.

We commence the proofs of our results by examining the residue classes of v modulo 24 (for sufficiently large v) and applying the PBD-construction. We present the first case in some detail but leave the reader to check the details in subsequent cases.

$\mathbf{v = 24s + 3}$ ($\mathbf{v \geq 99}$)

Applying the PBD-construction to a $(12s + 1, \{4, 13^*\}, 1)$-PBD ($s \geq 4$) we can obtain a 2-factorization of K_{24s+3} with the total number of cycles equal to

$\Sigma_{B \in \mathcal{B}}(c_B - |B|) + (12s + 1).$

The PBD contains $(12s^2 + s - 13)$ 4-blocks plus the single 13-block. For each 4-block B, c_B can take any of the values in $C^*(9)$, so each term $(c_B - |B|)$ can, independently, take the value 4, or 5, or 6, or 8. In the case of the 13-block the term $(c_B - |B|)$ can certainly take the values 13, or 103, or 104 (other values may be possible but we do not need them for our purposes). The total number of cycles achievable by the construction may therefore be any integer in the interval

$[4(12s^2 + s - 13) + 13 + (12s + 1), 8(12s^2 + s - 13) + 104 + (12s + 1)] =$
$[48s^2 + 16s - 38, 96s^2 + 20s + 1].$

Note that $M_v = 96s^2 + 20s + 1$; thus the PBD-construction covers (roughly speaking) the upper half of the interval $I(v)$ in the case $v = 24s + 3$ ($s \geq 4$).

$\mathbf{v = 24s + 9}$ ($\mathbf{v \geq 81}$)

A $(12s + 4, \{4, 13^*\}, 1)$-PBD ($s \geq 3$) contains $(12s^2 + 7s - 12)$ 4-blocks plus the single 13-block. The PBD-construction gives $[48s^2 + 40s - 31, 96s^2 + 68s + 12] \subseteq C(v)$, and we note that $96s^2 + 68s + 12 = M_v$.

$\mathbf{v = 24s + 15}$ ($\mathbf{v \geq 63}$)

A $(12s+7,\{4,7^*\},1)$-PBD $(s\geq 2)$ contains $(12s^2+13s)$ 4-blocks plus the single 7-block. Noting that $\{14,34,35\}\subseteq C^*(15)$, the PBD-construction gives $[48s^2+64s+14, 96s^2+116+35]\subseteq C(v)$, and we note $96s^2+116s+35=M_v$.

$v = 24s + 21$ $(v \geq 45)$

A $(12s+10,\{4,7^*\},1)$-PBD $(s\geq 1)$ contains $(12s^2+19s+4)$ 4-blocks plus the single 7-block. The PBD-construction gives $[48s^2+88s+33, 96s^2+164s+70]\subseteq C(v)$, and we note $96s^2+164s+70=M_v$.

For the next few residue classes of v modulo 24 we delete a point from the earlier PBDs, selecting that point so that it does not lie in the single large block. This process enables us to obtain $(u,\{3,4,13^*\},1)$-PBDs and $(u,\{3,4,7^*\},1)$-PBDs.

$v = 24s + 1$ $(v \geq 97)$

A $(12s,\{3,4,13^*\},1)$-PBD is formed $(s\geq 4)$ with $4s$ 3-blocks, $(12s^2-3s-13)$ 4-blocks and a single 13-block. The PBD-construction gives $[48s^2+12s-39, 96s^2]\subseteq C(v)$, and note $96s^2=M_v$.

$v = 24s + 7$ $(v \geq 79)$

A $(12s+3,\{3,4,13^*\},1)$-PBD is formed $(s\geq 3)$ with $(4s+1)$ 3-blocks, $(12s^2+3s-13)$ 4-blocks and a single 13-block. The PBD-construction gives $[48s^2+36s-33, 96s^2+48s+6]\subseteq C(v)$, and note $96s^2+48s+6=M_v$.

$v = 24s + 13$ $(v \geq 61)$

A $(12s+6,\{3,4,7^*\},1)$-PBD is formed $(s\geq 2)$ with $(4s+2)$ 3-blocks, $(12s^2+9s-2)$ 4-blocks and a single 7-block. The PBD-construction gives $[48s^2+60s+11, 96s^2+96s+24]\subseteq C(v)$, and note $96s^2+96s+24=M_v$.

$v = 24s + 19$ $(v \geq 43)$

A $(12s+9,\{3,4,7^*\},1)$-PBD is formed $(s\geq 1)$ with $(4s+3)$ 3-blocks, $(12s^2+15s+1)$ 4-blocks and a single 7-block. The PBD-construction gives $[48s^2+84s+29, 96s^2+144s+54]\subseteq C(v)$, and note $96s^2+144s+54=M_v$.

The final four residue classes of v modulo 24 are treated by deleting two points from our earlier PBDs; the two points are selected so that they do not lie in the single large block. This process enables us to obtain $(u,\{2^*,3,4,13^*\},1)$-PBDs and $(u,\{2^*,3,4,7^*\},1)$-PBDs. We use the modified PBD-construction which permits the occurrence of a single 2-block.

$v = 24s - 1$ $(v \geq 95)$

A $(12s-1, \{2^*, 3, 4, 13^*\}, 1)$-PBD is formed ($s \geq 4$) with one 2-block, $(8s-2)$ 3-blocks, $(12s^2-7s-12)$ 4-blocks and one 13-block. The PBD-construction gives $[48s^2+8s-42, 96s^2-20s+1] \subseteq C(v)$, and note $96s^2-20s+1 = M_v$.

$\mathbf{v = 24s+5}$ ($\mathbf{v \geq 77}$)

A $(12s+2, \{2^*, 3, 4, 13^*\}, 1)$-PBD is formed ($s \geq 3$) with one 2-block, $8s$ 3-blocks, $(12s^2-s-13)$ 4-blocks and one 13-block. The PBD-construction gives $[48s^2+32s-37, 96s^2+28s+2] \subseteq C(v)$, and note $96s^2+28s+2 = M_v$.

$\mathbf{v = 24s+11}$ ($\mathbf{v \geq 59}$)

A $(12s+5, \{2^*, 3, 4, 7^*\}, 1)$-PBD is formed ($s \geq 2$) with one 2-block, $(8s+2)$ 3-blocks, $(12s^2+5s-3)$ 4-blocks and one 7-block. The PBD-construction gives $[48s^2+56s+6, 96s^2+76s+15] \subseteq C(v)$, and note $96s^2+76s+15 = M_v$.

$\mathbf{v = 24s+17}$ ($\mathbf{v \geq 41}$)

A $(12s+8, \{2^*, 3, 4, 7^*\}, 1)$-PBD is formed ($s \geq 1$) with one 2-block, $(8s+4)$ 3-blocks, $(12s^2+11s-1)$ 4-blocks and one 7-block. The PBD-construction gives $[48s^2+80s+23, 96s^2+124s+40] \subseteq C(v)$, and note $96s^2+124s+40 = M_v$.

The twelve cases treated above deal with the upper part of $C(v)$ for $v \geq 41$ apart from v = 47, 49, 51, 53, 55, 57, 71, 73, and 75. We now deal with these special cases.

$\mathbf{v = 75}$

Take a 4-GDD of type 6^6 [cf. [2], §1.27, p.190] and extend every group by a new point ∞. Take the existing blocks and the extended groups to form the blocks of a $(37, \{4, 7\}, 1)$-PBD. This design has 90 4-blocks and six 7-blocks. The PBD-construction gives $[439, 925] \subseteq C(75)$, and note $M_{75} = 925$.

$\mathbf{v = 73}$

Take a 4-GDD of type 6^6 as above and take its blocks and groups to form the blocks of a $(36, \{4, 6\}, 1)$-PBD. This design has 90 4-blocks and six 6-blocks. Noting that $\{12, 23, 24\} \subseteq C^*(13)$, the PBD-construction gives $[432, 864] \subseteq C(73)$, and note $M_{73} = 864$.

$\mathbf{v = 71}$

Take a 4-GDD of type 6^6 as above and delete a single point. Take the resulting blocks and groups to form the blocks of a $(35, \{3, 4, 5^*, 6\}, 1)$-PBD. The design has ten 3-blocks, 80 4-blocks, one 5-block and five 6-blocks. Noting that $\{10, 14, 15\} \subseteq C^*(11)$, the PBD-construction gives $[420, 805] \subseteq C(71)$, and we have $M_{71} = 805$.

v = 57

Take a 4-GDD of type 7^4 (i.e., a transversal design TD(4,7), cf. [2], §1.27, p.190) and take its blocks and groups as blocks of a $(28, \{4, 7\}, 1)$-PBD. This design has 49 4-blocks and four 7-blocks. The PBD-construction gives $[252, 532] \subseteq C(57)$, and we note that $M_{57} = 532$.

v = 55

Take a 4-GDD of type 7^4 as above and delete a single point. Take the resulting blocks and groups to form the blocks of a $(27, \{3, 4, 6^*, 7\}, 1)$-PBD with seven 3-blocks, 42 4-blocks, one 6-block and three 7-blocks. The PBD-construction gives $[243, 486] \subseteq C(55)$, and we note that $M_{55} = 486$.

v = 53

Take a 4-GDD of type 7^4 as above and delete two points from the same group. Take the resulting blocks and groups as the blocks of a $(26, \{3, 4, 5^*, 7\}, 1)$-PBD with 14 3-blocks, 35 4-blocks, one 5-block and three 7-blocks. The PBD-construction gives $[234, 442] \subseteq C(53)$, and we note $M_{53} = 442$.

v = 51

Take a 4-GDD of type $3^4 6^2$ [cf. [2], §1.32, p.191] and extend every group by a new point ∞. Take the existing blocks and the extended groups as blocks of a $(25, \{4, 7\}, 1)$-PBD with 43 4-blocks and two 7-blocks. The PBD-construction gives $[211, 425] \subseteq C(51)$, and we note that $M_{51} = 425$.

v = 49

Take a 4-GDD of type $3^4 6^2$ as above, and take its blocks and groups as the blocks of a $(24, \{3, 4, 6\}, 1)$-PBD. This design has four 3-blocks, 39 4-blocks and two 6-blocks. The PBD-construction gives $[204, 384] \subseteq C(49)$, and note $M_{49} = 384$.

v = 47

Take a 4-GDD of type $3^4 6^2$ as above and delete a point from a group of cardinality three. Take the resulting blocks and groups as blocks of a $(23, \{2^*, 3, 4, 6\}, 1)$-PBD. This design has one 2-block, ten 3-blocks, 32 4-blocks and two 6-blocks. The modified PBD-construction (which permits a single 2-block) gives $[193, 345] \subseteq C(47)$, and note $M_{47} = 345$.

This completes constructions for the upper part of $C(v)$ for $v \geq 41$. The next section develops graph-theoretic methods to deal with the lower part of $C(v)$.

3. Graph theoretic constructions

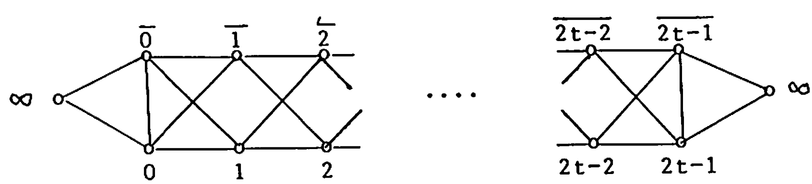
We deal first with the case $v \equiv 1 \pmod 4$; the case $v \equiv 3 \pmod 4$ is similar but the details are more complex. Both cases rely on the following 2-factorization of K_{2u+1}.

Represent K_{2u+1} on the vertex set $Z_{2u} \cup \{\infty\}$ and take F_u to be the Hamiltonian cycle

$(\infty\ 0\ 1\ 2u-1\ 2\ 2u-2\ 3 \ldots u+2\ u-1\ u+1\ u)$.

Let $F_u + i$ be the Hamiltonian cycle obtained from F_u by adding i to each element of F_u with the convention that $\infty + i = \infty$, and the arithmetic otherwise being modulo $2u$. Then the set $\mathcal{F} = \{F_u + i : i = 0, \ldots, u-1\}$ forms a 2-factorization of K_{2u+1}.

Piotrowski [6] observed that when u is even, say, $u = 2t$, then the graph formed by the edges of $F_{2t} + 0$ and $F_{2t} + t$ is isomorphic to the graph S_t shown below (Fig.1).



(The two end-vertices are identified.)

Fig.1

A proof of the isomorphism is given in [3]. An obvious consequence is that for each $i = 0, \ldots, t-1$ the graph formed by the pair of cycles $F_{2t} + i$ and $F_{2t} + t + i$ is also isomorphic to S_t. Thus K_{4t+1} can be partitioned into t copies of S_t. To avoid trivialities we assume subsequently that $t \geq 2$.

Lemma 3.1. *For each k satisfying $1 \leq k \leq t$, the graph S_t can be partitioned into two 2-factors together containing precisely $2k$ cycles.*

Proof. The case $k = 1$ simply reflects the original pair of Hamiltonian cycles. We obtain the result for $k \geq 2$ from the partition into the two 2-factors G and H given below. The first and last cycles of G have length 4 and $4(t-k)+5$, respectively, while those of H have lengths 3 and $4(t-k)+6$. If $k = 2$, only the first and last cycles are present in both G and H. If $k > 2$, the remaining cycles in both G and H are of length 4.

$G : (\infty\ 0\ 1\ \bar{0})(2\ 3\ \bar{2}\ \bar{1}) \ldots (2k-4\ 2k-3\ \overline{2k-4}\ \overline{2k-5})(\overline{2k-3}\ 2k-2\ \overline{2k-1}\ 2k\ \overline{2k+1} \ldots \overline{2t-1}\ 2t-1\ \overline{2t-2}\ 2t-3 \ldots 2k-1\ \overline{2k-2})$;

$H : (0\ \bar{0}\ \bar{1})(\bar{2}\ \bar{3}\ 2\ 1) \ldots (\overline{2k-4}\ \overline{2k-3}\ 2k-4\ 2k-5)(2k-3\ 2k-2\ 2k-1 \ldots 2t-1\ \infty\ \overline{2t-1}\ \overline{2t-2} \ldots \overline{2k-2})$. □

Lemma 3.2. *For each l satisfying $t \leq l \leq t^2$ there is a 2-factorization of K_{4t+1} into 2-factors together containing precisely $2l$ cycles, and hence* $\{2t, 2t+2, \ldots, 2t^2\} \subseteq C(4t+1)$.

Proof. The result follows immediately by decomposing K_{4t+1} into t copies of S_t and using Lemma 3.1 to partition each S_t independently in an appropriate manner to achieve the desired value of $2l$. □

Lemma 3.3. *The graph formed by the edges of $F_{2t}+0$ and $F_{2t}+1$ can be partitioned into two 2-factors together containing precisely three cycles.*

Proof. The result is obtained by the partition into the two 2-factors:
$\bar{G}$: $(0\ 1\ 2)(3\ 4t-2\ 5\ 4t-4\ 7\ 4t-6 \ldots 2t-3\ 2t+4\ 2t-1\ 2t+2\ 2t+1\ \infty\ 2t\ 2t+3\ 2t-2\ 2t+5\ 2t-4 \ldots 4t-3\ 4\ 4t-1)$;
$\bar{H}$: $(0\ \infty\ 1\ 4t-1\ 2\ 4t-2\ 4\ 4t-4\ 6 \ldots 2t-2\ 2t+2\ 2t\ 2t+1\ 2t-1\ 2t+3\ 2t-3\ 2t+5 \ldots 5\ 4t-3\ 3)$. □

Lemma 3.4. *For each l satisfying $t \leq l \leq t^2-2t+2$ there is a 2-factorization of K_{4t+1} into 2-factors together containing precisely $2l+1$ cycles, and hence*
$\{2t+1, 2t+3, \ldots, 2t^2-4t+5\} \subseteq C(4t+1)$.

Proof. We decompose K_{4t+1} into $F_{2t}+0, F_{2t+1}, F_{2t}+t, F_{2t}+(1+t)$ plus $t-2$ copies of S_t. We use Lemma 3.4 to partition the graph formed by $F_{2t}+0$ and $F_{2t}+1$ into three cycles, and Lemma 3.1 to partition the $t-2$ copies of S_t into a total of $2m$ cycles ($m \in [t-2, t(t-2)]$). The factors $F_{2t}+t$ and $F_{2t}+(1+t)$ are left unaltered. The total number of cycles in the resulting 2-factorization is $2m+5$, and taking $m = l-2$ gives the result. □

Theorem 3.5. *If $t \geq 2$ then $[2t, 2t^2-4t+6] \subseteq C(4t+1)$.*

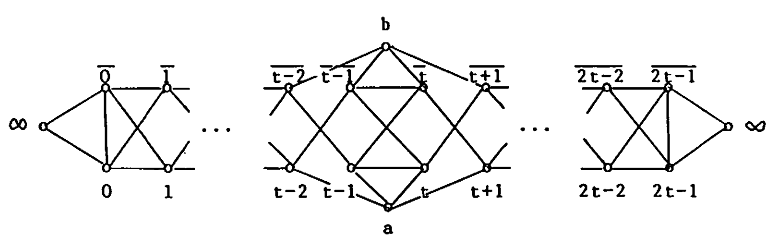
Proof. This follows immediately from Lemmas 3.2 and 3.4. Note also that if $v = 4t+1$ then $(v-1)/2 = 2t$. □

It is now easy to establish that for $v \geq 41$ and $v \equiv 1 \pmod 4$ we have $C(v) = I(v)$. This is obtained from Theorem 3.5 and the results of Section 2. All that is needed is to check that the upper endpoint $(2t^2-4t+6)$ of the interval in Theorem 3.5 exceeds the lower endpoints of the intervals established in Section 2. We leave this to the reader and proceed to the case $v \equiv 3 \pmod 4$.

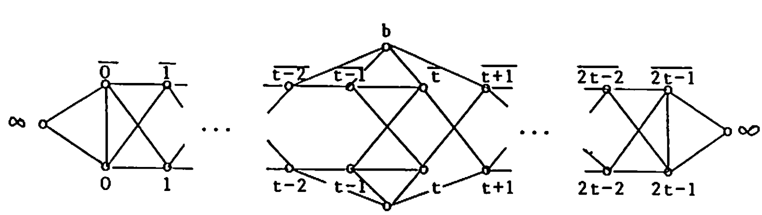
When $v = 4t + 3$ $(t \geq 2)$ we form the cycle F_{2t} as before on $Z_{4t} \cup \{\infty\}$. We then remove the edges $\{i_1, i_2\}$ and $\{j_1, j_2\}$ having $|i_1 - i_2| \equiv |j_1 - j_2| \equiv t + 1 \pmod{4t}$. A Hamiltonian cycle may then be formed on $Z_{4t} \cup \{\infty, a, b\}$, where a, b are two new points, by adding the edges $\{\{a, i_1\}, \{a, i_2\}, \{b, j_1\}, \{b, j_2\}\}$. Denote this cycle by F'_{2t}. A different Hamiltonian cycle, F''_{2t}, is obtained by instead adding the edges $\{\{a, i_1\}, \{a, j_1\}, \{b, i_2\}, \{b, j_2\}\}$. The following three lemmas are established in [3].

Lemma 3.6. *The 2t Hamiltonian cycles* $\{F'_{2t} + i : i = 0, \ldots, t\}$ *and* $\{F''_{2t} + i : i = t + 1, \ldots, 2t - 1\}$ *are pairwise edge-disjoint (arithmetic modulo 4t with* $\alpha + i = \alpha$ *for* $\alpha = \infty, a, b$*).*

Lemma 3.7. *a) The graph formed by the edges of* $F'_{2t} + 0$ *and* $F'_{2t} + t$ *is isomorphic to the graph* A_t *shown below (Fig.2);*
b) The graph formed by the edges of $F'_{2t} + 1$ *and* $F''_{2t} + t + 1$ *is isomorphic to the graph* B_t *shown below (Fig.2). Consequently, the graph formed by the edges of* $F'_{2t} + i$ *and* $F''_{2t} + t + i$ *is also isomorphic to* B_t *for* $i = 1, \ldots, t - 1$.



The graph A_t (the two end-vertices are identified)



The graph B_t (the two end-vertices are identified)

Fig.2

Lemma 3.8. *If $t \geq 2$ then K_{4t+3} can be partitioned into $(t+1)$ regular subgraphs $G_1, G_2, \ldots, G_{t+1}$ where G_1 is isomorphic to A_t, $G_2, G_3, \ldots, G_t$ are all isomorphic to B_t, and G_{t+1} is a 2-regular subgraph containing the triangle (∞ab) together with the cycles induced by the difference $t+1$ in Z_{4t}.*

Lemma 3.9. *For each k satisfying $1 \leq k \leq t+1$, the graph B_t can be partitioned into two 2-factors together containing precisely $2k$ cycles.*

Proof. The case $k = 1$ simply reflects the original pair of Hamiltonian cycles, so we now consider $k \geq 2$. If $t = 2$ the following 2-factorizations of B_2 generate four and six cycles, respectively:

a) (four cycles)

$G : (01\bar{2}\bar{1}b\bar{0})(2\bar{3}\infty 3a)$;

$H : (0\infty\bar{0}\bar{1}21a)(3\bar{3}b\bar{2})$.

b) (six cycles)

$G' : (012a)(\bar{0}\bar{1}\bar{2}b)(\infty 3\bar{3})$;

$H' : (1\bar{2}3a)(\bar{1}2\bar{3}b)(9\infty 0\bar{0})$.

For $t \geq 3$ and t odd the following 2-factorization of B_t achieves four cycles:

$G'' : (\infty\ 0\ 1 \ldots t-2\ a\ t+1\ \bar{t}\ t-1\ t\ \overline{t-1}\ b\ \overline{t-2}\ \overline{t-3} \ldots \bar{0})(\overline{t+1}\ t+2\ \overline{t+3}\ t+4 \ldots \overline{2t-2}\ 2t-1\ \overline{2t-1}\ 2t-2 \ldots \overline{t+4}\ t+3\ \overline{t+2})$;

$H'' : (0\ \bar{0}\ 1\ \bar{2}\ 3 \ldots t-2\ t-1\ a\ t\ \overline{t+1}\ b\ \bar{t}\ \overline{t-1}\ \overline{t-2}\ t-3\ \overline{t-4} \ldots \bar{1})(t+1\ t+2\ t+3 \ldots 2t-1\ \infty\ \overline{2t-1}\ \overline{2t-2} \ldots \overline{t+2})$.

When $t \geq 4$ and t is even, we may achieve four cycles by replacing the second cycle of G'' by

$(\overline{t+1}\ t+2\ \overline{t+3}\ t+4 \ldots \overline{2t-1}\ \ 2t-1\ \overline{2t-2}\ 2t-3 \ldots \overline{t+4}\ t+3\ \overline{t+2})$,

and the first cycle of H'' by

$(0\ \bar{0}\ 1\ \bar{2}\ 3 \ldots t-3\ \overline{t-2}\ \overline{t-1}\ \bar{t}\ b\ \overline{t+1}\ t\ a\ t-1\ t-2\ \overline{t-3}\ t-4 \ldots \bar{1})$.

For $t \geq 3$ and t odd, the following 2-factorization of B_t achieves six cycles:

$G^* : (\infty\ 0\ 1\ 2 \ldots t-3\ \overline{t-2}\ \overline{t-1}\ \overline{t-3} \ldots \bar{0})(t-2\ t-1\ \bar{t}\ \overline{t-1}\ b\ \overline{t+1}\ t\ a)(t+1\ t+2\ \overline{t+3}\ t+4\ \overline{t+5} \ldots 2t-1\ \overline{2t-1}\ 2t-2\ \overline{2t-3} \ldots \overline{t+2})$;

$H^* : (0\ \bar{1}\ 2\ \bar{3}\ 4 \ldots t-3\ t-2\ \overline{t-3}\ t-4\ \overline{t-5} \ldots \bar{0})(t-1\ t\ \overline{t-1}\ \overline{t-2}\ b\ \bar{t}\ t+1\ a)(t+2\ t+3\ t+4 \ldots 2t-1\ \infty\ \overline{2t-1}\ \overline{2t-2} \ldots \overline{t+1})$.

When $t \geq 4$ and t is even, we may achieve six cycles by replacing the last cycle of G^* by

$(t+1\ t+2\ \overline{t+3}\ t+4\ \overline{t+5}\ldots\overline{2t-1}\ 2t-1\ \overline{2t-2}\ 2t-3\ldots\overline{t+2})$ and the first cycle of H^* by
$(0\ \bar{1}\ 2\ \bar{3}\ 4\ldots\overline{t-3}\ t-2\ t-3\ \overline{t-4}\ t-5\ldots\bar{0})$.

For $t \geq 3$ the central section of B_t (spanning ten vertices) may be partitioned into the following edge-disjoint pairs of cycles:
$C_1 : (t-2\ t-1\ a)(t\ \overline{t+1}\ b\ \bar{t}\ \overline{t-1})$;
$C_2 : (\overline{t-2}\ \overline{t-1}\ b)(\bar{t}\ t+1\ a\ t\ t-1)$.

The two "ends" of B_t are each isomorphic to the following graph D_t (Fig.3).

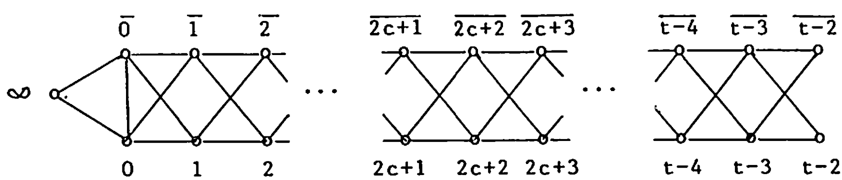


Fig.3. The graph D_t

For $t \geq 3$, D_t may be decomposed into two edge-disjoint cycles:
$E_1 : (0\ 1\ 2\ 3\ldots t-3\ \overline{t-2}\ \overline{t-3}\ldots\bar{0})$;
$E_2 : (\infty\ 0\ \bar{1}\ 2\ \bar{3}\ldots t-3\ t-2\ \overline{t-3}\ t-4\ \overline{t-5}\ldots\bar{0})$ (t odd)
or $(\infty\ 0\ \bar{1}\ 2\ \bar{3}\ldots t-2\ t-3\ \overline{t-4}\ t-5\ \overline{t-6}\ldots\bar{0})$ (t even).

Hence B_t may be resolved into two 2-factors comprising firstly C_1, E_1 and a copy of E_2; and secondly C_2, E_2 and a copy of E_1. These 2-factors together contain precisely eight cycles.

If t is odd and $t \geq 5$, then for any value of $c \in [0, (t-5)/2]$, D_t may be resolved into two sets of edge-disjoint cycles having a combined total of $2c+4$ cycles:
$E_1(c)$: $(0\ \bar{0}\ \bar{1})(1\ 2\ \bar{3}\ \bar{2})(3\ 4\ \bar{5}\ \bar{4})\ldots(2c-1\ 2c\ \overline{2c+1}\ \overline{2c})(2c+1\ 2c+2\ \overline{2c+3}\ 2c+4\ \overline{2c+5}\ldots\overline{t-2}\ \overline{t-3}\ t-4\ \overline{t-5}\ldots\overline{2c+2})$;
$E_2(c)$: $(\infty\ 0\ 1\ \bar{0})(2\ 3\ \bar{2}\ \bar{1})(4\ 5\ \bar{4}\ \bar{3})\ldots(2c\ 2c+1\ \overline{2c}\ \overline{2c-1})(2c+2\ 2c+3\ldots t-2\ \overline{t-3}\ \overline{t-4}\ldots\overline{2c+1})$.
(If $c = 0$ only the first and last cycles are present in each of $E_1(c)$ and $E_2(c)$.)

Hence, for any pair of values $c, c' \in [0, (t-5)/2]$, B_t may be resolved into two 2-factors comprising firstly $C_1, E_1(c)$ and a copy of $E_2(c')$; and secondly

$C_2, E_2(c)$ and a copy of $E_1(c')$. Furthermore, the collections $E_1(c)$ and $E_2(c)$ may be simultaneously replaced by the cycles E_1 and E_2 described immediately above. We thus obtain 2-factorizations of B_t (t odd and $t \geq 5$) having any even number of cycles from 10 to $2t+2$ inclusive.

If t is even and $t \geq 6$, then for any value of $c \in [0, (t-6)/2]$, D_t may be resolved into two sets of edge-disjoint cycles having a combined total of $2c+4$ cycles by taking $E_2(c)$ as before and replacing the last cycle of $E_1(c)$ by
$(2c+1\; 2c+2\; \overline{2c+3}\; 2c+4\; \overline{2c+5} \ldots \overline{t-3}\; \overline{t-2}\; t-3\; \overline{t-4}\; t-5 \ldots \overline{2c+2})$
to form $E_1'(c)$.

Hence, for any pair of values $c, c' \in [0, (t-6)/2]$, B_t may be resolved into two 2-factors comprising firstly $C_1, E_1'(c)$ and a copy of $E_2(c')$; and secondly, $C_2, E_2(c)$ and a copy of $E_1'(c')$. Furthermore, the collections $E_1'(c)$ and $E_2(c)$ may be simultaneously replaced by the cycles E_1 and E_2 described above. We thus obtain 2-factorizations of B_t (t even and $t \geq 6$) having any even number of cycles from 10 to $2t$ inclusive.

To complete the proof we show that if t is even and $t \geq 4$ then B_t has a 2-factorization with precisely $2t+2$ cycles. This is achieved by the following two 2-factors:
$G^{**} : (\infty\; 0\; \bar{0})(1\; 2\; \bar{1}\; \bar{2})(3\; 4\; \bar{3}\; \bar{4}) \ldots (t-3\; t-2\; \overline{t-3}\; \overline{t-2})(t-1\; \bar{t}\; t+1\; a)(\overline{t-1}\; t\; \overline{t+1}\; b)\; (t+2\; t+3\; \overline{t+2}\; \overline{t+3})(t+4\; t+5\; \overline{t+4}\; \overline{t+5}) \ldots (2t-2\; 2t-1\; \overline{2t-2}\; \overline{2t-1})$;
$H^{**} : (0\; 1\; \bar{0}\; \bar{1})(2\; 3\; \bar{2}\; \bar{3}) \ldots (t-4\; t-3\; \overline{t-4}\; \overline{t-3})(t-2\; t-1\; t\; a)(\overline{t-2}\; \overline{t-1}\; \bar{t}\; b)$
$(t+1\; t+2\; \overline{t+1}\; \overline{t+2})(t+3\; t+4\; \overline{t+3}\; \overline{t+4}) \ldots (2t-3\; 2t-2\; \overline{2t-3}\; \overline{2t-2})(2t-1\; \overline{2t-1}\; \infty)$.

Lemma 3.10. *If $t \geq 2$ then the graph A_t can be partitioned into two 2-factors containing a total of three cycles.*

Proof. If t is even, the following 2-factorization may be used.
$\bar{G} : (\infty\; 0\; \bar{1}\; \bar{0}\; 1\; 2\; \bar{3}\; \bar{2}\; 3\; 4\; \bar{5}\; \bar{4}\; 5 \ldots \overline{t-4}\; t-3\; t-2\; a\; t+1\; \bar{t}\; t-1\; \overline{t-2}\; b\; \overline{t-1}\; t\; \overline{t+1}\; \overline{t+2}\; t+3\; t+2\; \overline{t+3}\; \overline{t+4}\; t+5\; t+4\; \overline{t+5} \ldots 2t-2\; \overline{2t-1})$;
$\bar{H} : (t-1\; t\; a)(\infty\; \bar{0}\; 0\; 1\; \bar{2}\; \bar{1}\; 2\; 3\; \bar{4}\; \bar{3}\; 4 \ldots t-3\; \overline{t-2}\; \overline{t-3}\; t-2\; \overline{t-1}\; \bar{t}\; b\; \overline{t+1}\; t+2\; t+1\; \overline{t+2}\; \overline{t+3}\; t+4\; t+3\; \overline{t+4}\; \overline{t+5}\; t+6\; t+5\; \overline{t+6} \ldots \overline{2t-2}\; \overline{2t-1}\; 2t-1)$.
[If $t = 2$, $\bar{G}$ reduces to $(\infty 0 a 3 \bar{2} 1 \bar{0} b \bar{1} 2 \bar{3})$ and $\bar{H}$ to $(12a)(\infty \bar{0} 0 \bar{1} \bar{2} b \bar{3} 3)$.]

If t is odd, we may use
$\hat{G} : (\infty\; 0\; \bar{0}\; \bar{1}\; 2\; 1\; \bar{2}\; \bar{3}\; 4\; 3\; \bar{4} \ldots t-4\; \overline{t-3}\; t-2\; a\; t+1\; \bar{t}\; t-1\; \overline{t-2}\; b\; \overline{t-1}\; t\; \overline{t+1}\; t+2\; \overline{t+3}\; \overline{t+2}\; t+3\; t+4\; \overline{t+5}\; \overline{t+4}\; t+5 \ldots 2t-1\; \overline{2t-1})$;
$\hat{H} : (t-1\; t\; a)(\infty\; \bar{0}\; 1\; 0\; \bar{1}\; \bar{2}\; 3\; 2\; \bar{3}\; \bar{4}\; 5\; 4\; \bar{5} \ldots \overline{t-4}\; \overline{t-3}\; \overline{t-2}\; t-3\; t-2\; \overline{t-1}\; \bar{t}\; b\; \overline{t+1}\; \overline{t+2}\; t+1\; t+2\; t+3\; \overline{t+4}\; \overline{t+3}\; t+4\; t+5\; \overline{t+6}\; \overline{t+5}\; t+6 \ldots \overline{2t-2}\; 2t-1)$.

Theorem 3.11. *For* $t \geq 2$, $[2t+1, 2t^2+3] \subseteq C(4t+3)$.

Proof. From Lemmas 3.8, 3.9 and 3.10 we can find 2-factorizations of K_{4t+3} $(t \geq 2)$ containing precisely $\alpha + 2\sum_{i=1}^{t-1}\beta_i + \gamma_t + 1$ cycles, where $\alpha = 2$ or 3, each β_i may be chosen independently in the interval $[1, t+1]$ and γ_t is the number of cycles induced by the difference $t+1$ in Z_{4t}. Hence $[2t+1+\gamma_t, 2t^2+2+\gamma_t] \subseteq C(4t+3)$.

It is easy to see that $\gamma_t = 1,2,1,$or 4 depending as $t \equiv 0$, 1, 2, or 3(mod 4). Thus we certainly have $[2t+5, 2t^2+3] \subseteq C(4t+3)$. To deal with the remaining cases we recall that K_{4t+3} may be decomposed into the Hamiltonian cycles $\{F_{2t+1}+i : i = 0, \ldots, 2t\}$. This decomposition itself achieves $2t+1$ cycles.

If $t \geq 3$ we see that there are at least three disjoint consecutive pairs $\{F_{2t+1}+i, F_{2t+1}+(i+1)\}$. The graph formed by the edges of the consecutive pair $F_{2t+1}+0$ and $F_{2t+1}+1$ has a 2-factorization into three cycles as follows:
$F^0 : (0\ 1\ 2)(3\ 4t+1\ 4\ 4t-1\ 6 \ldots 2t+5\ 2t\ 2t+3\ 2t+2\ \infty\ 2t+1\ 2t+4\ 2t-1\ 2t+6 \ldots 5\ 4t)$;
$F^1 : (0\ \infty\ 1\ 4t+1\ 2\ 4t\ 4\ 4t-2\ 6 \ldots 2t+4\ 2t\ 2t+2\ 2t+1\ 2t+3\ 2t-1\ 2t+5 \ldots 5\ 4t-1\ 3)$.
Replacing $F_{2t+1}+0$ and $F_{2t+1}+1$ by F^0 and F^1 gives a 2-factorization with $2t+2$ cycles. Repeating the process for $F_{2t+1}+2$ and $F_{2t+1}+3$ gives $2t+3$ cycles, and a final repetition for $F_{2t+1}+4$ and $F_{2t+1}+5$ gives $2t+4$ cycles.

For the remaining case of $t = 2$, note γ_t=1. This establishes $[2t+2, 2t^2+3] \subseteq C(4t+3)$. The Hamiltonian decomposition completes the proof. □

Noting that if $v = 4t+3$ then $(v-1)/2 = 2t+1$, it is now easy to establish that, for $v \geq 43$ and $v \equiv 3 \pmod 4$, we have $C(v) = I(v)$. This is obtained from Theorem 3.11 and the results of Section 2. We again leave the reader to verify that the upper endpoint $(2t^2+3)$ of the interval in Theorem 3.11 exceeds the lower endpoints of the intervals established in Section 2. Combining this with the earlier result for $v \equiv 1 \pmod 4$ we can now state the following theorem.

Theorem 3.12. *If* $v \geq 41$ *is odd then* $C(v) = I(v)$.

4. Small values of v

In this section we determine $C(v)$ for all odd $v \leq 39$ with the exception of a small number of values which remain open (at most one undetermined case for each v). We also establish those values in $C^*(v)$ which were needed in Section 2. Trivially, we have $C(3) = \{1\}$ and $C(5) = \{2\}$.

v=7 The decomposition of K_7 into Hamiltonian cycles establishes $3 \in C(7)$. For a proof that $4 \in C(7)$ and $5 \notin C(7)$, see Lemma 2.1 of [4]. The 2-factorization generated by $F + i \pmod 6$ for $i = 0, 1, 2$ where $F : (\infty 14)(0235)$ establishes that $6 \in C^*(7)$. Thus $C(7) = \{3, 4, 6\}$ and $C^*(7) = \{6\}$.

v=9 Theorem 3.5 gives $\{4, 5, 6\} \subseteq C(9)$. The 2-factorization on Z_9: (123)(058476); (014657382); (078)(152436); (0354)(17268) proves $7 \in C(9)$. The 2-factorization on $Z_8 \cup \{\infty\}$ generated by $F+i \pmod 8$ for $i = 0, 1, 2, 3$ where $F : (\infty 15)(063427)$ establishes that $8 \in C^*(9)$. The 2-factorization on Z_9:
(018)(235)(467); (027)(143865); (036)(124857); (045)(137826) gives $9 \in C^*(9)$. The 2-factorization on Z_9: (078)(123)(456); (036)(147)(258); (015)(248376); (168)(027534) gives $10 \in C^*(9)$. Lemma 2.2 of [4] establishes $11 \notin C(9)$. Finally, the existence of a Kirkman triple system of order 9 yields $12 \in C^*(9)$.

Thus $C(9) = \{4, 5, 6, 7, 8, 9, 10, 12\}$ and $C^*(9) = \{8, 9, 10, 12\}$.

v=11 Theorem 3.11 establishes $[5, 11] \subseteq C(11)$. The 2-factorization on $Z_5 \times \{0, 1\} \cup \{\infty\}$ generated by $F + i \pmod 5$ for $i = 0, \ldots, 4$ where F: $(\infty 00')(1432'1'3'24')$ establishes $10 \in C^*(11)$. [Note: Here and elsewhere we write x and x' for $(x, 0)$ and $(x, 1)$, respectively, with modular arithmetic being performed on the first component and ∞ fixed.] If we replace the 2-factors $F + 0$ and $F + 2$ in this 2-factorization by
$F_0 : (\infty 00')(132'1'4)(23'4')$; and
$F_2 : (\infty 22')(014')(340'3'1')$
then we obtain $12 \in C(11)$. Repeating this process for the factors $F + 1$ and $F + 3$ (whose union is isomorphic to that of $F + 0$ and $F + 2$) gives $14 \in C^*(11)$.

Lemma 2.3 of [4] establishes $13 \in C(11)$ (using $G_1, G_2, G_3, G_4'', G_5''$). Finally, the 2-factorization on $Z_5 \times \{0, 1\} \cup \{\infty\}$ generated by $G + i$ (mod

5) for $i = 0, \ldots, 4$ where $G : (\infty 00')(141'4')(232'3')$ proves $15 \in C^*(11)$. Hence $C(11) = [5, 15] = I(11)$ and $\{10, 14, 15\} \subseteq C^*(11)$.

v=13 Following Theorem 2.5 of [4], consider the 2-factorization $\mathcal{Q} = \{Q_1, \ldots, Q_6\}$ of K_{13} on Z_{13} given by $Q_i = \{xy : d(xy) = i\}$ where $d(xy) = min\{|x - y|, 13 - |x - y|\}$. Let $G_{a,b}$ be the 4-regular subgraph of K_{13} with edge-set $\{xy : d(xy) = a \text{ or } b\}$ (so that $G_{a,b}$ is formed by the edges of Q_a and Q_b). The graphs $G_{1,2}$, $G_{3,6}$, and $G_{4,5}$ are isomorphic, and, as shown in [4], each can be independently decomposed into two 2-factors containing a total of j cycles for each $j \in \{2, 3, 4, 5\}$. Hence $[6, 15] \subseteq C(13)$. Our earlier Lemma 3.2 also gives $16, 18 \in C(13)$. The 2-factorization on Z_{13} below gives $17 \in C(13)$:

$F_1 : (1\ 7\ 10)(2\ 6\ 11)(4\ 5\ 9)(0\ 3\ 8\ 12)$;
$F_2 : (1\ 4\ 11)(2\ 3\ 9)(5\ 8\ 10)(0\ 6\ 12\ 7)$;
$F_3 : (1\ 3\ 5)(4\ 6\ 8)(0\ 2\ 10\ 12\ 11\ 7\ 9)$;
$F_4 : (0\ 4\ 10)(1\ 6\ 9)(2\ 5\ 12)(3\ 7\ 8\ 11)$;
$F_5 : (0\ 1\ 12\ 9\ 11\ 5\ 7\ 6\ 10\ 3\ 4\ 2\ 8)$;
$F_6 : (0\ 5\ 6\ 3\ 12\ 4\ 7\ 2\ 1\ 8\ 9\ 10\ 11)$.

Replacing F_5 and F_6 by
$F_5' : (0\ 1\ 2\ 8)(3\ 4\ 12\ 9\ 11\ 5\ 7\ 6\ 10)$ and
$F_6' : (2\ 4\ 7)(0\ 5\ 6\ 3\ 12\ 1\ 8\ 9\ 10\ 11)$
gives $19 \in C(13)$. Replacing F_5 and F_6 by
$F_5'' : (0\ 1\ 2\ 8)(5\ 6\ 7)(3\ 4\ 12\ 9\ 11\ 10)$ and
$F_6'' : (0\ 5\ 11)(2\ 4\ 7)(1\ 8\ 9\ 10\ 6\ 3\ 12)$
gives $21 \in C(13)$.

Lemma 2.4 of [4] gives a 2-factorization $H_1', H_2', H_3, H_4, H_5, H_6$ which proves $20 \in C(13)$. Similarly, $H_1'', H_2'', H_3'', H_4'', H_5', H_6'$ establishes $22 \in C(13)$ while $H_1, H_1', H_3'', H_4'', H_5', H_6'$ establishes $23 \in C^*(13)$. To complete the case of $v = 13$ we must prove that $12, 24 \in C^*(13)$. However, both of these cases are dealt with by the proof of Lemma 5.1 of [4].

Hence, finally, $C(13) = [6, 24] = I(13)$ and $\{12, 23, 24\} \subseteq C^*(13)$.

v=15 From the proof of Theorem 3.11, noting $\gamma_3 = 4$, we have $[7, 24] \subseteq C(15)$. From the proof of Theorem 4.1 of [4] we obtain $\{25, 26, 27, 28, 29, 31, 32, 33\} \subseteq C(15)$. From the factorization No.28 of [5] we obtain $34 \in C^*(15)$ and the existence of a Kirkman triple system of order 15 establishes $35 \in C^*(15)$. The 2-factorization on $Z_7 \times \{0, 1\} \cup \{\infty\}$

generated by $F+i \pmod 7$ for $i=0,\ldots,6$ where
$F:(\infty 00')(14'23'6'1'2'45365')$
gives $14 \in C^*(15)$. All that remains is to prove $30 \in C(15)$. To do this, take factorization No.11 of [5] which has 32 cycles and replace the first two lines of its tabulation (which comprise its second and third 2-factors) by
(0 3 6)(2 9 12)(8 11 14)(1 4 7 5 10 13); and
(0 4 9)(2 3 11)(6 10 12)(1 7 14 5 13 8).
Hence, finally, $C(15)=[7,35]=I(15)$ and $\{14,34,35\} \subseteq C^*(15)$.

v=17 Lemmas 3.2 and 3.4 above give $[8,22] \cup \{24,26,28,30,32\} \subseteq C(17)$. Next consider the 2-factorization of K_{17} on $Z_7 \times \{0,1\} \cup \{a,b,c\}$ given by $F+i \pmod 7$ for $i=0,\ldots,6$ where
$F:(a00')(b16')(c45')(1'3'4')(2352'6)$
(with the convention that $\alpha+i=\alpha$ for $\alpha=a,b$, or c) and by the additional 2-factor
$\bar{F}:(abc)(02'35'61'24'50'13'46')$.
This 2-factorization gives $37 \in C(17)$. Replacing $F+0$ and $\bar{F}$ by
$(a00')(b16')(2352'65'c43'4'1')$; and
$(abc)(02'35'46')(10'54'261'3')$
gives $36 \in C(17)$. Similarly replacing $F+0$ and $\bar{F}$ by
$(b16')(1'3'4')(0ac45'3262'50')$, and
$(02'354'21'65'cba0'13'46')$
gives $34 \in C(17)$. We may also replace $F+1$ and $F+2$ by
$(b20')(c56')(a11'3463'04'5'2')$, and
$(c60')(3'5'6')(a22'4'14503b1')$.
Isomorphic and independent replacements may also be applied to the pair $F+3$ and $F+4$, and to the pair $F+5$ and $F+6$. Thus we may also obtain $33,29,25 \in C(17)$.

Now consider the 2-factorization of K_{17} on $Z_7 \times \{0,1\} \cup \{a,b,c\}$ given by $G+i \pmod 7$ for $i=0,\ldots,6$ where
$G:(a35')(b53')(c66')(0421'0'2'14')$,
and by the additional 2-factor
$\bar{G}:(abc)(0123456)(0'3'6'2'5'1'4')$.
This 2-factorization gives $31 \in C(17)$. Replacing $G+0$ and $\bar{G}$ by
$(abc6'3'0'4'1'5'2'1065423)$, and
$(b53')(ac66'2'0'1'214'0435')$

gives $27 \in C(17)$. We may also replace the pair $G+1, G+3$ by
$(a46'b4'6)(c00')(1532'1'3'25')$,
$(a1'6b16')(c22')(3054'3'5'40')$
and then independently replace the pair $G+2, G+4$ in an isomorphic manner. This gives $23 \in C(17)$.

Next take the 2-factorization of K_{17} on $Z_7 \times \{0,1\} \cup \{a,b,c\}$ given by $H+i \pmod 7$ for $i = 0, \ldots, 6$ where
$H : (a36')(b63')(c55')(010'1')(242'4')$,
and by the additional 2-factor
$\bar{H} : (abc)(0362514)(0'3'6'2'5'1'4')$.
This 2-factorization gives $38 \in C(17)$. We may replace $H+0$ and $\bar{H}$ by
$(b63')(c55')(a3010'1'4'242'6')$, and
$(abc)(04152636'3'0'4'2'5'1')$.
This gives $35 \in C(17)$.

The following 2-factorization of K_{17} shows $39 \in C(17)$:
$(X\ 1\ 2\ 3\ Y)(4\ 10\ 14)(5\ 8\ 13)(6\ 9\ 15)(7\ 11\ 12)$,
$(1\ 5\ 4\ Y)(11\ 6\ 13\ X)(2\ 12\ 14)(3\ 9\ 10)(7\ 8\ 15)$,
$(6\ 1\ 7\ Y)(12\ 3\ 15\ X)(2\ 8\ 10)(4\ 9\ 13)(5\ 11\ 14)$,
$(2\ 7\ 5\ X)(13\ 3\ 14\ Y)(1\ 8\ 9)(4\ 11\ 15)(6\ 10\ 12)$,
$(2\ 13\ 15\ Y)(9\ 7\ 14\ X)(1\ 10\ 11)(3\ 5\ 6)(4\ 8\ 12)$,
$(3\ 4\ 7\ X)(9\ 2\ 11\ Y)(1\ 12\ 13)(5\ 10\ 15)(6\ 8\ 14)$,
$(4\ 2\ 6\ X)(5\ 9\ 12\ Y)(1\ 14\ 15)(3\ 8\ 11)(7\ 10\ 13)$,
$(1\ 3\ 7\ 6\ 4)(8\ X\ 10\ Y)(2\ 5\ 12\ 15)(9\ 11\ 13\ 14)$.

To prove $40 \in C(17)$ take the 2-factorization of K_{17} on $Z_{16} \cup \{\infty\}$ given by $I+i \pmod{16}$ for $i = 0, \ldots, 7$ where
$I : (\infty\ 0\ 8)(1\ 3\ 10\ 13)(2\ 5\ 9\ 11)(6\ 7\ 12)(4\ 14\ 15)$.
We have now established $C(17) = [8, 40] = I(17)$.

v=19 From Theorem 3.11 we have $[9, 35] \subseteq C(19)$. Applying the PBD-construction to the Steiner triple system of order 9 gives $45 \in C(19)$; applying the modified version in which three disjoint blocks of the system are used to generate 2-factorizations of K_7 which are not necessarily 2^*-factorizations, we also obtain $[36, 43] \subseteq C(19)$.

Next consider the 2-factorization of K_{19} on $Z_9 \times \{0,1\} \cup \{\infty\}$ given by $F+i \pmod 9$ for $i = 0, \ldots, 8$ where
$F : (\infty 506')(15'2')(20'4')(346)(71'3')(87'8')$.

This 2-factorization gives $54 \in C(19)$. We may replace the pair $F+0, F+1$ by the pair
$(\infty 506')(17'8'0'24'82'5')(36473'1')$, and
$(\infty 612'4'0'08'87')(26'3')(34571'5')$
which gives $48 \in C(19)$. We can also replace the same pair by
$(\infty 506')(17'8'82'5')(20'4')(36473'1')$, and
$(\infty 612'4'87')(00'8')(26'3')(34571'5')$
which gives $50 \in C(19)$. If we apply the penultimate replacement to $F+0, F+1$ and an isomorphic copy of the last replacement to $F+2, F+3$, we obtain $44 \in C(19)$; if we apply the last replacement to $F+0, F+1$ and an isomorphic copy of it to $F+2, F+3$, we obtain $46 \in C(19)$. Still using the same basic 2-factorization, we may replace the pair $F+0, F+3$ by
$(\infty 506')(15'2'20'4')(346)(71'3')(87'8')$,
$(\infty 830')(48'5')(53'7')(670)(12'1'24'6')$
which gives $52 \in C(19)$.

Next consider the 2-factorization of K_{19} on $Z_8 \times \{0,1\} \cup \{a,b,c\}$ given by $G+i \pmod 8$ for $i = 0, \ldots, 7$ where
$G : (a2'3)(b3'6)(c06')(124)(4'5'7')(50'71')$
and by the additional 2-factor
$\bar{G} : (abc)(044'0')(155'1')(266'2')(377'3')$.
This 2-factorization gives $53 \in C(19)$. The pair $G+0, G+1$ may be replaced by the pair
$(a32'6b3')(c06')(124)(4'5'7')(50'71')$,
$(a43'61'02')(c17')(235)(5'6'0')(b4'7)$
which gives $51 \in C(19)$. We may additionally replace $G+2, G+3$ isomorphically to $G+0, G+1$ to obtain $49 \in C(19)$. A further replacement of $G+4, G+5$ gives $47 \in C(19)$. Thus $C(19) = [9, 54] = I(19)$.

v = 21 Lemmas 3.2 and 3.4 give $[10,36] \cup \{38,40,42,44,46,48,50\} \subseteq C(21)$. We may apply the PBD-construction to the $(10,\{3,4\},1)$-PBD given in [2] (§4.20, p.216). This has nine 3-blocks and three 4-blocks. Note also that we may select three mutually disjoint 3-blocks. Applying the extended PBD-construction gives $[40,61] \setminus \{60\} \subseteq C(21)$.

Following Theorem 2.5 of [4], consider the 2-factorization $Q = \{Q_1, \ldots, Q_{10}\}$ of K_{21} on Z_{21} given by $Q_i = \{xy : d(xy) = i\}$ where $d(xy) = min(|x-y|, 21-|x-y|\}$. Let $G_{a,b}$ be the 4-regular subgraph of K_{21} with edge-set $\{xy : d(xy) = a \text{ or } b\}$ (so that $G_{a,b}$ is formed by the edges of Q_a and Q_b). The graph $G_{1,2}$ may be resolved into 2-factors either as

(0 1 2)(3 4 5)(6 7 8)(9 10 11)(12 13 14)(15 16 17)(18 19 20), and
(0 20 1 3 2 4 6 5 7 9 8 10 12 11 13 15 14 16 18 17 19)
or as
(0 1 2)(3 4 5)(6 7 8)(9 10 11)(12 13 14)(15 16 18 20 19 17), and
(0 20 1 3 2 4 6 5 7 9 8 10 12 11 13 15 14 16 17 18 19).

The graphs $G_{4,8}$ and $G_{5,10}$ are isomorphic to $G_{1,2}$ and may be resolved similarly and independently. The 2-factors Q_3, Q_6 and Q_9 each contain three cycles while Q_7 contains seven cycles. Thus we can obtain $7+7+7+3+3+3+7 = 37 \in C(21)$ as well as $8+8+7+3+3+3+7 = 39 \in C(21)$.

Lemma 5.2 of [4] establishes $\{60, 62, 63, 64, 66, 68, 69\} \subseteq C(21)$ and the existence of a Kirkman triple system of order 21 establishes $70 \in C(21)$. It only remains to prove $65, 67 \in C(21)$. To deal with these values, take the last 2-factorization in Lemma 5.2 of [4] (which establishes $69 \in C(21)$) and replace the first two 2-factors by

(1 2 4 11 16 5)(3 10 17)(6 9 18)(7 14 21)(8 12 13)(15 19 20),
(1 4 18 11 5 6)(2 8 20)(3 12 16)(7 10 19)(9 13 14)(15 17 21);

we obtain $67 \in C(21)$. If we replace the same pair of 2-factors by

(1 2 4 11 16 5)(3 10 19 20 15 17)(6 9 18)(7 14 21)(8 12 13),
(1 4 18 11 5 6)(7 10 17 21 15 19)(2 8 20)(3 12 16)(9 13 14),

we obtain $65 \in C(21)$. It now follows that $C(21) = [10, 70] = I(21)$.

v=23 From the proof of Theorem 3.11, noting $\gamma_5 = 2$, we have $[11, 54] \subseteq C(23)$. Starting with the (13,4,1)-BIBD we may delete two points to obtain an $(11, \{2^*, 3, 4\}, 1)$-PBD having one 2-block, six 3-blocks, and six 4-blocks. Applying the PBD-construction (allowing a single 2-block) gives $[53, 77] \setminus \{76\} \subseteq C(23)$, and note $M_{23} = 77$. Whether $76 \in C(23)$ remains open.

v=25 From Lemmas 3.2 and 3.4 we obtain $[12, 54] \cup \{56, 58, 60, \ldots, 72\} \subseteq C(25)$. Starting with the (13,4,1)-BIBD we may delete one point to obtain a $(12, \{3, 4\}, 1)$-PBD having four 3-blocks and nine 4-blocks. Applying the PBD-construction gives $[60, 96] \setminus \{95\} \subseteq C(25)$, and note $M_{25} = 96$. Noting that the four 3-blocks are pairwise disjoint we may obtain, from the extended PBD-construction, that $\{55, 57, 59\} \subseteq C(25)$. Whether $95 \in C(25)$ remains open.

v=27 From Theorem 3.11 we have $[13, 75] \subseteq C(27)$. Applying the PBD-construction to the (13,4,1)-BIBD which has 13 4-blocks we obtain $[65, 117] \setminus \{116\} \subseteq C(27)$ (in fact, $117 \in C^*(27)$), and note $M_{27} = 117$. The last 2-factorization of K_{27} given in the proof of Theorem 4.2 of [4] establishes $116 \in C^*(27)$. The 2-factorization of K_{27} on $Z_{13} \times \{0,1\} \cup \{\infty\}$ given by $F + i \pmod{13}$ for $i = 0, \ldots, 12$ where

$F : (\infty\ 0\ 0')(1\ 8\ 5'\ 7\ 11\ 6\ 9\ 10\ 12\ 11'\ 2'\ 3'\ 1'\ 4'\ 12'\ 6'\ 5\ 7'\ 4\ 8'\ 3\ 9'\ 2\ 10')$

gives $26 \in C^*(27)$. Thus $C(27) = [13, 117] = I(27)$ and $\{26, 116, 117\} \subseteq C^*(27)$.

v=29 From Lemmas 3.2 and 3.4 we obtain $[14, 76] \cup \{78, 80, 82, \ldots, 98\} \subseteq C(29)$. Starting with the (16,4,1)-BIBD, we may delete two points to obtain a $(14, \{2^*, 3, 4\}, 1)$-PBD with one 2-block, eight 3-blocks and eleven 4-blocks. Applying the PBD-construction (that allows for a single 2-block) gives $[82, 126] \setminus \{125\} \subseteq C(29)$, and note $M_{29} = 126$. Noting that there are at least four 3-blocks that are mutually disjoint and disjoint from the 2-block we may use the extended construction to obtain $\{77, 79, 81\} \subseteq C(29)$. Whether $125 \in C(29)$ remains open.

v=31 From Theorem 3.11 we have $[15, 101] \subseteq C(31)$. Starting with the (16,4,1)-BIBD we may delete one point to obtain a $(15, \{3, 4\}, 1)$-PBD with five 3-blocks and 15 4-blocks. Applying the PBD-construction gives $[90, 150] \setminus \{149\} \subseteq C(31)$, and note $M_{31} = 150$. Whether $149 \in C(31)$ remains open.

v=33 From Lemma 3.2 we have $[16, 102] \subseteq C(33)$. Applying the PBD-con- struction to the (16,4,1)-BIBD which has 20 4-blocks, we obtain $[96, 176] \setminus \{175\} \subseteq C(33)$, and note $M_{33} = 176$. Whether $175 \in C(33)$ remains open.

v=35 From Theorem 3.11 we have $[17, 131] \subseteq C(35)$. Take the (16,4,1)-BIBD and add a new point to each of the blocks in a parallel class to form a $(17, \{4, 5\}, 1)$-PBD which has 16 4-blocks and four 5-blocks. Applying the PBD-construction gives $[101, 185] \subseteq C(35)$. However, $M_{35} = 187$. To prove $187 \in C(35)$, consider the 2-factorization of K_{35} on $Z_{16} \times \{0, 1\} \cup \{\infty\}$ given by $F + i \pmod{17}$ for $i = 0, \ldots, 16$ where

$F : (\infty\ 0\ 0')(1\ 7\ 15\ 14)(3\ 5\ 10\ 13)(1'\ 6'\ 12)(2'\ 8'\ 9)(3'\ 10'\ 6)(4'\ 12'\ 16)(5'\ 7'\ 4)$

(9′ 13′ 11)(11′ 14′ 2)(15′ 16′ 8).
Whether $186 \in C(35)$ remains open.

v=37 From Lemma 3.2 we have $[18, 132] \subseteq C(37)$. Take a 4-GDD of type 5^4 [cf. [2], §1.32, p.191] and delete two points from a single group. Take the resulting blocks and groups to form the blocks of an $(18, \{3,4,5\}, 1)$-PBD having eleven 3-blocks, 15 4-blocks and three 5-blocks, and apply the PBD-construction. This gives $[126, 201] \subseteq C(37)$. However, $M_{37} = 216$. To deal with the outstanding values (except 215) we apply a frame construction.

Take a 3-frame of type 6^6, i.e. a 3-GDD of type 6^6 with the following properties:
a) the blocks can be partitioned into partial parallel classes each containing ten triples (and hence 30 points),
b) the 18 partial parallel classes can be partitioned into six sets each containing three partial parallel classes so that, within each set, the blocks are disjoint from precisely one of the six groups of the GDD.
Such an object exists [cf. [2], §6.13, p.225 and [7]]. Now take a new point, say ∞, and for each of the groups G_i $(i = 1, \ldots, 6)$ form a 2-factorizxation of K_7 on $G_i \cup \{\infty\}$. Put the three 2-factors of K_7 corresponding to G_i with the three partial parallel classes which are disjoint from G_i to form three 2-factors of K_{37}. The resulting 18 2-factors of K_{37} form a 2-factorization of K_{37}. Moreover, the six 2-factorizations of K_7 may be chosen (independently) to have three, four or six cycles, while each partial parallel class, having ten triples, generates ten cycles. Thus the 2-factorization of K_{37} may be selected to have $180 + 3a + 4b + 6c$ cycles for any nonnegative integers a, b, c satisfying $a + b + c = 6$. This gives $[198, 216] \setminus \{215\} \subseteq C(37)$. Whether $215 \in C(37)$ remains open.

v=39 From Theorem 3.11 we have $[19, 165] \subseteq C(39)$. Take a 4-GDD of type 5^4 and delete a point. Take the resulting blocks and groups to form the blocks of a $(19, \{3,4,5\}, 1)$-PBD having five 3-blocks, 21 4-blocks and three 5-blocks. Applying the PBD-construction gives $[133, 232] \subseteq C(39)$. However, $M_{39} = 247$.

To deal with the outstanding values (except 246), take the 3-frame of type 6^6 previously described. Take three new points, say, a, b, c, and for each group G_i form a 2-factorization of K_9 on $G_i \cup \{a, b, c\}$ which contains the triangle (abc). Leaving aside for a moment the 2-factor which contains (abc), put the remaining three 2-factors of K_9 corresponding to G_i with the three partial parallel classes which are disjoint from G_i, thereby forming three 2-factors of K_{39}. Repeating this for $i = 1, \ldots, 6$ gives 18 2-factors of K_{39}. These may be completed to a 2-factorization of K_{39} by taking an additional 2-factor formed by those 2-factors of $G_i \cup \{a, b, c\}$ which contained the triangle (abc).

The six 2-factorizations of K_9 may be chosen independently. A 2-factorization of K_9 whose 2-factors contain respectively n_1, n_2, n_3, n_4 cycles will be said to be of type $\{n_1, n_2, n_3, n_4\}$. For a 2-factorization containing a distinguished triangle (abc) we will identify the 2-factor containing (abc) by partitioning the appropriate n_i thus: $\{\alpha, \beta, \gamma, \delta + 1\}$. For our purposes it will suffice to observe that there are 2-factorizations of K_9 (containing a distinguished triangle) of types

a) $\{3, 3, 3, 2 + 1\}$

b) $\{3, 3, 2, 1 + 1\}$, and

c) $\{3, 2, 2, 1 + 1\}$.

(For verification, see Lemma 2.2 of [4] and our investigation of $C(9)$ above.)

The partial parallel classes of the frame generate a total of 180 cycles. The total number of cycles in the 2-factorization of K_{39} may be selected to be any number of the form $180 + 8d + 9e + 11f + 1$, where d, e, f are nonnegative integers satisfying $d + e + f = 6$. It follows that $[229, 247] \setminus \{246\} \subseteq C(39)$. Whether $246 \in C(39)$ remains open.

5. Conclusion

The results of the previous sections may be summarized in the following theorem.

Theorem 5.1. *For $v \geq 11$, $C(v) = I(v)$, apart from eight possible exceptions listed below.*

The eight values which remain in doubt are all of the form $M_v - 1$ and are given in the table below.

v	23	25	29	31	33	35	37	39
$M_v - 1$	76	95	125	149	175	186	215	246

Values outstanding

It is our belief that, in fact, $C(v) = I(v)$ for all $v \geq 11$.

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