Symmetric subdesigns of symmetric designs

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ABSTRACT. A symmetric design $(U, \mathcal{A})$ is a strong subdesign of a symmetric design $(V, \mathcal{B})$ if $U \subseteq V$ and $\mathcal{A}$ is the set of nonempty intersections $B \cap U$, where $B \in \mathcal{B}$. We demonstrate three constructions of symmetric designs, where this notion is useful, and produce two new infinite families of symmetric designs with parameters $v = (73^{m+1} - 64)/9$, $k = 73^m$, $\lambda = 9 \cdot 73^{m-1}$ and $v = 1 + 2(q+1)((q+1)^{2m} - 1)/(q+2)$, $k = (q+1)^{2m}$, $\lambda = (q+1)^{2m-1}(q+2)/2$, where m is a positive integer and $q = 2^p - 1$ is a Mersenne prime. The main tools in these constructions are generalized Hadamard matrices and balanced generalized weighing matrices.

1 Introduction

The *CRC Handbook of Combinatorial Designs*, the most comprehensive source of information on combinatorial designs, combines all known symmetric designs in twelve infinite families and several sporadic designs [5, I.5.6]. One more infinite family has been discovered by Fanning [9]. Parameters k and λ in Families 10 and 11 from [5] and the family of designs complementary to Fanning's are, respectively, q^m and q^{m-1}, q^m and $q^{m-1}(q-1)/2$, 16^m and $10 \cdot 16^{m-1}$, where m is a positive integer and q is a prime power. One of the goals of this paper is to show that these families share the following property: every but the first design in each family contains a multiple of a smaller symmetric design from the same family.

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Symmetric designs containing smaller symmetric designs were considered by Haemers and M. Shrikhande [10], Baartmans and M. Shrikhande [1], and Jungnickel [13]. They defined a symmetric design $(U, \mathcal{A})$ to be a subdesign of a symmetric design $(V, \mathcal{B})$ if $U \subset V$ and there is $\mathcal{B}_0 \subset \mathcal{B}$ such that $\mathcal{A} = \{B \cap U : B \in \mathcal{B}_0\}$. In this paper we will need a more restrictive notion. We will call a symmetric design $(U, \mathcal{A})$ a strong subdesign of a symmetric design $(V, \mathcal{B})$ if $U \subseteq V$ and $\mathcal{A}$ is the set of non-empty intersections $B \cap U$, where $B \in \mathcal{B}$. It is easy to show (Proposition 2.5) that if a symmetric 2-(v', k', λ') design has a strong symmetric 2-(v, k, λ) subdesign with $v > 1$, then there is a positive integer q such that $k' = qk$ and $\lambda' = q\lambda$. Another necessary condition is that k must divide $q\lambda$ (Corollary 2.8). If $(U, \mathcal{A})$ is a strong symmetric subdesign of a symmetric design $(V, \mathcal{B})$, then the structure induced by the larger design on the point-set $V \setminus U$ represents an affine resolvable pairwise balanced design (ARPBD). These designs were analyzed by Ionin and M. Shrikhande in [11] and by Bekker, Ionin and M. Shrikhande in [2].

The notions of strong symmetric subdesigns and ARPBDs lead to the following strategy for constructing symmetric designs. If we suspect that a given symmetric design is a strong subdesign of a larger symmetric design, we try to construct a respective ARPBD that when combined with the given design produces the larger symmetric design. The main tools in constructing these ARPBDs are generalized Hadamard matrices and balanced generalized weighing matrices. Definitions and necessary existence results for these matrices are given in Section 2.

In Sections 3 and 4, we show that each design in Families 10 and 11 from [5, I.5.6] is indeed a strong subdesign of the next design from the same family. Also in Section 3, we use a generalized Hadamard matrix GH(73, 8) found by de Launey and Dawson [7] to obtain a new family of symmetric designs with parameters $v = (73^{m+1} - 64)/9$, $k = 73^m$, and $\lambda = 9 \cdot 73^{m-1}$.

In Section 5, we generalize an idea from Fanning [9] to discover another new family of symmetric designs with parameters $v = 1 + 2(q+1)((q+1)^{2m+2} - 1)/(q+2)$, $k = (q+1)^{2m+2}$, and $\lambda = (q+1)^{2m+1}(q+2)/2$, where $q = 2^p - 1$ is a Mersenne prime.

2 Preliminaries

For basic properties of balanced incomplete block-designs (BIBD) see [3] or [5].

Definition 2.1 *A symmetric design* $\mathfrak{C} = (U, \mathcal{A})$ *is said to be a* strong subdesign *of a symmetric design* $\mathfrak{D} = (V, \mathcal{B})$ *if* $U \subseteq V$ *and* $\mathcal{A} = \{B \cap U : B \in \mathcal{B} \text{ and } B \cap U \neq \emptyset\}$. *We will write* $(v, k, \lambda) \subset (v', k', \lambda')$ *if there exists a symmetric* 2-(v', k', λ') *design having a strong symmetric* 2-(v, k, λ) *subdesign.*

Example 2.2 *Clearly,* $(1,1,\mu) \subset (v,k,\lambda)$ *for any symmetric* 2-(v,k,λ) *design. Note that* μ *does not have to be an integer.*

Example 2.3 *It follows from the standard procedure of doubling Hadamard matrices that* $(4n-1,2n,n) \subset (8n-1,4n,2n)$ *whenever there exists a Hadamard matrix of order* $4n$.

Example 2.4 $PG(n,q)^c \subset PG(n+1,q)^c$, *where the superscript* c *stands for the complement.*

Proposition 2.5 and Corollary 2.8 give necessary conditions for the parameters of a symmetric design and its non-trivial strong symmetric subdesign.

Proposition 2.5 *If* $(v,k,\lambda) \subset (v',k',\lambda')$ *and* $v > 1$, *then there exists a positive integer* q *such that* $k' = qk$ *and* $\lambda' = q\lambda$.

Proof: Let $\mathfrak{C} = (U,\mathcal{A})$ be a strong symmetric 2-(v,k,λ) subdesign of a symmetric 2-(v',k',λ') design $\mathfrak{D} = (V,\mathcal{B})$ and let $\mathcal{A}^*$ be the multiset $\{B \cap U \colon B \in \mathcal{B} \text{ and } B \cap U \neq \emptyset\}$. Then $\mathfrak{C}^* = (U,\mathcal{A}^*)$ is a quasi-symmetric 2-(v,k,λ') design with replication number k' and intersection numbers k and λ. Therefore $\mathfrak{C}^*$ is a multiple of $\mathfrak{C}$ and there exists a positive integer q such that $k' = qk$ and $\lambda' = q\lambda$. □

If $\mathfrak{C} = (U,\mathcal{A})$ is a strong symmetric subdesign of a symmetric design $\mathfrak{D} = (V,\mathcal{B})$, then the incidence structure induced by $\mathfrak{D}$ on the set $V \setminus U$ represents an *affine resolvable pairwise balanced design* [11].

Definition 2.6 *Let* λ *be a positive integer. An* affine resolvable pairwise balanced design *(ARPBD) of index* λ *is a triple* $\mathfrak{P} = (X,C,\mathcal{R})$, *where* X *is a non-empty finite set (of points),* C *is a collection of subsets of* X *(blocks), and* $\mathcal{R}$ *is a partition of* C *(resolution), satisfying the following conditions:*

(i) *any two points occur together in exactly* λ *blocks;*

(ii) *for any resolution class* R, *there is a positive integer* $\alpha(R)$ *(the replication number of* R*) such that every point occurs in exactly* $\alpha(R)$ *blocks from* R*;*

(iii) *the cardinality of each block and the cardinality of the intersection of two distinct blocks depend only on their respective resolution classes.*

As shown in [11, Theorems 2.2 and 2.3], condition (iii) can be replaced by the equality $|\mathcal{B}| = |X| + |\mathcal{R}| - 1$.

The proofs of the following two propositions are modifications of proofs given in [2, Theorems 2.1 and 2.2].

Proposition 2.7 *For positive integers $v > 1$ and $q > 1$, if $(v, k, \lambda) \subset (v', qk, q\lambda)$, then there exists an ARPBD of index $q\lambda$ whose resolution consists of v classes of cardinality q and replication number $q\lambda/k$ and one class of cardinality $v' - qv$ and replication number $q - q\lambda/k$.*

Proof: Suppose there exists a strong symmetric 2-(v, k, λ) subdesign $\mathfrak{C} = (U, \mathcal{A})$ of a symmetric 2-$(v', qk, q\lambda)$ design $\mathfrak{D} = (V, \mathcal{B})$. Let $X = V \setminus U$. If B_1 and B_2 are distinct blocks from $\mathcal{B}$, then $B_1 \cap X \neq B_2 \cap X$, since otherwise we would have had $|B_1 \cap X| = (q-1)\lambda$, $(q-1)\lambda + k = qk$, $k = \lambda$.

Let $\mathcal{C} = \{B \cap X : B \in \mathcal{B}\}$. Define partition $\mathcal{R}$ of $\mathcal{C}$ by declaring $B_1 \cap X$ and $B_2 \cap X$ $(B_1, B_2 \in \mathcal{B})$ equivalent if and only if $B_1 \cap U = B_2 \cap U$, so $\mathcal{R} = \{R_1, ..., R_v, R_{v+1}\}$, where $|R_i| = q$ for $i = 1, ..., v$. Clearly, $(X, \mathcal{C}, \mathcal{R})$ is a pairwise balanced design of index $q\lambda$ with the cardinality of each block and the cardinality of the intersection of two distinct blocks depending only on their respective classes. For $x \in X$ and $i = 1, ..., v, v+1$, let $\alpha_i(x)$ be the number of blocks from R_i that contain x. Fixing $i \in \{1, ..., v\}$ and counting pairs (x, A), where $A \in R_i$ and $x \in A \cap X$, we obtain:

$$\sum_{x \in X} \alpha_i(x) = q(q-1)k. \tag{1}$$

Fixing $i \in \{1, ..., v\}$ and counting triples (x, A, B), where $A, B \in R_i$, $A \neq B$, and $x \in A \cap B \cap X$, we obtain:

$$\sum_{x \in X} \alpha_i(x)(\alpha_i(x) - 1) = q(q-1)(q\lambda - k). \tag{2}$$

Eqs. (1) and (2) imply:

$$\sum_{x \in X} (\alpha_i(x))^2 = q^2(q-1)\lambda. \tag{3}$$

Fixing $i, j \in \{1, ..., v\}$, $i \neq j$ and counting triples (x, A, B), where $A \in R_i$, $B \in R_j$, and $x \in A \cap B \cap X$, we obtain:

$$\sum_{x \in X} \alpha_i(x)\alpha_j(x) = q^2(q-1)\lambda. \tag{4}$$

Eqs. (3) and (4) imply

$$\sum_{x \in X} (\alpha_i(x) - \alpha_j(x))^2 = 0,$$

so $\alpha_i(x) = \alpha(x)$ is the same for $i = 1, ..., v$. Let $x \in X$ and $y \in U$. Since x and y occur together in $q\lambda$ blocks from $\mathcal{B}$ and y occurs in k blocks from $\mathcal{A}$, we have $k\alpha(x) = q\lambda$, so $\alpha(x) = q\lambda/k$ does not depend on $x \in X$. From $v\alpha(x) + \alpha_{v+1}(x) = qk$ and $(v-1)\lambda = k(k-1)$, we derive that $\alpha_{v+1}(x) = q - q\lambda/k$. $\square$

Corollary 2.8 *If* $(v,k,\lambda) \subset (v',qk,q\lambda)$*,* $v>1$*, and* $q>1$*, then* k *divides* $q\lambda$*.*

Proposition 2.9 *For positive integers* $v>1$ *and* $q>1$*, if there exists a symmetric* 2-(v,k,λ) *design and an ARPBD of index* $q\lambda$ *whose resolution consists of* v *classes of cardinality* q *and replication number* $q\lambda/k$ *and one class of replication number* $q-q\lambda/k$*, then* $(v,k,\lambda) \subset (v',qk,q\lambda)$*, where* $v' = 1 + k(qk-1)/\lambda$*.*

Proof: Let $(X,\mathcal{C},\mathcal{R})$ be an ARPBD satisfying the condition of the proposition and $(U,\mathcal{A})$ be a symmetric 2-(v,k,λ) design. We assume that $X \cap U = \emptyset$. Let $\mathcal{R} = \{R_1,...,R_v,R_{v+1}\}$, where $|R_i| = q$ for $i = 1,...,v$, and $\mathcal{A} = \{A_1,...,A_v\}$. For any $B \in \mathcal{C}$, define

$$B^* = \begin{cases} B \cup A_i & \text{if } B \in R_i,\ i=1,...,v, \\ B & \text{if } B \in R_{v+1}. \end{cases}$$

Put $V = X \cup U$ and $\mathcal{B} = \{B^*\colon B \in \mathcal{C}\}$. We claim that $(V,\mathcal{B})$ is a symmetric 2-$(v',qk,q\lambda)$ design. Indeed, $|\mathcal{B}| = |\mathcal{C}| = |X|+|\mathcal{R}|-1 = |X|+v = |V|$. If $x \in X$, then x occurs in $\frac{q\lambda}{k}(v-1)+q = qk$ blocks from $\mathcal{B}$, the same is true for $y \in U$. Clearly, any two distinct points from X as well as any two distinct points from U occur together in $q\lambda$ blocks from $\mathcal{B}$; if $x \in X$ and $y \in U$, then x and y occur together in $k(q\lambda)/k = q\lambda$ blocks from $\mathcal{B}$. □

In this paper we will consider three different constructions of symmetric designs from their strong symmetric subdesigns. The main tools in these constructions will be generalized Hadamard matrices and balanced generalized weighing matrices.

Definition 2.10 *A* generalized Hadamard matrix *GH*(q,s) *over a group* G *of order* q *is a matrix* $H = [h_{ij}]$ *of order* qs *with entries from* G *such that for any two distinct rows* i *and* l*, the multiset*

$$\{h_{lj}^{-1}h_{ij}\colon 1 \le j \le qs\}$$

contains s *copies of every element of* G*.*

Remark 2.11 *If* q *is a prime power and* G *is the additive group of the field* $GF(q)$*, then the following generalized Hadamard matrices* $GH(q,s)$ *over* G *are known to exist (the list is not complete):*

(i) $GH(q,1)$ *[8]; (ii)* $GH(q,2)$ *for odd* q *[12]; (iii)* $GH(q,4)$ *for odd* q *[6];(iii)* $GH(q,q-1)$ *if* $q-1$ *is also a prime power [15]; (iv)* $GH(q,8)$ *if* $q>19$ *is a prime [7].*

It is also known [16] that if $GH(q,s)$ *and* $GH(q,t)$ *over a group* G *exist, then there exists a* $GH(q,qst)$ *over* G*.*

Definition 2.12 *A* balanced generalized weighing matrix $BGW(v,k,\lambda)$ *over a finite group G is a matrix $W=[w_{ij}]$ of order v with entries from the set $G\cup\{0\}$ (we assume that $0\notin G$) such that* (i) *each row and each column of W contain exactly k non-zero entries and* (ii) *for any two distinct rows i and l, the multiset*

$$\{w_{lj}^{-1}w_{ij}\colon 1\le j\le v, w_{ij}\ne 0, w_{lj}\ne 0\}$$

contains exactly $\lambda/|G|$ copies of every element of G.

Remark 2.13 *Replacing every non-zero entry in a $BGW(v,k,\lambda)$ by 1 produces the incidence matrix of a symmetric 2-(v,k,λ) design. It is known [5, IV.4.4] (see also Lemma 5.9 and Remark 5.10 below) that a $BGW(v,k,\lambda)$ over G exists for $v=(q^{m+1}-1)/(q-1)$, $k=q^m$, $\lambda=q^{m-1}(q-1)$, and $G=\mathbb{Z}_t$, where q is a prime power, m is a positive integer, and t is a divisor of $q-1$.*

Remark 2.14 (Notations) *In subsequent constructions we will employ the following notations.*

(i) *If π is a permutation of rows or columns of a matrix M, then πM is the resulting matrix.*

(ii) *If $P=[\pi_{ij}]$ is a generalized Hadamard matrix or a balanced generalized weighing matrix over a group of permutations of rows or columns of a matrix M, then $P\otimes M$ is the matrix obtained by replacing each π_{ij} in P by a block $\pi_{ij}M$ (if $\pi_{ij}=0$, then $\pi_{ij}M$ is the zero matrix of the same size as M).*

(iii) *For any m by n matrix M and for any positive integer t, $t\times M$ denotes a tm by n matrix obtained by replicating each row of M consecutively t times and $M\times t=(t\times M^T)^T$.*

(iv) *We will use $\mathbf{j}$ and $\mathbf{0}$ for the all-one and all-zero row and/or column vectors as well as J and O for the all-one and all-zero matrices. The sizes of these vectors and matrices will be clear from the context. We will use $\langle\cdot,\cdot\rangle$ for the inner product of two rows or columns of the same size.*

3 Construction 1

The following theorem is due to Rajkundlia [14, Construction 3.7].

Theorem 3.1 *Let q and λ be positive integers, $\lambda\equiv 1 \pmod q$. If there exist symmetric 2-$(q\lambda+1,\lambda,\frac{\lambda-1}{q})$ and 2-$(q^2\lambda+q+1,q\lambda+1,\lambda)$ designs and generalized Hadamard matrices $GH(q\lambda+1,1)$ and $GH(q\lambda+1,q)$ over* *a group G, then for any positive integer m,*

$$\left(\frac{(q\lambda+1)^{m+1}-q\lambda-1}{\lambda}+1,(q\lambda+1)^m,\lambda(q\lambda+1)^{m-1}\right)$$
$$\subset\left(\frac{(q\lambda+1)^{m+2}-q\lambda-1}{\lambda}+1,(q\lambda+1)^{m+1},\lambda(q\lambda+1)^{m}\right).$$

We will consider two applications of this theorem.

If $\lambda = 1$ and q is a prime power, then the required symmetric designs are PG$(1, q)$ and PG$(2, q)$. If $q + 1$ is also a prime power, then matrices GH$(q + 1, 1)$ and GH$(q + 1, q)$ over the additive group of GF$(q + 1)$ exist, and Theorem 3.1 yields Family 10 from [5, I.5.6.].

Theorem 3.2 *If q and $q+1$ are prime powers, then for any positive integer m there exists a symmetric 2-$((q+1)^{m+1}-q,(q+1)^m,(q+1)^{m-1})$ design.*

If $\lambda = 9$ and $q = 8$, then the symmetric designs required by Theorem 3.1 are PG(2,8) and PG(3,8). Since there exist matrices GH(73,1) and GH(73,8) [7, Theorem 1.2] over the additive group of GF(73), we obtain another family of symmetric designs.

Theorem 3.3 *For any positive integer m, there exists a symmetric 2-$((73^{m+1}-64)/9,73^m,9\cdot 73^{m-1})$ design.*

Remark 3.4 *These designs are new except $m = 1$, when we obtain a projective geometry $PG(3,8)$.*

4 Construction 2

The following theorem modifies Brouwer's construction of a family of symmetric designs [4].

Theorem 4.1 *Suppose that $(v_0, k, \lambda) \subset (v_1, qk, q\lambda)$, where $q > 1$ is a positive integer and $(q-1)k = 2q\lambda$. Suppose further that there exists a generalized Hadamard matrix $GH(q, 1)$ and a balanced generalized weighing matrix $BGW(q+1, q, q-1)$ over the group of order 2. Then, for any positive integer m,*

$$(v_{m-1},q^{m-1}k,q^{m-1}\lambda)\subset(v_m,q^mk,q^m\lambda),$$

where $v_m = 1 + k(q^m k - 1)/\lambda$.

Proof: It suffices to show that $(v_1, qk, q\lambda) \subset (v_2, q^2k, q^2\lambda)$. Let $\mathfrak{D}_0$ be a strong symmetric 2-(v_0, k, λ) subdesign of a symmetric 2-$(v_1, qk, q\lambda)$ design $\mathfrak{D}_1$. Using $(q - 1)k = 2q\lambda$, we obtain that $v_0 = 1 + 2q(k - 1)/(q - 1)$,

$v_1 = qv_0 + q + 1$, and $v_2 = q^2v_0 + (q+1)^2$. By Proposition 2.7, there exists an affine resolvable pairwise balanced design $\mathfrak{P}$ of index $q\lambda$ having resolution classes R_i, $1 \le i \le v_0$ of cardinality q and replication number $\frac{q-1}{2}$ and resolution class R_∞ of cardinality $q+1$ and replication number $\frac{q+1}{2}$. Let $V = \{1, ..., d\}$, $d = v_1 - v_0$ be the point set of this ARPBD. Let E_i, $1 \le i \le v_0$ be the blocks vs. points incidence matrix of R_i and E_∞ be the blocks vs. points incidence matrix of R_∞, so E_i is a q by d matrix and E_∞ is a $q+1$ by d matrix.

Let H be a generalized Hadamard matrix $\mathrm{GH}(q,1)$ over a group Q of order q. We assume that Q acts as a regular group of permutations on the set of rows of each matrix E_i, $1 \le i \le v_0$, so q^2 by qd matrices $H \otimes E_i$ are defined.

Let $W = [w_{ij}]$ be a balanced generalized weighing matrix $\mathrm{BGW}(q+1, q, q-1)$ over a group G of order 2. We will assume that the diagonal entries of W are equal to 0 and the off-diagonal entries in the last row and column are equal to the neutral element of G. If ω is the non-neutral element of G, we define $\omega E_\infty = J - E_\infty$. For $1 \le l \le q+1$, form q by qd matrix T_l whose consecutive rows are the lth rows of the block-matrices $[w_{11}E_\infty \ \ldots \ w_{1q}E_\infty]$, $[w_{21}E_\infty \ \ldots \ w_{2q}E_\infty]$, ... , $[w_{q1}E_\infty \ \ldots \ w_{qq}E_\infty]$. Finally, define a $q+1$ by qd matrix $T_\infty = [E_\infty \ \ E_\infty \ \ldots \ E_\infty]$.

Arrange matrices $H \otimes E_1, \ldots, H \otimes E_{v_0}, T_1, \ldots, T_{q+1}, T_\infty$ consecutively to obtain a v_2 by qd matrix P and divide all but the last $q+1$ rows of P into groups of q consecutive rows each. Since $v_1 = qv_0 + q + 1$, we assign a row of the incidence matrix of $\mathfrak{D}_1$ to each of these groups so that distinct rows are assigned to distinct groups and then adjoin the assigned row to every row in the group. Adjoining 0 to each of the last $q+1$ rows of P, we obtain a square matrix of order $v_2 = qd + v_1$. It is readily verified that this is the incidence matrix of a symmetric 2-$(v_2, q^2k, q^2\lambda)$ design which therefore contains $\mathfrak{D}_1$ as a strong subdesign. □

The only current application of this theorem is the following. For odd prime power q, we have $(1, 1, \frac{q-1}{2q}) \subset (2q+1, q, \frac{q-1}{2})$, and there exist matrices $\mathrm{GH}(q,1)$ over the additive group of $\mathrm{GF}(q)$ and $\mathrm{BGW}(q+1, q, q-1)$ over the group of order 2, so we obtain Family 11 from [5, I.5.6.].

Theorem 4.2 *For any odd prime power q and positive integer m,*

$$\left(\frac{2q(q^m-1)}{q-1} + 1, q^m, \frac{q^{m-1}(q-1)}{2}\right)$$
$$\subset \left(\frac{2q(q^{m+1}-1)}{q-1} + 1, q^{m+1}, \frac{q^m(q-1)}{2}\right).$$

5 Construction 3

In this section we will construct a new family of symmetric designs that includes Fanning's family [9]. The construction will be based on three lemmas, two of them showing how generalized Hadamard matrices and generalized balanced weighing matrices can be used for construction of infinite series of quasi-derived and quasi-residual designs. First we recall several definitions from design theory.

Definition 5.1 *A 2-(v,k,λ) design is called* quasi-derived *if* $\lambda = k-1$.

Definition 5.2 *A 2-(v,k,λ) design is called* quasi-residual *if* $k+\lambda = r$, *where r is the replication number of the design.*

Definition 5.3 *A 2-(v,k,λ) design is called α-resolvable if there exists a partition of its block-set (α-resolution) such that every point is replicated exactly α times in the blocks of each resolution class.*

Remark 5.4 *Of course, any 2-(v,k,λ) design has a trivial r-resolution, where r is the replication number of the design.*

Lemma 5.5 *Let A be the points vs. blocks incidence matrix of a 2-(v,k,λ) design ($k>1$) with replication number r. For $m=1,2,\dots$, let H_m be a generalized Hadamard matrix $GH(v,v^{m-1})$ over a regular group of permutations of rows of A. Put $A_0 = A$ and define inductively for $m \geq 1$ block-matrices $A_m = [H_m \otimes A \quad v \times A_{m-1}]$. Then A_m is the points vs. blocks incidence matrix of an r-resolvable 2-$(v^{m+1}, v^m k, \lambda_m)$ design, where $\lambda_m = \lambda(v^m k - 1)/(k-1)$.*

Proof: The statement is true for $m=0$, so let $m \geq 1$ and A_{m-1} be the incidence matrix of an r-resolvable 2-$(v^m, v^{m-1}k, \lambda_{m-1})$ design. The replication number of this design is $r_{m-1} = \lambda(v^m-1)/(k-1)$. Clearly, the column sum of A_m is $v^m k$. Let $s = v^m$ and let $\mathbf{a}_{11},\dots,\mathbf{a}_{1v};\ \dots\ ;\mathbf{a}_{s1},\dots,\mathbf{a}_{sv}$ be the consecutive rows of A_m. If $j \neq l$, then $\langle \mathbf{a}_{ij}, \mathbf{a}_{il}\rangle = v^m\lambda + r_{m-1} = \lambda_m$; if $i \neq h$, then $\langle \mathbf{a}_{ij}, \mathbf{a}_{hl}\rangle = v^{m-1}r_0 + (v^m - v^{m-1})\lambda + \lambda_{m-1} = \lambda_m$.

Dividing the columns of A_m into groups of b consecutive columns each, where b is the number of columns of A, we obtain an r-resolution. □

Remark 5.6 *If the above matrix A is the incidence matrix of a quasi-derived design, then A_m is also the incidence matrix of a quasi-derived design.*

Lemma 5.7 *Let B be the points vs. blocks incidence matrix of a 2-(v,k,λ) design. Suppose that this design has an α-resolution consisting of t classes* *of cardinality s. Let σ be a cyclic permutation of order s that acts on each of these classes and let G be the cyclic group generated by σ. Finally, let W be a balanced generalized weighing matrix $BGW(w, l, \mu)$ over G. If $\mu t\alpha^2 = s\lambda l$, then $W \otimes B$ is the incidence matrix of a 2-$(vw, kl, \lambda l)$ design.*

Proof: Clearly, the column sum of $W \otimes B$ is equal to kl. The rows of $W \otimes B$ are naturally divided into w groups of v consecutive rows each. If $\mathbf{x}$ and $\mathbf{y}$ are distinct rows from the same group, then $\langle \mathbf{x}, \mathbf{y} \rangle = \lambda l$. Consider two different groups of rows. Omitting the columns of zero matrices in these groups, we obtain a block-matrix

$$\begin{bmatrix} \pi_1 B & \pi_2 B & \cdots & \pi_\mu B \\ \rho_1 B & \rho_2 B & \cdots & \rho_\mu B \end{bmatrix},$$

where $\pi_i, \rho_i \in G$. We can permute the columns of this matrix to obtain

$$\begin{bmatrix} \rho_1^{-1}\pi_1 B & \rho_2^{-1}\pi_2 B & \cdots & \rho_\mu^{-1}\pi_\mu B \\ B & B & \cdots & B \end{bmatrix},$$

where the upper row of matrices contains μ/s copies of πB for each $\pi \in G$. If $\mathbf{x}$ is a row from the upper group and $\mathbf{y}$ is a row from the lower group, then there exist rows $\mathbf{b}$ and $\mathbf{c}$ in B such that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{\mu}{s} \sum_{\pi \in G} \langle \pi \mathbf{b}, \mathbf{c} \rangle = \frac{\mu}{s} \langle \sum_{\pi \in G} \pi \mathbf{b}, \mathbf{c} \rangle = \frac{\mu}{s} \langle \alpha \mathbf{j}, \mathbf{c} \rangle = \frac{\mu t \alpha^2}{s} = \lambda l.$$

□

Remark 5.8 *If the above matrix B is the incidence matrix of a quasi-residual design, then so is $W \otimes B$.*

For the subsequent construction, we will need balanced generalized weighing matrices of specific format that is provided by the next lemma.

Lemma 5.9 *Let q be a prime power and $GF(q) = \{a_1, a_2, ..., a_q\}$. Let S be a q by q matrix with entries from GF(q) and (i, j)-entry equal $a_i - a_j$, $i, j = 1, 2, ..., q$. For $m = 1, 2, ...$, let H_m be a generalized Hadamard matrix $GH(q, q^{m-1})$ over a regular group of permutations of rows of S. Let matrices S_m and W_m, $m = 0, 1, 2, ...$ be defined inductively by*

$$S_0 = S, \quad S_m = \begin{array}{|c|} \hline H_m \otimes S \\ \hline S_{m-1} \times q \\ \hline \end{array}$$

and

$$W_0 = \begin{array}{|c|c|} \hline S & \mathbf{j} \\ \hline \mathbf{j} & 0 \\ \hline \end{array}, \quad W_m = \begin{array}{|c|c|} \hline S_m & q \times W_{m-1} \\ \hline \mathbf{j} & \mathbf{0} \\ \hline \end{array}.$$

Then W_m is a balanced generalized weighing matrix $BGW(q^{m+1} + q^m + ... + q + 1, q^{m+1}, q^m(q-1))$ over $GF(q)^*$.

Proof: For any rows $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ with entries from GF(q), we will denote by $\mathbf{y}^{-1}\mathbf{x}$ the multiset $\{y_j^{-1}x_j \colon 1 \le j \le n, x_j \ne 0, y_j \ne 0\}$. The rows of S_m are naturally divided into groups of q consecutive rows. Note that the multiset

$$\{(a_k - a_j)^{-1}(a_i - a_j) \colon 1 \le j \le q, a_i - a_j \ne 0, a_k - a_j \ne 0\}$$

contains each non-neutral element of GF$(q)^*$ once. From this, it follows by induction on m that the if $\mathbf{x}$ and $\mathbf{y}$ are distinct rows of S_m from the same group, then $\mathbf{y}^{-1}\mathbf{x}$ consists of q^m copies of each non-neutral element of GF$(q)^*$, and if $\mathbf{x}$ and $\mathbf{y}$ are rows of S_m from different groups, then $\mathbf{y}^{-1}\mathbf{x}$ consists of $q^{m-1}(q-1)$ copies of every element of GF$(q)^*$. Another induction on m implies now that W_m is a BGW$(q^{m+1}+q^m+...+q+1, q^{m+1}, q^m(q-1))$ over GF$(q)^*$. □

Remark 5.10 *If G is a cyclic group of order t, where t divides $q-1$, then G can be considered as a factor-group of GF$(q)^*$. Applying the canonical homomorphism from GF$(q)^*$ to G to each entry of matrices W_m constructed in the above lemma, we obtain balanced generalized matrices of the same format over G.*

We can now introduce Construction 3.

Theorem 5.11 *Let $q = 2^p - 1$ be a prime. For any positive integer m,*

$$\left(\frac{2(q+1)((q+1)^{2m}-1)}{q+2}+1, (q+1)^{2m}, \frac{(q+1)^{2m-1}(q+2)}{2}\right)$$
$$\subset \left(\frac{2(q+1)((q+1)^{2m+2}-1)}{q+2}+1, (q+1)^{2m+2}, \frac{(q+1)^{2m+1}(q+2)}{2}\right).$$

Proof: Let X be the incidence matrix of a symmetric 2-$(q, \frac{q-1}{2}, \frac{q-3}{4})$ design. Let $Y = \begin{bmatrix} \mathbf{j} \\ X \end{bmatrix}$, $E = [X \quad X]$, and $F = [Y \quad J - Y]$. Then E is the points vs. blocks incidence matrix of a quasi-derived $2-(q, \frac{q-1}{2}, \frac{q-3}{2})$ design with replication number $q-1$ and F is the points vs. blocks incidence matrix of a quasi-residual 1-resolvable $2-(q+1, \frac{q+1}{2}, \frac{q-1}{2})$ design whose resolution consists of q classes of cardinality 2. Let H_1 be a generalized Hadamard matrix GH$(q,1)$ over a group Q of order q. We will consider Q as a regular group of permutations of the rows of E and apply Lemma 5.5 to obtain the incidence matrix $E_1 = [H_1 \otimes E \quad q \times E]$ of a quasi-derived 2-$(q^2, \frac{q(q-1)}{2}, \frac{q(q-1)}{2} - 1)$ design. Since $E = [X \quad X]$, this design is $(\frac{q-1}{2})$-resolvable.

Let W be a balanced generalized weighing matrix BGW$(q+1, q, q-1)$ over the cyclic group of order 2. Using Lemma 5.7, we obtain the incidence matrix $F_1 = W \otimes F$ of a quasi-residual 2-$((q+1)^2, \frac{q(q+1)}{2}, \frac{q(q-1)}{2})$ design.

Let $A = J - F_1$, so A is the incidence matrix of a quasi-derived 2-$((q+1)^2, \frac{(q+1)(q+2)}{2}, \frac{(q+1)(q+2)}{2} - 1)$ design. Applying Lemma 5.5, we construct the incidence matrix A_m of a quasi-derived

$$2\text{-}\left((q+1)^{2m+2}, \frac{(q+1)^{2m+1}(q+2)}{2}, \frac{(q+1)^{2m+1}(q+2)}{2} - 1\right)$$

design.

Let $B = J - E_1$, so B is the incidence matrix of a quasi-residual $(\frac{q+1}{2})$-resolvable 2-$(q^2, \frac{q(q+1)}{2}, \frac{(q+1)(q+2)}{2})$ design. Its resolution consists of $2q+2$ classes of cardinality q. Since $(q+1)^2$ is a prime power and q divides $(q+1)^2 - 1$, Lemma 5.9 and Remark 5.10 supply a balanced generalized weighing matrix

$$\mathrm{BGW}\left(\frac{(q+1)^{2m+2}-1}{q(q+2)}, (q+1)^{2m}, q(q+2)(q+1)^{2m-2}\right)$$

over the cyclic group G of order q which we will denote W_m. By Lemma 5.7, $B_m = W_m \otimes B$ is the incidence matrix of a quasi-residual

$$2\text{-}\left(\frac{q((q+1)^{2m+2}-1)}{q+2}, \frac{q(q+1)^{2m+1}}{2}, \frac{(q+1)^{2m+1}(q+2)}{2}\right)$$

design. We claim that the matrix

$$D_m = \begin{array}{|c|c|} \hline A_m & \mathbf{j} \\ \hline B_m & \mathbf{0} \\ \hline \end{array}$$

is the incidence matrix of a symmetric

$$2\text{-}\left(\frac{2(q+1)((q+1)^{2m+2}-1)}{q+2} + 1, (q+1)^{2m+2}, \frac{(q+1)^{2m+1}(q+2)}{2}\right)$$

design. Since A_m and B_m are the incidence matrices of a quasi-derived and a quasi-residual design with proper parameters, we have only to check that $\langle \mathbf{a}, \mathbf{b} \rangle = \frac{(q+1)^{2m+1}(q+2)}{2}$, where $\mathbf{a}$ is a row of A_m and $\mathbf{b}$ is a row of B_m. Clearly,

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{(q+1)^{2m}} \langle \mathbf{a}_i, \mathbf{b}_i \rangle,$$

where each $\mathbf{a}_i$ is a row of A and each $\mathbf{b}_i$ is a row of $\pi_i B$, $\pi_i \in G$. We will represent $\mathbf{a}_i$ as $\mathbf{a}_i = [\mathbf{a}_{i1} \quad \mathbf{a}_{i2} \ldots \mathbf{a}_{i,q+1}]$, where each $\mathbf{a}_{ij}$ is a row of size $2q$. One of these rows equals $\mathbf{j}$, every other is of form $[\mathbf{y} \quad \mathbf{j} - \mathbf{y}]$ or $[\mathbf{j} - \mathbf{y} \quad \mathbf{y}]$, where $\mathbf{y}$ is a row of Y. We will represent $\mathbf{b}_i$ as $\mathbf{b}_i = [\mathbf{b}_{i1} \quad \mathbf{b}_{i2} \ldots \mathbf{b}_{i,q+1}]$ with each $\mathbf{b}_{ij}$ is of form $\pi_i \mathbf{e}_{ij}$, where $\mathbf{e}_{ij}$ is a row of $J - E$. Observe that

$$\langle \mathbf{a}_{ij}, \mathbf{b}_{ij} \rangle = \begin{cases} q+1 & \text{if } \mathbf{a}_{ij} = \mathbf{j}, \\ \frac{q+1}{2} & \text{if } \mathbf{a}_{ij} \neq \mathbf{j}. \end{cases}$$

Therefore,

$$\sum_{i=1}^{(q+1)^{2m}} \langle \mathbf{a}_i, \mathbf{b}_i \rangle = (q+1)^{2m}(q+2)\left(\frac{q+1}{2}\right) = \frac{(q+1)^{2m+1}(q+2)}{2}.$$

To complete the proof note that the format of the matrices W_m supplied by Lemma 5.9 is such that the symmetric design determined by D_{m-1} is a strong subdesign of the symmetric design determined by D_m. □

Remark 5.12 *Symmetric designs with incidence matrices D_m are new except $m = 0$ (Brouwer [4]) and $q = 3$ (Fanning [9]).*

6 An open problem

Symmetric designs of Family 12 from [5, I.5.6] discovered by Spence, Jungnickel and Pott have parameters $k = q^{2m-1}p^{s-1}$ and $\lambda = q^{2m-2}p^{s-1}(p^{s-1} - 1)/(p-1)$, where m is a positive integer, p is a prime, and $q = (p^s-1)/(p-1)$ is a prime power. Two consecutive (with respect to m) designs in this family satisfy the necessary conditions imposed by Proposition 2.5 and Corollary 2.8. Is the smaller design a subdesign of the larger one? The smallest case: true or false that $(16, 6, 2) \subset (160, 54, 18)$?

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