On Ovoids in Orthogonal Spaces of Type $O_5(q)$

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Abstract. It is known that the ovoids in $O_5(q)$, $q \leq 7$, are classical ovoids. Using algebraic and computational techniques, we classify ovoids in $O_5(9)$ and $O_5(11)$ with the aid of a computer. We also study the ovoids which contain an irreducible conic and classify them in $O_5(13)$. Our results show that there is only one nonclassical ovoid (a member from a family of Kantor) up to isomorphism in $O_5(9)$ and all the ovoids in $O_5(11)$ are classical.

1. Notation

More details of the following definitions can be found in Artin [1], Kantor [6] and Taylor [12]. An *orthogonal space* is a pair (V, Q) such that V is a finite dimensional vector space of dimension n over $GF(q)$ and $Q : V \to GF(q)$ is a quadratic form, i.e.

$$Q(\lambda x) = \lambda^2 Q(x)$$

and

$$Q(x + y) = Q(x) + Q(y) + (x, y),$$

for all $\lambda \in F; x, y \in V$, where $(\ ,\ )$ is a bilinear form. The quadratic form Q is *nondegenerate* if $(x, y) = 0$ for $Q(x) = 0$ and all $y \in V$ implies $x = 0$. The *points* of an orthogonal space (V, Q) are one-dimensional subspaces and $\langle v \rangle$ is called a *singular* point if $Q(v) = 0$. Two points $\langle v_1 \rangle$ and $\langle v_2 \rangle$ are *perpendicular* if $(v_1, v_2) = 0$. Two sets of singular points X and Y are *totally nonperpendicular* if $(x, y) \neq 0$ for any $\langle x \rangle \in X$ and for any $\langle y \rangle \in Y$. A subspace S is *totally singular* if $Q(v) = 0$ for all $v \in S$. When $n = 2m$ and Q is nondegenerate, there are two types of quadratic forms up to equivalence. One form is called *hyperbolic* which produces maximal totally singular subspaces with dimension m. The other form is called *elliptic* and it produces maximal singular subspaces with dimension $m-1$. When Q is nondegenerate, (V, Q) is called an $O_{2m-1}(q)$ space if $n = 2m-1$, or an $O_{2m}^{\pm}(q)$ space if $n = 2m$, using superscript + or − according as Q is hyperbolic or elliptic.

An *r-cap* in an orthogonal space is a set of pairwise nonperpendicular singular points with cardinality r. An *ovoid* $\mathcal{O}$ in an orthogonal space of type $O_{2m+2}^{+}(q)$, $O_{2m}^{-}(q)$ or $O_{2m+1}(q)$ is a set of singular points such that every maximal totally singular subspace contains just one point in $\mathcal{O}$, or equivalently, $\mathcal{O}$ is a (q^m+1)-cap. Using $O_{2m}^{+}(q)$ ovoids, $O_{2m-2}^{+}(q)$ ovoids can be produced as follows. Let $\mathcal{O}$ be an ovoid in an $O_{2m}^{+}(q)$ space. If $\langle x \rangle$ is any singular point not in $\mathcal{O}$ then $x^{\perp} \cap \mathcal{O}$ projects onto an ovoid of $x^{\perp}/\langle x \rangle$. This process is commonly referred to as "slicing".

The *orthogonal group* (also called *the group of isometries*) is the subgroup of $GL(V)$ which fixes the quadratic form. The *generalized orthogonal group* (also called *the group of similarities*) is the subgroup of $GL(V)$ which fixes the quadric (the set of singular points). Two n−caps are *isomorphic* if there is a group element in the generalized orthogonal group that takes one cap to the other.

2. Background and the Statements of the Results

The $O_5(q)$ spaces are well known as a family of classical generalized quadrangles, Payne and Thas [8]. As mentioned in Section 1, any ovoid in an $O_5(q)$ space has q^2+1 points. Thus any hyperplane intersecting $O_5(q)$ in an orthogonal space of type $O_4^{-}(q)$ is an ovoid in $O_5(q)$. Such an ovoid is called a *classical ovoid.* When q is odd and not a prime, there is an infinite family of nonclassical ovoids due to Kantor [6]. There are two other known infinite families of nonclassical ovoids in $O_5(q)$. One family is when $q = 3^{2e-1}, e \geq 2$ due to Kantor [6] and the other family is when $q = 3^e, e \geq 3$ due to Thas and Payne [11]. A recent result by Penttila and Williams [9] shows another nonclassical ovoid in $O_5(3^5)$.

Ovoids do not exist in $O_{2n+1}(q)$, $n > 3$ [2, 5, 10]. Ovoids do exist in $O_5(q)$ as mentioned above. The $O_7(q)$ spaces behave differently. Thas [10] has shown that ovoids do not exist in $O_7(2^e)$. Ovoids do exist in $O_7(3^e)$, Kantor [6]. O'Keefe and Thas [7] show that if every ovoid in $O_5(q)$, $q > 3$ and odd, is classical then ovoids do not exist in $O_7(q)$. The orthogonal space $O_5(q)$, $q \leq 7$ contains only classical ovoids [7] and hence there are no ovoids in the corresponding $O_7(q)$. Thus one of the major open problems about ovoids is to show the existence or nonexistence of ovoids in $O_7(q)$ for the remaining open cases.

Thus we have two interesting questions. Are there any other nonclassical ovoids in $O_5(q)$? Is every ovoid in $O_5(p)$, p a prime, classical? Our results produce the answers for these questions in the smallest two open cases, namely $O_5(9)$ and $O_5(11)$. The work involved in obtaining the following results is described in Section 4.

2.1 Result There is only one nonclassical ovoid up to isomorphism in $O_5(9)$. This ovoid is a member of a family of ovoids in $O_5(q)$ constructed by Kantor [6].

2.2 Result Any ovoid in $O_5(11)$ is classical.

The following result is a direct consequence of Result 2.2 and the theorem due to O'Keefe and Thas mentioned above.

2.3 Result $O_7(11)$ does not contain ovoids.

An orthogonal space of type $O_3(q)$ is called an *irreducible conic.*

2.4 Result Let $\mathcal{O}$ be an ovoid which contains an irreducible conic in $O_5(13)$. Then $\mathcal{O}$ is classical.

3. Preliminaries

There are two orbits of irreducible conics under the generalized orthogonal group in $O_5(q)$, q odd. We call them Type 1 and Type 2 according to the following fact. The orthogonal complements of Type 1 and Type 2 conics in $O_5(q)$ are of type $O_2^+(q)$ and $O_2^-(q)$ respectively.

3.1 Lemma Let C be an irreducible conic in $O_5(q)$, where q is odd. There are exactly $(q+\epsilon)/2$ classical ovoids in $O_5(q)$ which contain C, where ϵ is -1 or $+1$ according as C is of Type 1 or Type 2.

Proof: Let α be the number of classical ovoids containing C, let β be the number of orthogonal spaces of type $O_4^+(q)$ containing C, and let γ be the number of hyperplanes containing C and intersecting $O_5(q)$ in a degenerate orthogonal space. By counting the singular points of $O_5(q)$, we obtain

$$\alpha(q^2-q)+\beta(q^2+q)+\gamma q^2+q+1=(q+1)(q^2+1),$$

and $\alpha+\beta+\gamma=q+1$. If C is of Type 1, then $\gamma=2$, and so $\alpha=(q-1)/2$. If C is of Type 2, then $\gamma=0$, and so $\alpha=(q+1)/2$. ∎

Let $\mathcal{O}$ be an ovoid in an $O^+_{2m}(q)$ space. Let V be the underlying vector space over $F = GF(q)$ and let Q be the quadratic form for $O^+_{2m}(q)$. Fix a singular point $\langle u \rangle \in \mathcal{O}$. We select vectors $v_1, v_2, ..., v_{m-1}$ such that $\langle u, v_1, v_2, ..., v_{m-1} \rangle$ is a maximal totally singular subspace. For any $a_1, a_2, ..., a_{m-1} \in GF(q)$, $\langle v_1 + a_1 u, v_2 + a_2 u, ..., v_{m-1} + a_{m-1} u \rangle^{\perp} \cap \mathcal{O}$ is a cap of size 2 with one of the points being $\langle u \rangle$. (As mentioned in the terminology of Section 1, we slice the ovoid $m-1$ times).

3.2 Proposition (Gunawardena [4]) Define a function $\phi : F^{m-1} \rightarrow \mathcal{O} \setminus \{\langle u \rangle\}$ such that $\phi(a_1, a_2, ..., a_{m-1}) = \langle v_1 + a_1 u, v_2 + a_2 u, ..., v_{m-1} + a_{m-1} u \rangle^{\perp} \cap \mathcal{O} \setminus \{\langle u \rangle\}$. Then ϕ is a bijection.

The above bijection helps us create an efficient computer algorithm to list all the ovoids in a given set of singular points. The following algorithm is taken from [4]. Suppose we have a cap C and a set S consisting of singular points such that S and C are totally nonperpendicular. This algorithm will list all the possible ovoids in $C \cup S$ that contain C.

Input: $C, S_1, S_2, \ldots, S_k$: $S_1, S_2, \ldots, S_k$ are subsets of S and they are created as follows.

Let $u \in C$ and select vectors $v_1, v_2, \ldots, v_{m-1}$ such that $\langle u, v_1, v_2, \ldots, v_{m-1} \rangle$ is a maximal totally singular subspace. Construct subsets $S_1, S_2, \ldots, S_k$ (where $k = q^{m-1} + 1 - |C|$) of S as follows.

1. $S_i = \langle \{v_1 + a_{i1} u, v_2 + a_{i2} u, \ldots, v_{m-1} + a_{i(m-1)} u\} \rangle^{\perp} \cap S$, for some $a_{i1}, a_{i2}, \ldots, a_{i(m-1)} \in GF(q)$.
2. $S_i \cap S_j = \emptyset$ if $i \neq j$.
3. $S_i \cap C = \emptyset$ for each i.

The above sets $S_1, S_2, \ldots, S_k$ partition S. This means that any ovoid in $C \cup S$ should contain exactly one singular point from each S_i. Thus, we have the following recursive algorithm.

Ovoid($C, \{S_1, S_2, \ldots, S_k\}$)
begin
If ($|C| = q^{m-1} + 1$), print C which is an ovoid and return.
Select i such that $|S_i| = min\{|S_j| : 1 \leq j \leq k\}$.
If $|S_i| = 0$ return.
For each $\langle x \rangle \in S_i$, do
begin
$\overline{C} \leftarrow C \cup \langle x \rangle$.

For each $j \neq i$,
$\overline{S}_j \leftarrow \{\langle y \rangle \in S_j : (y, x) \neq 0\}$.
Call ovoid($\overline{C}$, $\{\overline{S}_1, \overline{S}_2, ..., \overline{S}_{i-1}, \overline{S}_{i+1}, ..., \overline{S}_k\}$).
end
end

3.3 Proposition (O'Keefe and Thas [7]) If every ovoid of $O_5(q)$, where q is odd and $q \neq 3$ is classical then $O_7(q)$ has no ovoid.

The following proposition exhibits a member of an infinite family of ovoids in $O_5(q)$ constructed by Kantor.

3.4 Proposition (Kantor [6]) There is a nonclassical ovoid in $O_5(9)$. Let $K = GF(9)$ and ϵ be a nonsquare in K. Equip $V = K^5$ with the quadratic form $Q(x_1, x_2, x_3, x_4, x_5) = x_1x_5 + x_2x_4 + x_3^2$. The ovoid consists of the points $\langle 0, 0, 0, 0, 1 \rangle$ and $\langle 1, y, z, -\epsilon y, -z^2 + \epsilon y^2 \rangle$, $y, z \in K$.

4. $O_5(9)$, $O_5(11)$, $O_7(11)$ and $O_5(13)$ Spaces

Our approach to classify the ovoids in $O_5(9)$ is as follows. We create a list of ovoids which contains all the ovoids in $O_5(9)$ up to isomorphism. We show that every ovoid in this list contains an irreducible conic. Then we classify the ovoids that contain an irreducible conic. Let $F = GF(9) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ with addition and multiplication tables as follows.

+	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	6
2	2	0	1	5	3	4	8	6	7
3	3	4	5	6	7	8	0	1	2
4	4	5	3	7	8	6	1	2	0
5	5	3	4	8	6	7	2	0	1
6	6	7	8	0	1	2	3	4	5
7	7	8	6	1	2	0	4	5	3
8	8	6	7	2	0	1	5	3	4

·	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	1	6	8	7	3	5	4
3	0	3	6	2	5	8	1	4	7
4	0	4	8	5	6	1	7	2	3
5	0	5	7	8	1	3	4	6	2
6	0	6	3	1	7	4	2	8	5
7	0	7	5	4	2	6	8	3	1
8	0	8	4	7	3	2	5	1	6

Let $V = F^6$. Define $Q : V \rightarrow F$ such that

$$Q(x_1, x_2, ..., x_6) = x_1x_6 + x_2x_5 + x_3x_4.$$

Thus, (V, Q) is an $O_6^+(9)$ space and for $x = (x_1, x_2, ..., x_6), y = (y_1, y_2, ..., y_6)$, we have

$$(x, y) = x_1y_6 + x_6y_1 + x_2y_5 + x_5y_2 + x_3y_4 + x_4y_3.$$

Now consider the $O_5(9)$ space obtained by intersecting the $O_6^+(9)$ space with the hyperplane $x_3 = x_4$. $O_5(9)$ contains 820 singular points and any ovoid contains 82 singular points. There are two orbits on the set of 3-caps under the generalized orthogonal group. It is known that any ovoid in $O_5(9)$ contains both types of these 3-caps [5]. Hence, we can begin with any 3-cap. We fixed the 3-cap $C = \{\langle u\rangle, \langle x_1\rangle, \langle x_2\rangle\}$, where u =(0 0 0 0 0 1), x_1 =(1 0 0 0 0 0) and x_2 =(1 0 1 1 0 2).

We created the set S which is the set of all singular points totally nonperpendicular to C. From [3], we have a complete classification of 4-caps which span subspaces of type $O_4^+(q)$ or $O_4^-(q)$ in orthogonal spaces under the orthogonal group. There were thirteen nonisomorphic 4-caps which contained C. We ordered these 4-caps as $C_1, C_2, \ldots, C_{13}$. For each C_i we created a set S_i as follows. Let X_i be the set of all singular points totally nonperpendicular to C_i. Let Y_i be the set of all the singular points such that if $y \in Y_i$ then $y \cup C$ is isomorphic to C_j for some j, $1 \leq j < i$. Now $S_i = X_i \setminus Y_i$. To list the ovoids in the sets $C_i \cup S_i$, we applied the algorithm given in Section 3 with the maximal totally singular subspace $\langle\{u, v_1, v_2\}\rangle$, where u =(0 0 0 0 0 1), v_1 =(0 0 0 0 1 0), v_2 =(0 0 0 1 0 0). The algorithm listed 252 ovoids of $O_5(9)$. By a computer, we found that each ovoid in this list contained an irreducible conic. Next we selected two irreducible conics CO_1 and CO_2, which contain $\langle u\rangle$, such that CO_1 is Type 1 and CO_2 is Type 2. Let Z_1 and Z_2 be the sets of all the singular points that are totally nonperpendicular to CO_1 and CO_2 respectively. Using our algorithm, we listed all the ovoids in $CO_1 \cup Z_1$ and $CO_2 \cup Z_2$. There were 44 ovoids which contained CO_1 and 5 ovoids which contained CO_2. The five ovoids which contained CO_2 were classical by Lemma 3.1. Using a probabilistic algorithm, we checked the isomorphisms among the nonclassical ovoids that contained CO_1 and found them in one orbit under the stabilizer of CO_1 in the generalized orthogonal group. Therefore, we have a unique nonclassical ovoid in $O_5(9)$ space and it is mentioned in Proposition 3.4.

In $O_5(11)$, we used 4-caps and followed the above mentioned procedure to list all the ovoids up to isomorphism. Every ovoid found was classical. Thus by Proposition 3.3, there are no ovoids in $O_7(11)$.

In $O_5(13)$, we found that the total number of ovoids that contained a given Type 1 conic was six and the total number of ovoids that contained a Type 2 conic was seven. By Lemma 3.1, these ovoids must be classical.

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