Negative Results for Orthogonal Triples of Latin Squares of Order 10

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Abstract

We consider whether an order ten Latin square with an order four Latin subsquare can belong to an orthogonal triple of Latin squares. We eliminate 20 of 28 possibilities for how this could occur by considering the structure of possible mates. Our technique supplements the small collection of existing tools for obtaining negative results regarding the existence of collections of orthogonal Latin squares.

1 Latin Squares

A *latin square* of order n is an n by n array containing the symbols 0 through $n-1$ so that each symbol appears exactly once in each row and exactly once in each column. A *transversal* of a latin square consists of n positions of the square chosen so that there is exactly one entry from each row and column, and so that each symbol appears exactly once. A transversal is uniquely denoted by recording the permutation obtained by listing the symbols chosen from each of the columns. A latin square of order four is pictured in Figure 1. The boxed entries constitute the transversal represented by 0 2 3 1 .

Two latin squares are *orthogonal* if when they are superimposed, each ordered pair of symbols occurs exactly once. An *orthogonal triple* of latin squares is a set of three latin squares which are pairwise orthogonal. One

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$$\begin{array}{cccc} \boxed{0} & 1 & 2 & 3 \\ 1 & 0 & \boxed{3} & 2 \\ 2 & 3 & 0 & \boxed{1} \\ 3 & \boxed{2} & 1 & 0 \end{array}$$

Figure 1: A latin square of order four

outstanding open question is whether there exists an orthogonal triple of latin squares of order ten. A large amount of attention has been paid to this question and much of it is summarized in [6] and [7]. The four mutually orthogonal latin squares of order ten with a hole of order two [3], and a turn-square having a mate which shares four parallel transversals [4] are as close as people have come to constructing a triple (turn-squares are introduced in [12], and one having 5504 transversals, a remarkable number, is given).

For any pair of orthogonal latin squares, the cells containing a particular symbol of one square delimit a transversal of the other. Thus, a mate of a latin square of order n can be described simply by presenting an ordered list of n disjoint transversals, where the ith transversal gives the location of symbol i in the mate. This is how we describe mates in this paper.

It is convenient to visualize the n permutations representing the transversals of an orthogonal mate (obtained as described above) as the n rows of a latin square. There is a nice characterization of when an arbitrary collection of latin squares corresponds to this "transversal representation" for an orthogonal set of latin squares:

Theorem 1.1 *A collection of r mutually orthogonal latin squares of order n exists if and only if there exists a collection of r latin squares of order n having the property that the rows of each square correspond to transversals for each of the other squares.*

This "transversal representation" of mutually orthogonal latin squares has proven especially useful to us when designing computer programs for exploring squares and their mates. Further, it is used exclusively throughout the discussion in the rest of this paper.

Various constructions are known for finding large sets of mutually orthogonal latin squares. A unified approach to the small cases (up to order 32) is discussed in [2]. A table of the maximum sets discovered as of 1991 going up to order 200 is in [7, pp. 166-167]. A more extensive and up to date table is given in the CRC Handbook of Combinatorial Designs [1] (the WWW page for this book records updates occurring since the book was published). An explanation of how this table was generated with a wealth of information on existing constructions appears in [5].

In contrast, there are few tools for showing that large collections cannot be created. Many of the available techniques are summarized in [6, pp. 445-456] and [7, pp. 23-32]. One such result is a theorem of Mann (Theorem 1.2) from which it can be seen for example that an order ten latin square with an order five latin subsquare has no orthogonal mates.

Theorem 1.2 ***Mann's Theorem*** *[9]. If L is a latin square*

1. *of order $4n+2$ with an order $2n+1$ subsquare which has at most n squares having symbols distinct from a set of $2n+1$ of the symbols, or*

2. *of order $4n+1$ with an order $2n$ subsquare which has at most $n/2-1$ squares having symbols distinct from a set of $2n$ of the symbols.*

then L has no orthogonal mate. □

A corollary of a theorem of Parker (Theorem 1.3) indicates that an order ten latin square and a mate cannot have mutually orthogonal subsquares of order four (a slightly stronger theorem appears in the paper as well, but it provides no further information for order ten). This is also obvious from our discussions regarding the structure of potential mates.

Theorem 1.3 ***Parker's Theorem*** *[10]. If a set of t mutually orthogonal latin squares of order n has a set of t mutually orthogonal subsquares of order r with $r < n$, then $n \geq (t+1)r$.* □

Further, a second theorem of Parker (Theorem 1.4) indicates that two mutually orthogonal latin squares of order ten with mutually orthogonal order three subsquares cannot appear together in an orthogonal triple.

Theorem 1.4 ***Another of Parker's Theorems*** *[11]. If a set of $r-1$ mutually orthogonal latin squares of order n has a set of $r-1$ mutually orthogonal subsquares of order r, $r < n$, and there exists a latin square of order n orthogonal to all $r-1$, then $n \geq r^2$. Further, if $n > r^2$, then $n \geq r^2 + r$.* □

The known constructions result in squares with a lot of structure and "large" (relative to the order) latin subsquares are common. So it seems natural, given that an order ten square with a mate has no order five latin

0	4	5	3	2	1	6	7	8	9		0	1	7	2	4	6	5	8	9	3
1	0	4	5	3	2	7	6	9	8		1	6	3	7	0	5	9	2	8	4
2	1	0	6	7	3	8	9	5	4		2	9	6	3	5	8	7	0	4	1
3	2	1	0	6	7	9	8	4	5		3	8	5	1	7	9	4	6	0	2
9	3	2	1	0	8	5	4	7	6		4	7	2	0	9	3	6	5	1	8
8	9	3	2	1	0	4	5	6	7		5	3	4	9	2	7	8	1	6	0
4	5	6	7	8	9	0	1	2	3		6	2	9	8	3	1	0	4	5	7
5	6	7	8	9	4	1	0	3	2		7	4	1	5	8	0	2	9	3	6
6	7	8	9	4	5	2	3	0	1		8	5	0	4	6	2	1	3	7	9
7	8	9	4	5	6	3	2	1	0		9	0	8	6	1	4	3	7	2	5

Figure 2: A latin square of order ten and transversals corresponding to an orthogonal mate

subsquare, to ask if there exists an orthogonal triple containing a square with an order four latin subsquare. Such a square may have a mate as evidenced by Figure 2 (the order four subsquare occurs in the lower right hand corner and is based on the symbols $0-3$). In fact, such squares are not particularly rare. A quick check (about two hours) on the computer yielded 83 such squares with mates, but none were contained in an orthogonal triple of latin squares.

By examining the structure of the possible mates of an order ten square with an order four subsquare, we eliminate 20 of 28 prospective pairs of mate patterns as possibilities for an orthogonal triple. Possibly with a bit more ingenuity, the remaining cases can be eliminated. If not, they provide guidance towards a search for potential triples.

Section 2 describes the notation we use when drawing our pictures. Then in Section 3, we discuss the structures of transversals of a square with an order four subsquare, and enumerate the potential mates. Once we have this machinery, the proofs in Section 4 are very simple. We conclude in Section 5 with some suggestions for future research.

2 Notation

We assume we start with an order ten square L which has an order four latin subsquare in the bottom right hand corner. Without loss of generality, the order four subsquare is based on the symbols 0 through 3. We distinguish between these symbols and those in the range 4-9 in a transversal or row of L by colour; *white* cells indicate an entry in the range 0-3, and *grey* cells indicate 4-9. The twelve cells with values in the range 4-9 in the upper

Figure 3: White, grey, and black cells

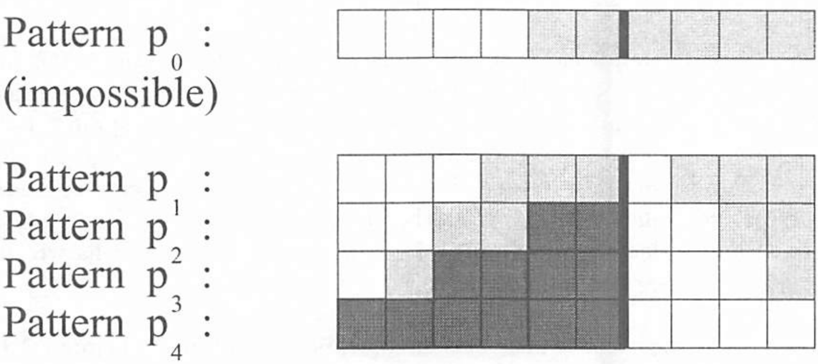


Figure 4: Patterns for transversals of L

left hand order six subsquare of L are singled out for special attention by colouring them *black*. Figure 3 depicts these three types of cells. It should be noted that in the transversals and the orthogonal mates of L, a cell is white, grey, or black if the corresponding cell in L is that colour.

Columns are divided into two blocks consisting of the first six columns, and then the last four columns. This division is indicated in our figures with a thick black line. Within a block of columns, the symbols indicated may be arranged in any order. For example, the pattern p_2 (refer to figure 4) indicates that there are two symbols chosen from 0 through 3 in the first six columns, and four symbols from 4-9. The permutations 9 8 3 4 6 0 1 5 2 7 and 7 1 8 6 2 9 3 0 5 4 both fit this pattern, but 0 5 9 1 4 3 6 7 2 8 does not but is a realization of pattern p_1 instead (see Figure 4 again). We name the possible patterns for transversals as p_i, $i = 0, 1, 2, 3, 4$ where p_i is the pattern with i white cells in the last four columns. These are pictured in Figure 4. The number of black cells is justified later in Lemma 3.1.

Our starting square L is as pictured in Figure 6. We number the rows and columns of L with 0 through 9. The four blocks of L are called A (the order four subsquare is in the lower right hand corner), B (4 by 6), C (6 by 4), and D (6 by 6) as indicated in the figure.

3 Transversals and Mates of L

We start by justifying the number of black cells given in the transversals in Figure 4, and also show that transversal pattern p_0 is impossible.

Lemma 3.1 *The number of black cells in a transversal of L which has w white cells in the first six columns is $6 - 2w$.*

Proof. To understand the proof, it helps to refer to the picture of L in Figure 6. A transversal of L with w white cells in the first six columns must have $4 - w$ white cells in the last four columns (each symbol 0-3 appears once in each transversal), and consequently, there are w grey cells in the last four columns. Hence there are $2w$ nonblack cells chosen from the first six rows; w white ones from the first six columns, and w grey ones chosen from the last four columns. This leaves $6 - 2w$ black cells which must also be chosen from the first six rows. □

Because p_0 has four white cells in the first six columns. Lemma 3.1 indicates that there should be -2 black cells. Since a negative value is not realizable, there are no transversals of L fitting pattern p_0.

The mates are classified according to the number of transversals of L of each type which correspond to the symbols in that mate. The seven possibilities for the sets of transversals which correspond to mates are tabulated in Figure 5 and their structures are pictured in Figures 6, 7, 8, and 9. Viable mates have n_i transversals of type p_i where $n_i \geq 0$, $n_1 + n_2 + n_3 + n_4 = 10$ (the mate has ten rows), and $n_1 + 2n_2 + 3n_3 + 4n_4 = 16$ (the number of symbols chosen from block A is 16 in total). The mate pictured in Figure 2 has pattern X (pictured in Figure 9).

4 Illegal Combinations of Mates

In this section, we show that 20 of the potential 28 pairings of the mate structures indicated in the previous section are impossible. This first theorem eliminates 18 possibilities.

Theorem 4.1 *The following combinations of latin square patterns do not appear together in an orthogonal triple with a square of pattern L:*

1. *R with R, S, T, U, V, or W,*
2. *S with S, T, U, V, or W,*

Figure 5: The seven possibilities for mates of L

Name	n_1	n_2	n_3	n_4
R	8	0	0	2
S	7	0	3	0
T	7	1	1	1
U	6	2	2	0
V	6	3	0	1
W	5	4	1	0
X	4	6	0	0

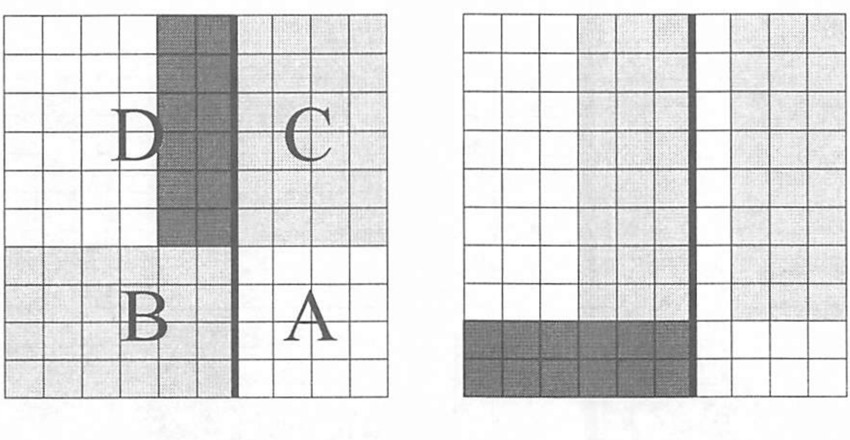
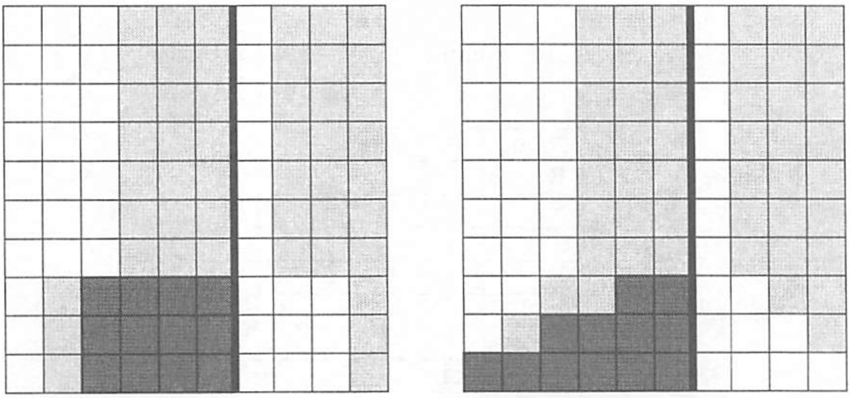
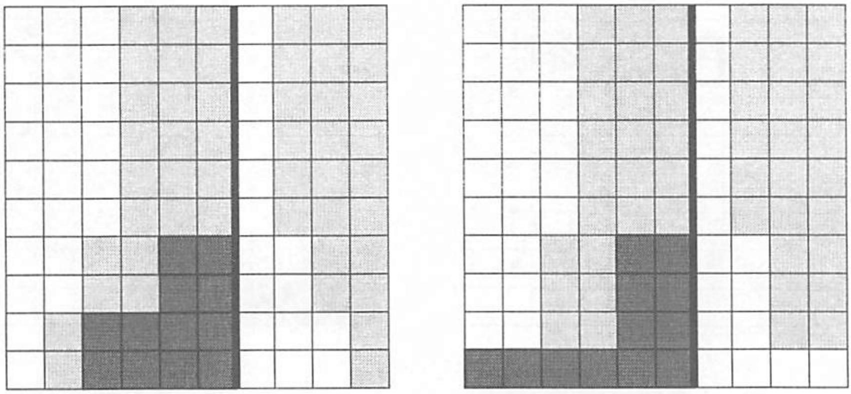


Figure 6: Latin square L and mate pattern R.



S T

Figure 7: Mate patterns S and T.



U V

Figure 8: Mate patterns U and V.

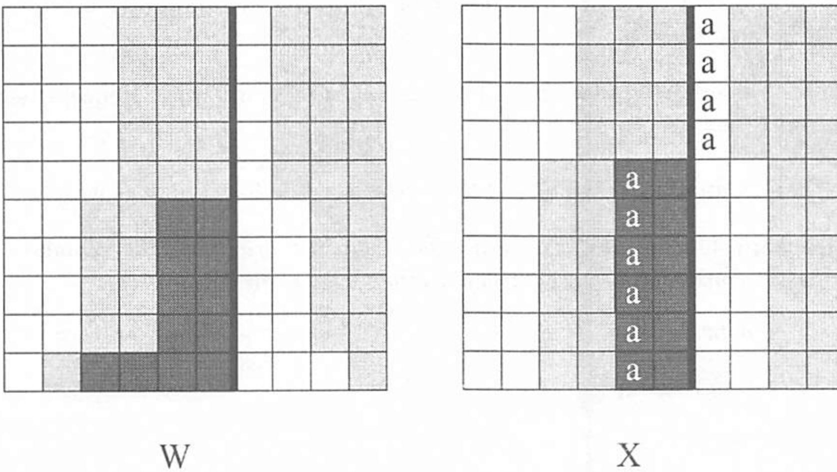


Figure 9: Mate patterns W and X.

3. T with T, U, V, or W,

4. U with V, or

5. V with V or W.

Proof. By Theorem 1.1, it suffices to show that the rows of the first square listed in each part above cannot correspond to disjoint transversals of the other squares listed in the theorem. Look at the rows containing black symbols in the first square listed. We cannot find corresponding transversals in the second square because the black cells span too few rows, or as for example with S and U, too few black cells span an adequate number of rows. □

The result in our next lemma is used to eliminate two further cases.

Lemma 4.2 *If a mate orthogonal to X (Figure 9) and L (Figure 6) has a transversal of type p_4 then it has no further transversals of types p_4 or p_3 (Figure 4 indicates the transversal types).*

Proof. Figure 9 has the cells of X corresponding to a p_4 transversal marked with the letter "a". A shortage of white cells available in the last four columns in rows disjoint from where the black cells are chosen makes another p_4 or p_3 impossible. □

Corollary 4.3 *An orthogonal triple of latin squares containing L and X cannot have R or T.*

Proof. Pattern R has two p_4's. Pattern T has a p_4 and a p_3. Hence, these are ruled out by Lemma 4.2. □

We summarize the results of this section in the following theorem.

Theorem 4.4 ***Main Theorem.*** *The only potentially feasible combinations for constructing an orthogonal triple with L are either*

1. *X with S, U, V, W, or X,*
2. *U with U or W, or*
3. *W with W.* □

5 Future Work

The computer has been one tool used to provide negative results. However, exhaustive computer searches such as that of Lam, Swiercz and Thiel [8] who claim that there are no projective planes of order ten (or equivalently, no set of nine mutually orthogonal latin squares of order ten) are very laborious, prone to error, hard to check, and they take too much time on larger problems. Consequently, more tools for proving negative results are urgently needed.

The most obvious next step in extending the current work is to eliminate the remaining eight cases from consideration. Either this, or find an orthogonal triple fitting one of these patterns. Another possibility is to generalize these results to squares of arbitrary orders.

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