# Well-Covered Graphs

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ABSTRACT. The problem of determining which graphs have the property that every maximal independent set of vertices is also a *maximum* independent set was proposed by M.D. Plummer in 1970 [28]. This was partly motivated by the observation that whereas determining the independence number of an arbitrary graph is NP-complete, for a well-covered graph one can simply apply the greedy algorithm. Although a good deal of effort has been expended in an attempt to obtain a complete characterization of such graphs, that result appears as elusive as ever. In this paper, intended to serve as an introduction to the problem, several of the main attacks will be highlighted with particular emphasis on the approach involving the girth of such graphs.

## An Overview

This expository paper is by no means complete and the interested reader is referred to the excellent survey paper by M.D. Plummer [29] for much more detailed information. Also, the paper by Y. Caro [4], has many references (and thus more recent ones). As indicated, M.D. Plummer [28] coined the term *well-covered* for those graphs in which every maximal independent set of vertices is maximum. To appreciate the reason for such a choice recall that a *point cover* is a set of points (vertices) such that every edge is incident with some point in the cover. Gallai [18] observed that the complement of such a set would necessarily be independent. Hence, if one minimizes the order of a point cover, the remaining vertices will form a maximum independent set. Much of the work on well-covered graphs has been from this complementary point of view, namely, consideration of independent sets of vertices.

As simple examples of well-covered graphs, consider a path on 4 vertices or a 7-cycle. Every maximal independent set is of size 2 for the former and of size 3 for the latter. On the other hand, a path on 3 vertices admits

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maximal independent sets of either one or two vertices and thus is not well-covered. J. Staples was the first to study these graphs in detail [34, 35]. One of her results was a characterization of the well-covered graphs for which the independence number is exactly half the number of vertices in the graph. These include the bipartite well-covered ones. O. Favaron [10] obtained a similar characterization and G. Ravindra [32] also characterized the bipartite case.

Quite unaware of the problem, A. Finbow and the author [11] became involved through the connection of another question. Consider a 2-person game in which the players alternate removing a vertex and all of its neighbours from a graph. The player last able to move wins. For example, if the graph were a path on 5 vertices then player 1 could ensure a win by choosing the central vertex and removing it and its neighbours leaving player 2 facing two isolated vertices. The main result of [11] was a characterization of the graphs of girth 8 or more with the property that *regardless* of how the players moved, the same player would always win. That is, the parity of the number of moves was always the same. For instance, consider a star on an even number of vertices (and thus an odd number of leaves). Either player 1 chooses the central vertex (removing it and all others) or chooses a leaf (removing it and its unique neighbour, the central vertex). In the latter case there are now an even number of isolated vertices left as moves meaning the total number (of moves) is odd. In either situation the number of moves is odd.

In general, if a graph has the property that every vertex is either a leaf or has an odd number of leaves attached, then the number of moves in this game will always be of the same parity. Furthermore if the girth is 8 or more, these are the *only* graphs (called *parity* graphs) with this property.

Now observe that the union of the moves made by the players forms an independent set and, when the game is over, a *maximal* independent set. Hence, a special case of this game is the situation where the number of moves is not only of the same parity but actually a constant. In order for this to occur every vertex must be either a leaf or have exactly one leaf attached. Thus a corollary of the above characterization is the determination of the well-covered graphs of girth 8 or more.

We observe that these graphs are such that the independence number is exactly half the order of the graph and thus this result overlaps with, but is not contained in, the work of J. Staples [34] and O. Favaron [10].

The next results on well-covered graphs were announced in 1987 although, in one case, appeared in print much later. S. Campbell [1] studied cubic graphs with connectivity at most 2 and, along with M.D. Plummer [2], proved that there are only four 3-connected, cubic, planar well-covered graphs. This work was later extended by S. Campbell along with M. Ellingham and G. Royle [3] to include *all* cubic well-covered graphs. Still later,

J. Ramey [31] managed to completely characterize all well-covered graphs of *maximum degree* 3.

Meanwhile, A. Finbow and the author, along with R. Nowakowski, continuing a girth approach, characterized the well-covered graphs of girth 5 or more [15]. As indicated previously, this result was actually announced at a conference in early 1987. In addition, the same three authors, using a similar approach, characterized the well-covered graphs in which there are no 4-cycles nor 5-cycles (but triangles are allowed) [16]. In order to understand the description of these graphs we require the following definition. A vertex $v$ in a graph $G$ is called *extendable* if and only if $G - v$ is also well-covered and the independence number of $G - v$ is the same as $G$. For instance, any vertex of a graph which is a 5-cycle (or a 3-cycle or a complete graph) is extendable. On the other hand, no vertex in a 4-cycle is.

The leaves of a path on 4 vertices are not extendable whereas the other two vertices are. Extendable vertices play a very important role as they can be used as attachment points to join well-covered graphs to create larger ones. For instance, two 5-cycles can be joined by an edge or a 5-cycle and a complete graph can be joined by an edge. In fact, one can start with any collection of $K_2$'s and 5-cycles and designate one vertex of each $K_2$ as well as any two non-adjacent vertices of each 5-cycle as attachment points and then form a connected graph by arbitrarily joining attachment points (see Figure 1).

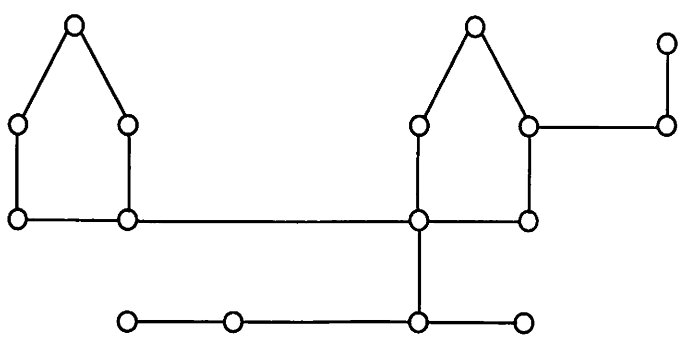


**Figure 1.** A well-covered graph of girth 5

The graph so formed will be well-covered. In [15] it is shown that any well-covered graph of girth 5 or more that has an extendable vertex must in fact belong to this family. The rather surprising result is that there are only six other well-covered graphs having no extendable vertices but having girth at least 5 (see Figure 2).

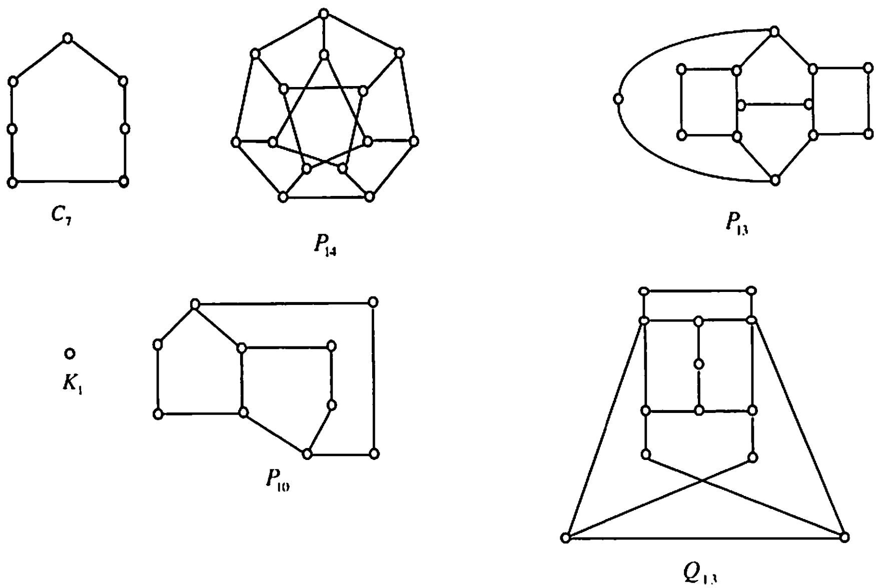
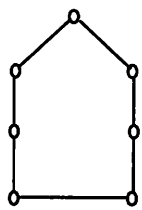



**Figure 2.**
Well-covered graphs (with no extendable vertices)
of girth 5 or more

Hence the $K_2$ and the 5-cycle are essentially the two basic building blocks used to form the family. Observe that in this setting, the characterization for girth 8 or more [11] could be rephrased as being a collection of $K_2$'s with one vertex of each selected as an attachment point and then arbitrarily joining attachment points (as long as girth is 8 or more). In [16], this approach is employed again, this time to graphs with no 4-cycles or 5-cycles. In this case, there is a collection of $K_2$'s, with exactly one attachment point, and 3-cycles, with either one or two attachment points. Once more one can create a well-covered graph by arbitrarily joining attachment points. It is shown that any well-covered graph with an extendable vertex must in fact belong to this family. Again, it turns out there are only two other graphs (having no extendable vertices), besides the trivial $K_1$, in the collection (see Figure 3).



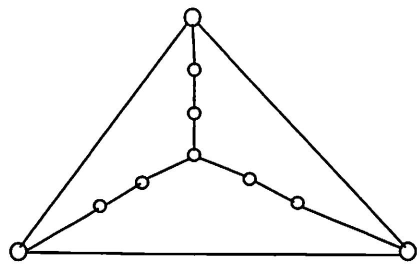




**Figure 3.**
Well-covered graphs (with no extendable vertices)
with no 4-cycles nor 5-cycles

Attempts to extend this attack to include all well-covered graphs with no 4-cycles have met with limited success. Although it is likely that there are a reasonably small number of basic building blocks (each has at least one extendable vertex) it is *not* obvious how to determine them! In [19] one attempt to do so resulted in the discovery of over a dozen new ones (besides the $K_2$, 3-cycle and 5-cycle seen before). Furthermore, in [20] it is shown that an extendable vertex in a well-covered graph without 4-cycles is either part of one of the already known special $K_2$'s, 3-cycles or 5-cycles or is a vertex of a special induced subgraph on 8 vertices called $S_8$ (see Figure 4).

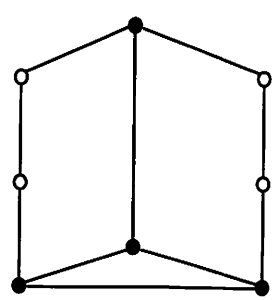



**Figure 4**

The black vertices indicate those that are extendable.

Although imposing either a maximum degree or girth constraint have been the most fruitful in attacking the well-covered problem, these are certainly not the only approaches. In addition, as indicated earlier, there are a number of more specialized results (for example, [8, 13,14,21,23,24,25,26,27, 30,38]).

### Complexity issues and some related problems

The question of how easy or difficult it is to recognize if a graph has the well-covered property has been addressed by a number of individuals. Independently it was shown [7,33] that deciding if a given graph is well-covered is CO-NPC. More recently, this has been refined to the result that deciding if a $K_{1,n}$-free graph (for $n \geq 4$) is well-covered is CO-NPC [6]. On the other hand, D. Tankus and M. Tarsi [36,37] have established a polynomial algorithm to decide if a claw-free graph is well-covered (see [21,38] for partial characterizations). In a more general setting, the original problem that A. Finbow and the author had considered, namely, characterizing parity graphs (every maximal independent set is of the same parity) was again studied in [12] where a characterization of such graphs of girth 6 or more was presented. This idea was further extended in [5] to graphs whose vertices are labeled by the elements of a finite abelian group $A$, where a graph $G$ with a given labeling is called *$A$-well-covered* if maximal independent sets have the same sum of labels in $A$. In [5], Y. Caro, M. Ellingham and J. Ramey give a polynomial time algorithm to decide $A$-well-coveredness provided the maximum degree of $G$ is no more than $c(\log|G|)^{1/3}$. Hence, this also shows, as a special case, that it is polynomial to decide if a graph $G$ is well-covered in the case that the maximum degree is bounded as indicated. Y. Caro [4] further shows, as a corollary of a more general result, that even if one knows that $G$ is a *parity* graph it is CO-NPC to decide whether $G$ is *well-covered.* This question had been posed in [12]. Another direction that has been examined is the following. Let $M(t)$ be the class of graphs having maximal independent sets of exactly $t$ distinct sizes. Then $M(1)$ is the collection of well-covered graphs. In [17], the $M(2)$ graphs of girth 8 or more are characterized. Again in [4] Y. Caro has shown that recognizing membership in $M(t)$ is CO- NPC even in the class of $K_{1,4}$-free graphs.

In still another direction [22], C. Whitehead and the author have examined graphs in which every maximal $k$-packing is of one size. Recall that a set of vertices $P$ is a *$k$-packing* if the distance between any two vertices of $P$ is at least $k+1$. In [22] such graphs, in the case that their girth is at least $4k+4$, are characterized. Note that for $k=1$, these correspond to the well-covered graphs.

N. Dean and J. Zito [9] considered still another generalization of well-covered. In particular, a graph is called *$k$-extendable* if every independent set of vertices of size $k$ can be extended to a maximum independent set. One of a number of interesting results in [9] is a corollary that establishes the existence of a polynomial time algorithm to test perfect graphs of bounded clique size to determine if they are well-covered.

## Conclusions

It would certainly seem that a complete characterization of well-covered graphs, at least in the sense of easy to recognize, is still not on the immediate horizon. However, determining various subclasses, that are interesting in their own right, should still be possible. In addition, the concept of all maximal (or minimal) sets with property $P$ being of one size (such a graph could be called *P-greedy*) may well prove to be a useful approach to a variety of other problems as a simple greedy algorithm would determine the maximum (or minimum) set with property $P$.

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