Orbit Structures of Automorphism Groups of Designs

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ABSTRACT. In the last two decades, mathematicians discuss various transtivities of automorphism groups of designs (i.e. points, blocks and flag transtivities), from all of these study, we know that

$$0 \leq O^{\#}(G, \mathbf{B}) - O^{\#}(G, \mathbf{X}) \leq |\mathbf{B}| - |\mathbf{X}|$$

for $2 - (v, k, \lambda)$ designs. (See [BMP]).

In this paper, we discussed the orbit structure of general combinatorial designs $\mathbf{D}(\mathbf{X}, \mathbf{B})$, obtained the equalities

$$O^{\#}(G, \mathbf{F}) = \sum_{i=1}^{u} O^{\#}(H(x_i), X_i) = \sum_{j=1}^{l} O^{\#}((H(B_j), B_j),$$

where $H(x_i)$ and $H(B_j)$ are the stabilizers of the point x_i and the block B_j respectively, $u = O^{\#}(G, \mathbf{X})$, $l = O^{\#}(G, \mathbf{B})$.

A design $\mathbf{D}(\mathbf{X}, \mathbf{B})$ with parameters $t - (v, k, \lambda)$ is an incidence structure, such that

(1) $\mathbf{X}$ is a v-set,

(2) $\mathbf{B}$ is a collection of k-subsets of $\mathbf{X}$, and

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(3) each t-subset of $\mathbf{X}$ is contained in exactly λ members of $\mathbf{B}$. The elements of $\mathbf{X}$ are called *points*, and the elements of $\mathbf{B}$, *blocks*. We shall assume that all the parameters are positive integers, and that $v > k \geq t$ (to avoid trivial situations). Also, the members of $\mathbf{B}$ must be distinct; in other words, repeated blocks are not allowed.

For each positive integer s, with $0 \leq s < t$, a t-design $\mathbf{D}(\mathbf{X}, \mathbf{B})$ is also an s-design. If the given design has parameters $t-(v, k, \lambda)$, then its parameters as an s-design are $s-(v, k, \lambda_s)$, where

$$\lambda_s = \lambda \frac{(v-s)(v-s-1)\dots(v-t+1)}{(k-s)(k-s-1)\dots(k-t+1)}.$$

In particular, we set $\lambda_0 = b$ and $\lambda_1 = r$. This means that r is the number of blocks containing any given point. Thus we have the general equation

$$(v-i)\lambda_{i+1} = (k-i)\lambda_i \qquad (0 \leq i < t),$$

and the important case of $i = 0$ may be written as

$$vr = bk.$$

(see [HP] and [BW])

We define the *flag set* $\mathbf{F}$ of a design $\mathbf{D}(\mathbf{X}, \mathbf{B})$ as the set of all pairs (x, B), where $x \in \mathbf{X}$, $B \in \mathbf{B}$, and $x \in B$. Clearly, the number of elements of $\mathbf{F}$ is bk or vr.

A design $\mathbf{D}(\mathbf{X}, \mathbf{B})$ with parameters $t-(v, k, \lambda)$ is *symmetric* if it is incomplete (i.e., $b \neq \left[\binom{v}{k}\right]$), $t \geq 2$, and $b = v$.

An *automorphism* of a design $\mathbf{D}(\mathbf{X}, \mathbf{B})$, is a permutation σ of $\mathbf{X}$ such that $B \in \mathbf{B}$ implies that $\sigma(B) \in \mathbf{B}$. Furthermore, according to the definition of flags, for $F = (x, B) \in \mathbf{F}$, $\sigma(F) \in \mathbf{F}$ must also be satisfied. Clearly, the automorphisms of $\mathbf{D}(\mathbf{X}, \mathbf{B})$ form a group which acts of X from the left. Since an automorphism takes blocks to blocks and flags to flags, the group also has a permutation representation on the block set $\mathbf{B}$ and the flag set $\mathbf{F}$. They are denoted by $(G, \mathbf{X})$, $(G, \mathbf{B})$, and $(G, \mathbf{F})$, respectively.

Let $O^{\#}(G, \mathbf{F})$, $O^{\#}(G, \mathbf{X})$, and $O^{\#}(G, \mathbf{B})$ denote the number of orbits of $(G, \mathbf{F})$, $(G, \mathbf{X})$, and $(G, \mathbf{B})$,respectively. For each $x_i \in \mathbf{X}$, we let $X_i = \{B \in \mathbf{B} \mid x_i \in B\}$. Clearly, $|X_i| = r$, independent of the choice of $x_i \in \mathbf{X}$, and $|B| = k$ for all $B \in \mathbf{B}$.

Proposition 1. *If G is point transitive, i.e., (G, X) is transitive, then $O^{\#}(G, \mathbf{F}) \leq r$. Furthermore, $O^{\#}(G, \mathbf{F}) = r$ if and only if $H(x) = H(F)$ for any $x \in \mathbf{X}$ and $F \in \mathbf{F}$.*

Proof: It is easy to compute $|\mathbf{F}| = vr = bk$ and $v = |\mathbf{X}| = [G\colon H(x)]$, for all $x \in \mathbf{X}$ since G is point transitive. Let $F_1, F_2, \dots, F_w$ be a set of representatives of orbits for $(G, \mathbf{F})$ such that $F_j = (x, B_j)$. Hence we obtain

$$H(x) \supseteq H(F_j);$$

in fact, $H(F_j) = H(x) \cap H(B_j)$, for $B_j \in \mathbf{B}$, $j = 1, 2, \ldots, w$.

From the structure of a design, we have

$$\begin{aligned} vr = |\mathbf{F}| &= \sum_{j=1}^{w} [G\colon H(F_j)] = \sum_{j=1}^{w} [G\colon H(x)][H(x)\colon H(F_j)] \\ &= [G\colon H(x)] \sum_{j=1}^{w} [H(x)\colon H(F_j)] = v \sum_{j=1}^{w} [H(x)\colon H(F_j)], \end{aligned}$$

so

$$r = \sum_{j=1}^{w} [H(x)\colon H(F_j)].$$

Therefore, $r \geq O^{\#}(G, \mathbf{F})$ since $[H(x)\colon H(F_j)] \geq 1$ for $j = 1, 2, \ldots, w$. Moreover,

$$\begin{aligned} & r = O^{\#}(G, \mathbf{F}) \\ \Leftrightarrow\ & [H(x)\colon H(F_j)] = 1 \\ \Leftrightarrow\ & H(x) = H(F_j). \end{aligned}$$

□

Corollary. *If G is point regular, then G is flag semiregular and $r = O^{\#}(G, \mathbf{F})$.*

Proof: Since $H(x) = 1$, $H(F) = 1$; and since $H(x) = H(F)$, $r = O^{\#}(G, \mathbf{F})$. □

Similarly, we have the following:

Proposition 1′. *If G is block transitive, then $k \geq O^{\#}(G, \mathbf{F})$. Furthermore, $k = O^{\#}(G, \mathbf{F})$ if and only if $H(B) = H(F)$ for any $B \in \mathbf{B}$ and $F \in \mathbf{F}$.*

Corollary. *If G is block transitive, then G is flag semiregular and $k = O^{\#}(G, \mathbf{F})$.*

Let $x_1, x_2, \ldots, x_u$ be a set of representatives of the orbits for the permutation representation $(G, \mathbf{X})$, $u = O^{\#}(G, \mathbf{X})$, $B_1, B_2, \ldots, B_l$ be a set of representatives of the orbits for $(G, \mathbf{B})$, $l = O^{\#}(G, \mathbf{B})$. The permutation representations $(H(x_i), X_i)$, $i = 1, 2, \ldots, u$, indicates that $H(x_i)$ acts on X_i, which is a subset of $\mathbf{B}$, and $O^{\#}(H(x_i), X_i)$ is the number of orbits of $(H(x_i), X_i)$; similarly, $O^{\#}(H(B_j), B_j)$ is the number of orbits of $(H(B_j), B_j)$, where $H(B_j)$ acts on B_j, a subset of $\mathbf{X}$, for $j = 1, 2, \ldots, 1$. We conclude:

Theorem. (Point, Block and Flag Orbit Theorem)

$$O^{\#}(G,\mathbf{F})=\sum_{i=1}^{u}O^{\#}(H(x_i),X_i)=\sum_{j=1}^{l}O^{\#}(H(B_j),B_j).$$

Proof: We first prove that $O^{\#}(G,\mathbf{F})=\sum_{i=1}^{u}O^{\#}(H(x_i),X_i)$.

Let $t_i=O^{\#}(H(x_i),X_i)$, and let $B_{i1},B_{i2},\ldots,B_{it}$ be a set of representatives of the orbits for $(H(x_i),X_i)$, $i=1,2,\ldots,u$.

We need to prove that the pairs (x_i,B_{ij}), $j=1,2,\ldots,t_i$, $i=1,2,\ldots,u$, is a set of representatives of the orbits for $(G,\mathbf{F})$; in other words, we need to prove that, for any two pairs (x_i,B_{ij}) and $(x_{i'},B_{i'j'})$,

$$\sigma(x_i,B_{ij})\neq(x_{i'},B_{i'j'}),$$

for all $\sigma\in G$. Here $j=1,2,\ldots,t_i$, $j'=1,2,\ldots,t_{i'}$, and i', $i=1,2,\ldots,u$.

Case 1. If $i\neq i'$, clearly $\sigma(x_i,B_{ij})\neq(x_{i'},B_{i'j'})$, for all $\sigma\in G$ since x_i and $x_{i'}$ are not in the same orbit of $(G,\mathbf{X})$.

Case 2. If $i=i'$, and $j\neq j'$, we also have $\sigma(x_i,B_{ij})\neq(x_i,B_{ij'})$, for all $\sigma\in G$. $j,j'=1,2,\ldots,t_i$, and $i=1,2,\ldots,u$. Otherwise, there exists $\sigma\in G$ such that

$$\sigma(x_i,B_{ij})=(x_i,B_{ij'}),$$

which implies that $\sigma(x_i)=x_i$, i.e., $\sigma\in H(x_i)$, and $\sigma B_{ij}=B_{ij'}$. This contradicts the fact that B_{ij} and $B_{ij'}$ are in different orbits of $(H(x_i),X_i)$.

For any flag $F\in\mathbf{F}$, where $F=(x,B)$ and $x\in B$, there is x_i and $\sigma\in G$ such that $\sigma x_i=x$, then

$$F=(x,B)=(\sigma x_i,B)=\sigma(x_i,\sigma^{-1}B).$$

Since $x\in B$, $\sigma^{-1}x\in\sigma^{-1}B$, which implies that $x_i\in\sigma^{-1}B$, and $\sigma^{-1}B\in X_i$. Hence, there exists $\tau\in H(x_i)$ and $B_{ij}\in X_i$ such that

$$\tau\sigma^{-1}B=B_{ij}.$$

Therefore

$$F=\sigma(x_i,\sigma^{-1}B)=\sigma\tau^{-1}(x_i,\tau\sigma^{-1}B)=\sigma\tau^{-1}(x_i,B_{ij}).$$

Moreover,

$$O^{\#}(G,\mathbf{F})=\sum_{i=1}^{u}t_i=\sum_{i=1}^{u}O^{\#}(H(x_i),X_i).$$

Similarly, we take the dual of points and blocks and conclude,

$$O^{\#}(G,\mathbf{F})=\sum_{j=1}^{l}O^{\#}(H(B_j),B_j).$$

□

Corollary 1. *If G is point semiregular, then*

$$O^{\#}(G,\mathbf{F}) = rO^{\#}(G,\mathbf{X}).$$

If G is block semiregular, then

$$O^{\#}(G,\mathbf{F}) = kO^{\#}(G,\mathbf{B}).$$

Proof: It follows from

$$O^{\#}(H(x_i),X_i) = r, \text{ for } i = 1,2,\ldots,u,$$

and

$$O^{\#}(H(B_j),B_j) = k, \text{ for } j = 1,2,\ldots,1.$$

□

Corollary 2. *If G is point transitive, G is flag transitive if and only if $H(x_0)$ is transitive on X_0 for some $x_0 \in \mathbf{X}$. If G is block transitive, G is flag transitive if and only if $H(B_0)$ is transitive on B_0 for some $B_0 \in \mathbf{B}$.*

Proof: Since G is point transitive, $O^{\#}(G,\mathbf{X}) = 1$. G is block transitive, then $O^{\#}(G,\mathbf{B}) = 1$. □

As an example,, Let $\mathbf{D}(\mathbf{X},\mathbf{B})$ be a $2-(3^2,3,1)$ design, where

$$\begin{aligned}
&\mathbf{X} = \{0,1,2,\ldots,8\}, \text{ so } |\mathbf{X}| = 9;\\
&\mathbf{B} = \{B_1 = \{0,1,5\},\ B_5 = \{2,7,8\},\ B_9 = \{3,4,6\},\\
&\qquad B_2 = \{0,2,6\},\ B_6 = \{1,3,8\},\ B_{10} = \{4,5,7\},\\
&\qquad B_3 = \{0,3,7\},\ B_7 = \{1,2,4\},\ B_{11} = \{5,6,8\},\\
&\qquad B_4 = \{0,4,8\},\ B_8 = \{1,6,7\},\ B_{12} = \{2,3,5\}\},
\end{aligned}$$

so $|\mathbf{B}| = 12$;

$$\begin{aligned}
\mathbf{F} = \{&(0,B_1),(1,B_1),(5,B_1),(0,B_2),(2,B_2),(6,B_2),\ldots,\\
&(2,B_{12}),(3,B_{12}),(5,B_{12})\},
\end{aligned}$$

so $|\mathbf{F}| = 3\cdot 12 = 36$.

The automorphism group G of the $2-(3^2,3,1)$ design is

$$\begin{aligned}
(G,\mathbf{X}) = \{&\sigma_0 = \mathbf{I} = (0)(1)\ldots(8),\\
&\sigma_1 = (015)(287)(346),\ \sigma_1^{-1} = (051)(278)(364),\\
&\sigma_2 = (026)(183)(457),\ \sigma_2^{-1} = (062)(138)(475),\\
&\sigma_3 = (037)(142)(568),\ \sigma_3^{-1} = (073)(124)(586),\\
&\sigma_4 = (048)(167)(253),\ \sigma_4^{-1} = (084)(176)(235)\}
\end{aligned}$$

According to Burnside's Lemma,

$$O^{\#}(G,\mathbf{F}) = \frac{1}{|G|}\sum_{\sigma\in G} |\{F \mid \sigma(F) = F \text{ for all } F \text{ in } \mathbf{F}\}| = \frac{1}{9}(36 + 0\cdot 8) = 4,$$

since $|\mathbf{F}| = 36$ and σ_i fixes no flags for $i = 1, \dots, 8$.

Similarly,

$$u = O^{\#}(G,\mathbf{X}) = \frac{1}{|G|}\sum_{\sigma\in G} |\{(x \mid \sigma(x) = x \text{ for all } x \text{ in } \mathbf{X}\}| = \frac{1}{9}(9+0\cdot 8) = 1.$$

since $|\mathbf{X}| = 9$ and σ_i fixes no points for $i = 1, \dots, 8$, so we know that $(G,\mathbf{X})$ is transitive.

$$l = O^{\#}(G,\mathbf{B}) = \frac{1}{|G|}\sum_{\sigma\in G} |\{B \mid \sigma(B) = B \text{ for all } B \text{ in } \mathbf{B}\}| = \frac{1}{9}(12+3\cdot 8) = 4$$

since $|\mathbf{B}| = 12$ and

$$\begin{aligned} \sigma_1 &\text{ fixes blocks } B_1, B_5 \text{ and } B_9,\\ \sigma_2 &\text{ fixes blocks } B_2, B_6 \text{ and } B_{10}, \dots,\\ \sigma_4^{-1} &\text{ fixes blocks } B_4, B_8 \text{ and } B_{12}. \end{aligned}$$

Since $(G,\mathbf{X})$ is transitive, we can choose any point to be a representative of its orbit, without loss of generality, we choose the point $x_1 = 1$, then $H(x_1) = \langle\sigma_0\rangle$, $X_1 = \{B \mid 1 \in B, B \in \mathbf{B}\} = \{B_1, B_6, B_7, B_8\}$. Since σ_0 is the identity element, the number of orbits

$$O^{\#}(H(x_1), X_1) = 4.$$

Since $|X_1| = 4$, we have

$$O^{\#}(G,\mathbf{F}) = \sum_{i=1}^{u} O^{\#}(H(x_i), X_i) = O^{\#}(H(x_1), X_1) = 4.$$

We know that the number of orbits of $(G,\mathbf{B})$ is 4. Next we need to find the set of representative of orbits of $(G,\mathbf{B})$.

Since

$$\begin{aligned} \sigma_1(B_1) &= B_1, & \sigma_1^{-1}(B_1) &= B_1,\\ \sigma_2(B_1) &= B_5, & \sigma_2^{-1}(B_1) &= B_9,\\ \sigma_3(B_1) &= B_9, & \sigma_3^{-1}(B_1) &= B_5,\\ \sigma_4(B_1) &= B_9, & \sigma_4^{-1}(B_1) &= B_5, \end{aligned}$$

it follows that $\{B_1, B_5, B_9\}$ is one orbit of $(G, \mathbf{B})$. Using this method, we get 4 orbits of $(G, \mathbf{B})$. They are

$$U_1 = \{B_1, B_5, B_9\}, \qquad U_2 = \{B_2, B_6, B_{10}\},$$
$$U_3 = \{B_3, B_7, B_{11}\}, \qquad U_4 = \{B_4, B_8, B_{12}\}.$$

We choose $U = \{B_1, B2, B_3, B_4\}$ as the set of representative of orbits for $(G, \mathbf{B})$.

We know that

$$H(B_1) = \langle \sigma_1 \rangle, \qquad H(B_2) = \langle \sigma_2 \rangle,$$
$$H(B_3) = \langle \sigma_3 \rangle, \qquad H(B_4) = \langle \sigma_4 \rangle.$$

Since $H(B_j)$ is transitive on B_j, this implies that $O^{\#}(H(B_j), B_j) = 1$, $j = 1, 2, 3, 4$.

Hence, we have

$$O^{\#}(G, \mathbf{F}) = \sum_{i=1}^{4} O^{\#}(H(B_i), B_i) = 4.$$

□

As an application of the **point, block and flag orbit theorem** to the cohomology of permutation representation on designs, we assume that $(G, \mathbf{X})$ is a permutation representation of a group G, $\mathbf{X}$ is a nonempty set, A is a G-module, and $C^n(\mathbf{X}; G, A)$ is the nth cochain group. We have the well-known theorem of Ernst Snapper (see [S1]):

Theorem. (Snapper)

$$C^{-1}(\mathbf{X}; G, A) = C^0(\mathbf{X}; G, A) \cong A^{H(X_1)} \oplus \ldots \oplus A^{H(X_n)},$$

where $\oplus$ designates the direct sum of Z-modules. If the action of G on A is trivial then

$$C^{-1}(\mathbf{X}; G, A) = C^0(\mathbf{X}; G, A) \cong A \oplus \ldots \oplus A,$$

u times, where $\{x_1, x_2, \ldots, x_u\}$ is a set of representatives of $(G, \mathbf{X})$, and $u = O^{\#}(G, \mathbf{X})$.

According to the theorem, we have

$$C^0(\mathbf{F}; G, A) \cong A^{H(F_1)} \oplus \ldots \oplus A^{H(F_w)},$$
$$C^0(\mathbf{X}; G, A) \cong A^{H(x_1)} \oplus \ldots \oplus A^{H(x_u)},$$

and

$$C^0(\mathbf{B}; G, A) \cong A^{H(B_1)} \oplus \ldots \oplus A^{H(B_l)};$$

where $w = O^{\#}(G, \mathbf{F})$, $u = O^{\#}(G, \mathbf{X})$, and $l = O^{\#}(G, \mathbf{B})$; $\mathbf{X}$, $\mathbf{B}$, and $\mathbf{F}$ are respectively point, block, and flag set of the design $\mathbf{D}(\mathbf{X}, \mathbf{B})$.

When we study the inflation from the cochain group of points $C^0(\mathbf{X}; G, A)$ to the cochain group of flags $C^0(\mathbf{F}; G, A)$, we need to discuss the diagram

$$\begin{array}{ccc} A^{H(F_1)} \oplus \ldots \oplus A^{H(F_w)} & \xrightarrow{p_0} & C^0(\mathbf{F}; G, A) \\ a_0' \uparrow & & \uparrow a_0 \\ A^{H(x_1)} \oplus \ldots \oplus A^{H(x_u)} & \overrightarrow{p_0'} & C^0(\mathbf{X}; G, A) \end{array}$$

Obviously, we need to know the relationship between $w = O^{\#}(G, \mathbf{F})$ and $u = O^{\#}(G, \mathbf{X})$, before we give the definition of the mapping a_0'. This motive forces us to discover the point, block, flag orbit theorem, where we have

$$w = O^{\#}(G, \mathbf{F}) = \sum_{i=1}^{u} O^{\#}(H(x_i), X_i).$$

Based on this result, we define the mapping a_0' as the generalized inclusion mapping such that the above diagram is commutative.

For inflation and deflation between the cohomology group of points $H^n(\mathbf{X}; G, A)$, the cohomology group of blocks $H^n(\mathbf{B}; G, A)$, and the cohomology group of flags $H^n(\mathbf{F}; G, A)$, we have studied all of them in detail. (see [W3]).

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