SEQUENCINGS OF DICYCLIC GROUPS II*

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ABSTRACT: Recent examples of perfect 1-factorizations arising from dicyclic groups have led to the question of whether or not dicyclic groups have symmetric sequencings. For every positive integer $n \geq 2$, there is a dicyclic group of order 4n. It is known that if $n \geq 3$ is odd, then the dicyclic group of order 4n has a symmetric sequencing. In this paper a new proof is given for the odd case; a consequence being that in this situation sequencings abound. A generalization of the original proof is exploited to show that if $n \geq 4$ is even and is not twice an odd number, then the dicyclic group of order 4n has a symmetric sequencing.

1. Introduction. Suppose G is a finite group of order n with identity e. A sequencing of G is an ordering

$$s:\ e, s_2, s_3, \ldots, s_n$$

of all elements of G such that the partial products

$$t:\ e, es_2, es_2s_3, \ldots, es_2s_3\cdots s_n$$

are distinct and hence also all of G. Sequenceable Abelian groups have been characterized [9] as those Abelian groups with a unique element of order 2. Several infinite families of non-Abelian groups have been shown sequenceable [1, 3, 5, 8, 14] including some of odd order. It is known [9] that the non-Abelian groups of orders 6 and 8 are not sequenceable, but in 1983, Keedwell [16] conjectured that all finite non-Abelian groups of order n, $n \geq 10$ are sequenceable. Recently [4, 5] this conjecture has been verified for the 86 non-Abelian groups of order n, $10 \leq n \leq 32$.

Sequencings have arisen in several mathematical and statistical situations [3, 9, 15]. Apart from Keedwell's conjecture, there is interest in finding a certain type of sequencing for dicyclic groups for a reason now to be explained.

* This paper is an expansion of a talk given at the First Vermont Summer Workshop on Combinatorics, University of Vermont, June 1987.

DEFINITION 1. Suppose $n \geq 2$ is a positive integer. The dicyclic group Q_{2n} is the group of order $4n$ defined by

$$Q_{2n} = \{a^i b^j : 0 \leq i \leq 2n-1,\ 0 \leq j \leq 1,\ a^{2n} = e,\ b^2 = a^n,\ ba = a^{2n-1}b\}$$

Reference [18] contains two examples of perfect 1-factorizations of the complete graph K_{14} such that the full symmetry group of the 1-factorization is the dicyclic group Q_6 of order 12 (1-factorizations of K_{2n} are said to be perfect if every 2-factor union of distinct 1-factors is a Hamiltonian circuit of the graph). Relatively little is known about the existence of perfect 1-factorizations on K_{2n} [2, 13, 19]. It turns out that the above mentioned examples can be interpreted in terms of what is called an "even-starter" induced 1-factorization. It would be of considerable interest if either of these examples could be shown to be part of an infinite family of perfect 1-factorizations. As a first step toward attempting to find such a family, it would be useful to know the answer to the

QUESTION. If $n \geq 2$, can one exhibit a 1-factorization of K_{4n+2} whose symmetry group contains Q_{2n}?

This paper settles most of the previously undecided cases in the affirmative.

DEFINITION 2. Suppose G is a group of order $2n$ with identity e and unique element z of order 2. A sequencing $e, s_2, \ldots, s_n, \ldots, s_{2n}$ will be called a symmetric sequencing iff $s_{n+1} = z$ and for $1 \leq i \leq n-1$, $s_{n+1+i} = (s_{n+1-i})^{-1}$.

If G has a symmetric sequencing and z is the unique element of order 2 in G, then z is in the center of G. Thus, symmetric sequencings

$$s:\ e, s_2, \ldots, s_n, z, s_n^{-1}, \ldots, s_3^{-1}, s_2^{-1}$$

have the associated partial product sequence

$$t:\ e, t_2, \ldots, t_n, t_n z, t_{n-1} z, \ldots, t_2 z, z.$$

In this paper, when $G = Z_{2n}$, the cyclic group of order $2n$, symmetric sequencings will be written additively. Thus, a symmetric sequencing of Z_{2n}, would be expressed

$$\begin{aligned} &s:\ 0, s_2, s_3, \ldots, s_n, n, -s_n, -s_{n-1}, \ldots, -s_3, -s_2 \\ (1)\quad &t:\ 0, t_2, t_3, \ldots, t_n, t_n+n, t_{n-1}+n, \ldots, t_2+n, n. \end{aligned}$$

The following should now be clear.

LEMMA 1. If s is a symmetric sequencing of Z_{2n}, then

i) $2 \leq i \leq n$ implies $s_i = -s_{2n+2-i}$,

ii) $1 \leq i \leq n$ implies $t_i = t_{2n+1-i} + n$.

DEFINITION 3. Suppose G is a group of order 2n with identify e and unique element z of order 2. Call

$$E = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_{n-1}, y_{n-1}\}\}$$

a <u>left even starter</u> for G iff

i) every nonidentity element of G except one, denoted m, occurs as an element of some pair of E,

ii) every nonidentity element of G except z occurs in

$$\{x_i^{-1}y_i,\ y_i^{-1}x_i : 1 \leq i \leq n-1\}.$$

If E is a left even starter for G, define $E^* = E \cup \{e, m\}$ and $Q^* = \{\{x, xz\} : x \in G\}$, with the understanding that Q^* contains unordered pairs so that $\{x, xz\} = \{xz, xzz\}$. Think of the elements of G as labeling the complete graph $K_{|G|}$. It is clear that when $m \neq z$, E^* and Q^* are disjoint 1-factors of $K_{|G|}$ and one may consider the 2-factor union of E^* and Q^*. Note that if G is Abelian, the adjective "left" may be omitted.

THEOREM 2.[1]. The group G has symmetric sequencing iff G has a left even starter E such that $E^* \cup Q^*$ is a Hamiltonian circuit of $K_{|G|}$.

It is easy to see [1] that if E is a left even starter for G, then E induces a 1-factorization F(E) on $K_{|G|+2}$. Now, results of Ihrig [11, 12] show that the perfect 1-factorizations of K_{14} mentioned above can be represented as arising via even starters on Q_6. It follows that Q_6 has a symmetric sequencing and hence the basic question of this paper can be rephrased.

QUESTION. If $n \geq 3$, does Q_{2n} have a symmetric sequencing?

It is known that for $n = 2$, Q_{2n} does not have a symmetric sequencing.

One more idea must be introduced.

DEFINITION 4. Suppose H is a finite group of order n with identity e. A <u>2-sequencing</u> of H is an ordering $e, k_2, \ldots, k_n$ of certain elements of H (not necessarily distinct) such that

i) the associated partial products $e, ek_2, ek_2k_3, \ldots, ek_2k_3 \cdots k_n$ are distinct and hence all of H,

ii) if $y \in H$ and $y \neq y^{-1}$, then

$|\{i: 2 \le i \le n \text{ and } (k_i = y \text{ or } k_i = y^{-1})\}| = 2$

iii) if $y \in H$ and $y = y^{-1}$, then

$|\{i: 1 \le i \le n \text{ and } k_i = y\}| = 1.$

It is known that Q_{2n} has a unique element $z = a^n$ of order 2 and that $Q_{2n}/Z_2 \cong D_n$, the dihedral group of order 2n. The following result is verified in [3].

THEOREM 3. Q_{2n} has a symmetric sequencing iff D_n has a 2-sequencing.

Again the basic question can be rephrased.

QUESTION For $n \ge 3$, does D_n have a 2-sequencing?

In [3] it is shown that if $n \ge 3$ is odd, D_n has a 2-sequencing.

It should be noted that 2-sequencings have been studied recently by statisticians [7, 17] who call the notion a "terrace" and use terraces in the design of certain experiments where a given plot can be assumed to be equally affected by all neighboring plots. The problem of sequencing dihedral groups has been considered [4, 8, 10], but only limited results are currently available. It appears to be much easier to find 2-sequencings for dihedral groups.

The basic aim of this paper is to 2-sequence dihedral groups D_n, $n \ge 4$ and n even. It is verified that this would follow from the existence of a certain type of 2-sequencing of Z_n, n odd, by means of what is essentially a single construction and a doubling process. Unfortunately, the best that can be done at present is to show that the required type of 2-sequencing exists on Z_{2n}, n odd.

A companion paper [6] overcomes the difficulties that remain in 2-sequencing D_n, n even, by relaxing the conditions associated with the "certain" type of 2-sequencing of Z_n, n odd, but at the cost of increasing the number of constructions required.

2. The Odd Case Revisited.

As was stated in the introduction, [3] contains a proof of the statement that for $n \ge 3$, n odd, D_n has a 2-sequencing. There is an illuminating alternate argument for this that will now be presented.

LEMMA 4. The following results hold for any dicyclic group Q_{2n} and any integers i and j.

i) $a^j b = b a^{-j}$

ii) $(a^jb)(a^{j+n}b) = e$

iii) $a^ix = a^jb$ implies $x = a^{j-i}b$

$(a^jb)x = a^i$ implies $x = a^{j-i+n}b$

iv) $a^ix = a^j$ implies $x = a^{j-i}$

v) $(a^ib)x = a^jb$ implies $x = a^{i-j}$

PROOF. The computations are straightforward.

DEFINITION 5. If G is a finite group with unique element z of order 2, let $\Sigma(G)$ denote the family of all symmetric sequencings of G and let $\Omega(G)$ denote the associated family of partial product sequences.

By [9], $\Sigma(Z_{2n}) \neq \phi$. As before, elements s of $\Sigma(Z_{2n})$ will be expressed as in (1). Elements C of $\Sigma(Q_{2n})$ and D of $\Omega(Q_n)$ will be listed

C: $e, c_2, \ldots, c_{2n}, \quad z, \quad c_{2n}^{-1}, \ldots, c_3^{-1}, c_2^{-1}$

D: $e, d_2, \ldots, d_{2n}, d_{2n}z, d_{2n-1}z, \ldots, d_2z, \; z$

THEOREM 5. If $n \geq 3$ is odd, then there is a 1-1map $\hat{f}$ from $\Sigma(Z_{2n})$ into $\Sigma(Q_{2n})$ and an associated 1-1 map $\hat{g}$ from $\Omega(Z_{2n})$ into $\Omega(Q_{2n})$.

PROOF. Suppose $s \in \Sigma(Z_{2n})$ with associated partial sum sequence t. Define $\hat{g}(t) = D$ as follows.

$$(2) \qquad d_j = \begin{cases} a^{t_{2i+1}} & , j = 4i+1, \; 0 \leq i \leq n-1 \\ a^{-t_{2i+1}}b & , j = 4i+2, \; 0 \leq i \leq n-1 \\ a^{-t_{2i}}b & , j = 4i-1, \; 1 \leq i \leq n \\ a^{t_{2i}} & , j = 4i, \; 1 \leq i \leq n \end{cases}$$

An example and picture will be useful. Consider the following symmetric sequencing of Z_{10}.

s: 0, 4, 3, 1, 8, 5, 2, 9, 7, 6

t: 0, 4, 7, 8, 6, 1, 3, 2, 9, 5

Lift to $\hat{g}(t) = D$ as in figure 1.

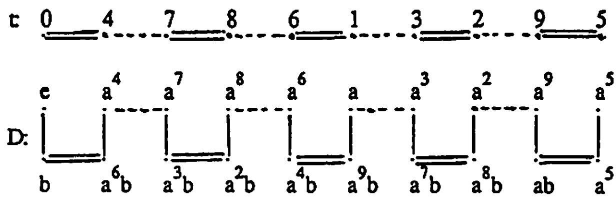

Figure 1

It is clear from (2) that D defines a Hamiltonian path through Q_{2n}. It is also clear that if $s_1, s_2, \varepsilon\ \Sigma(Z_{2n})$ such that $s_1 \neq s_2$, then $\hat{g}(t_1) \neq \hat{g}(t_2)$ and $\hat{f}(s_1) \neq \hat{f}(s_2)$.
Thus, if $\hat{f}$: $\Sigma(Z_{2n}) \rightarrow \Sigma(Q_{2n})$, it is 1-1 and so is $\hat{g}$.

Interest now centers on $C = \hat{f}(s)$. It can be verified algebraically that C has the right properties. This will be done first and then a more intuitive argument will be given.

It is easy to conclude that given D of (2), the associated C has the following properties.

(3)

i) $0 \leq i \leq n-1$ implies $c_{4i+1} = a^{s_{2i+1}}$

ii) $0 \leq i \leq n-1$ implies $c_{4i+3} = a^{s_{2i+2}}$

iii) $0 \leq i \leq n-1$ implies $c_{4i+2} = a^{-2t_{2i+1}}b$

iv) $1 \leq i \leq n$ implies $c_{4i} = a^{-2t_{2i}+n}b$

Claim 1. C contains all elements of Q_{2n}.

It is clear from (i) and (ii) of (3) that C contains all elements of the form a^j. Now consider the elements that arise via parts (iii) and (iv) of (3). Define

$$F_n = \{\{x, x+n\}: x \in Z_{2n}\}.$$

$$A_1 = \{t_{2i+1}: 0 \leq i \leq n-1\}$$

$$A_2 = \{t_{2i}: 1 \leq i \leq n\}$$

By the properties of t, A_1 and A_2 are choice functions on F_n such that A_1 chooses x iff A_2 chooses $x + n$. Since $2x = 2(x+n)$ in Z_{2n}, $0 \leq i \leq n-1$ implies

$$2i \in 2A_1 \cap 2A_2.$$

Thus $2A_1 = 2A_2 = \{2i: 0 \leq i \leq n-1\} = -2A_1 = -2A_2$. This means

$$-2A_2 + n = \{2i+1: 0 \leq i \leq n-1\}$$

and C contains all elements of the form a^jb. Note how this part of the argument fails if n is even.

Claim 2. i) $c_{2n+1} = z$,

ii) if $1 \le i \le 2n - 1$, then $c_{2n+1+i} = (c_{2n+1-i})^{-1}$.

If n is odd, then $2n + 1 \equiv 3 \pmod 4$ and if $4i + 3 = 2n + 1$, then $i = (n - 1)/2$. Therefore by (3)

$$c_{2n+1} = a^{Sn+1} = z.$$

In order to verify (ii), consider the pairs of positive integers k, j such that $k + j = 4n + 2$. The argument breaks into several cases depending on the congruence classes mod 4 of k and j . A single case will give the flavor of all possibilities.

Suppose $4n + 2 = (4i + 2) + 4(n - i)$. Then by (3)

$$c_{4i+2} = a^{-2t_{2i+1}}b$$

and $c_{4(n-i)} = a^{-2t_{2(n-i)}+n}b$.

Since $2(n-i) + (2i + 1) = 2n + 1$, Lemma 1 implies

$$t_{2(n-i)} = t_{2i+1} + n$$

and thus

$$c_{4(n-i)} = a^{-2t_{2i+1}+n}b = (c_{4i+2})^{-1}.$$

This completes the argument for Theorem 5.

A more intuitive argument might proceed as follows (with several glances at figure 1). By Lemma 4

$$a^i x = a^j \text{ implies } x = a^{j-i}$$

$$(a^{-i}b)x = a^{-j}b \text{ implies } x = a^{-i+j}.$$

Thus the elements of s become the exponents of "a" in alternate positions of C. Clearly all a^k appear and the order of appearance is the same as that of the elements k in s. It follows that $c_{2n+1} = z$ and for j even, $c_{2n+1+j} = (c_{2n+1-j})^{-1}$.

Now if t_i is in position i of t, then $t_i + n$ is in position $2n + 1 - i$. Since

$$[(2n + 1) - i] - i = 2(n - i) + 1$$

there are an odd number of positions encountered as one moves from position i to position $(2n + 1) - i$. What elements of C arise as one "crosses from one side to the other" (see figure 1) at these positions?

Again by Lemma 4,

$$a^{t_i}x = a^{-t_i}b \quad \text{implies} \quad x = a^{-2t_i}b$$
$$(a^{-t_i}b)x = a^{t_i} \quad \text{implies} \quad x = a^{-2t_i+n}b$$

$$a^{t_i+n}x = a^{(-t_i+n)}b \quad \text{implies} \quad x = a^{-2t_i}b$$
$$(a^{-(t_i+n)}b)x = a^{t_i+n} \quad \text{implies} \quad x = a^{-2t_i+n}b$$

Thus position i and position $(2n + 1) - i$ yield the same pair of inverse solutions with the same orientation. The odd number of positions from i to $(2n + 1) - i$ means that the crossings at these positions are in different directions so that both possibilities arise in C. The fact that n is odd insures (as argued previously) that all a^ib arise and it is not hard to see that the inverse a^ib pairs occur in symmetric positions.

DEFINITION 6. If G is a finite group, let $\sigma_2(G)$ denote the family of all 2-sequencings of G and let $\omega_2(G)$ denote the associated family of partial product sequences.

There is a result analogous to Theorem 5 connecting 2-sequencings of Z_n and D_n when n is odd.

DEFINITION 7. Suppose $n \geq 2$ is a positive integer. The <u>dihedral group</u> D_n is the group of order 2n defined by

$$D_n = \{a^ib^j : 0 \leq i \leq n-1,\ 0 \leq j \leq 1,\ a^n = e,\ b^2 = e,\ ba = a^{n-1}b\}.$$

LEMMA 6. The following results hold for any dihedral group and any integers i and j.

i) $a^jb = ba^{-j}$

ii) $(a^jb)(a^jb) = e$

iii) $a^ix = a^jb$ implies $x = a^{j-i}b$

$(a^jb)x = a^i$ implies $x = a^{j-i}b$

iv) $a^ix = a^j$ implies $x = a^{j-i}$

v) $(a^ib)x = a^jb$ implies $x = a^{i-j}$

PROOF. The computations are straightforward.

THEOREM 7. If $n \geq 3$ is odd, then there is a 1-1 map f from $\sigma_2(Z_n)$ into $\sigma_2(D_n)$ and an associated 1-1 map g from $\omega_2(Z_n)$ into $\omega_2(D_n)$.

PROOF. The argument is very similar to that given for Theorem 5. Suppose $s \in \sigma_2(Z_n)$ with associated partial sum sequence t. Define $g(t) = D$ as follows.

$$(4) \qquad d_j = \begin{cases} a^{t_{2i+1}} & , j = 4i + 1 \quad , 0 \le i \le (n-1)/2 \\ a^{-t_{2i+1}}b & , j = 4i + 2 \quad , 0 \le i \le (n-1)/2 \\ a^{-t_{2i}}b & , j = 4i - 1 \quad , 1 \le i \le (n-1)/2 \\ a^{t_{2i}} & , j = 4i \quad , 1 \le i \le (n-1)/2 \end{cases}$$

Again, it is clear from (4) that D defines a Hamiltonian path through D_n and that f and g are 1-1. Computation shows that $C = f(s)$ has the following properties.

$$(5) \qquad \begin{array}{l} \text{i) } 0 \le i \le (n-1)/2 \text{ implies } c_{4i+1} = a^{s_{2i+1}} \\ \text{ii) } 0 \le i \le (n-3)/2 \text{ implies } c_{4i+3} = a^{s_{2i+2}} \\ \text{iii) } 0 \le i \le (n-1)/2 \text{ implies } c_{4i+2} = a^{-2t_{2i+1}}b \\ \text{iv) } 1 \le i \le (n-1)/2 \text{ implies } c_{4i} = a^{-2t_{2i}}b \end{array}$$

This time it is only necessary to show that C is a 2-sequencing. Parts (i) and (ii) of (5) show that the elements of form a^i in C don't violate the requirements for a 2-sequencing of D_n. Since n is odd, (iii) and (iv) of (5) give all elements in D_n of the form a^ib and the result follows.

The last part of the above argument fails if n is even because $-2Z_n \ne Z_n$ in that case.

It is now easy to see that when $n \ge 3$ is odd, the following diagrams commute (in all cases, π is the natural projection of the "first half" of the appropriate ordered sets).

$$\begin{array}{ccc} \Omega(Z_{2n}) & \xrightarrow{\hat{g}} & \Omega(Q_{2n}) \\ \pi \downarrow & & \downarrow \pi \\ \omega_2(Z_n) & \xrightarrow{g} & \omega_2(D_n) \end{array} \qquad \begin{array}{ccc} \Sigma(Z_{2n}) & \xrightarrow{\hat{f}} & \Sigma(Q_{2n}) \\ \pi \downarrow & & \downarrow \pi \\ \sigma_2(Z_n) & \xrightarrow{f} & \sigma_2(D_n) \end{array}$$

As noted previously, $\Sigma(Z_{2n}) \ne \phi$, so if $n \ge 3$ is odd, it is immediate that $\sigma_2(D_n) \ne \phi$. Available data suggest that $|\Sigma(Z_{2n})|$ increases rapidly with n.

Sequencings of dihedral groups have been hard to find and the above result is no help in finding any.

THEOREM 8. If $n \ge 3$ is odd, then $f(\sigma_2(Z_{2n}))$ contains no sequencings of D_n.

PROOF. Suppose $s \in \sigma_2(Z_{2n})$ and $f(s) = C \in \sigma_2(D_n)$.

By the definition of the embedding process,

$$c_1, c_3, c_5, \ldots, c_{2n-1} = a^{s_1}, a^{s_2}, a^{s_3}, \ldots, a^{s_n}.$$

Thus, if f(s) is a sequencing of D_n, then s must be a sequencing of Z_n and this is impossible for n odd.

It will be instructive to consider another embedding process that generalizes the original proof of existence of 2-sequencings of D_n, n odd. This is an idea that can be successfully modified to handle the even case.

DEFINITION 8. Suppose G is a group of odd order and PS = $\{\{x, -x\}: x \neq 0\}$. Let $\sigma_2^*(G) \subset \sigma_2(G)$ denote the family of all 2-sequencings s of G such that both

$$\{s_2, s_4, \ldots, s_{n-1}\} \text{ and } \{s_3, s_5, \ldots, s_n\}$$

can be viewed as choice functions on the pairs of PS. Elements of $\sigma_2^*(G)$ will be called starter-translate 2-sequencings. Let $\omega_2^*(G)$ denote the associated family of partial product sequences.

THEOREM 9. If $n \geq 3$ is odd, then there is a 1-1 map ϕ from $\sigma_2^*(Z_n)$ into $\sigma_2(D_n)$ and an associated 1-1 map θ from $\omega_2^*(Z_n)$ into $\omega_2(D_n)$.

PROOF. Suppose s ε $\sigma_2^*(Z_n)$ with associated partial sum sequence t. Define $\theta(t) = D$ as follows.

$$(6) \qquad d_j = \begin{cases} e & , j = 1 \\ b & , j = 2 \\ a^{t_{2i}}b & , j = 4i - 1 \qquad , 1 \leq i \leq (n-1)/2 \\ a^{t_{2i+1}} & , j = 4i \qquad , 1 \leq i \leq (n-1)/2 \\ a^{t_{2i}} & , j = 4i + 1 \qquad , 1 \leq i \leq (n-1)/2 \\ a^{t_{2i+1}}b & , j = 4i + 2 \qquad , 1 \leq i \leq (n-1)/2 \end{cases}$$

As before, it is not hard to see that D defines a Hamiltonian path through D_n and that θ and ϕ are 1-1.

The following facts can be used to verify that $C = \phi(s)$ ε $\sigma_2(D_n)$.

$$(7) \qquad \begin{array}{llll} \text{i)} & (a^{t_{2i}}b)x & = a^{t_{2i+1}} & \text{implies } x = a^{-s_{2i+1}}b \\ \text{ii)} & (a^{t_{2i}})x & = a^{t_{2i+1}}b & \text{implies } x = a^{s_{2i+1}}b \\ \text{iii)} & (a^{t_{2i+1}}b)x & = a^{t_{2i+2}}b & \text{implies } x = a^{-s_{2i+2}} \\ \text{iv)} & (a^{t_{2i+1}})x & = a^{t_{2i}} & \text{implies } x = a^{-s_{2i+1}} \end{array}$$

An example can be constructed using the following member of $\omega_2^*(Z_{11})$.

$$t: \ 0, 2, 4, 10, 9, 5, 8, 7, 3, 6, 1$$

Once again, this method does not yield any sequencings. This follows from (iii) and (iv) of (7) and the fact that for n odd, Z_n is not sequenceable.

THEOREM 10. If $n \geq 3$ is odd, then $\phi\left(\sigma_2^* (Z_n)\right)$ contains no sequencings of D_n.

3. Sequencings of Certain Dihedral Groups. The construction to be described depends on Theorem 2 and basic properties of dihedral groups.

It is important to have more detailed information about the left even starter mentioned in Theorem 2. This information, taken from [1], is given below in the form in which it is used in this paper.

LEMMA 11. Suppose s is a symmetric sequencing of Z_{2n} as in (1). If n is odd, then

i) $E = \{\{t_2, t_3\}, \ldots, \{t_{n-1}, t_n\}, \{t_{n+2}, t_{n+3}\}, \ldots, \{t_{2n-1}, t_{2n}\}\}$

is an even starter for Z_{2n} (note that $t_1 = 0$ and t_{n+1} are not in any pair of E),

ii) $E + n = \{\{0, t_2\}, \ldots, \{t_{n-2}, t_{n-1}\}, \{t_{n+1}, t_{n+2}\}, \ldots, \{t_{2n-2}, t_{2n-1}\}\}$

is a translate of E.

If n is even, then

iii) $E = \{\{t_2, t_3\}, \ldots, \{t_{n-2}, t_{n-1}\}, \{t_{n+1}, t_{n+2}\}, \ldots, \{t_{2n-1}, t_{2n}\}\}$

is an even starter for Z_{2n} (note that $t_1 = 0$ and t_n are not in any pair of E),

iv) $E + n = \{\{0, t_2\}, \ldots, \{t_{n-1}, t_n\}, \{t_{n+2}, t_{n+3}\}, \ldots, \{t_{2n-2}, t_{2n-1}\}\}$

is a translate of E.

One additional property will be required of the sequencings used in the construction.

DEFINITION 9. If s is either a 2-sequencing of Z_n or a symmetric sequencing of Z_{2n}, then s will be called special iff there is a $y \neq 0$ such that s and the associated partial sum sequence t begin as follows:

$$s_1, s_2, s_3 = 0, 2y, -y$$

$$t_1, t_2, t_3 = 0, 2y, y.$$

THEOREM 12. If $n \geq 3$ and Z_{2n} has a special symmetric sequencing, then D_{2n} has a 2-sequencing.

PROOF. The argument divides into two similar cases. Suppose first that $n \geq 3$ is odd and s, written as in (1), is a special symmetric sequencing of Z_{2n}. The aim is to define a 2-sequencing

$$C:\ e, c_2, c_3, \ldots, c_{2n}, \ldots, c_{4n}$$

of D_{2n} with associated partial product sequence

$$D:\ e, d_2, d_3, \ldots, d_{2n}, \ldots, d_{4n}.$$

Define D as follows.

$$d_j = \begin{cases} e & , j = 1 \\ a^{t_{2n-2i}} & , j = 4i+2 \quad , 0 \le i \le (n-1)/2 \\ a^{t_{2n-2i-1}}b & , j = 4i+3 \quad , 0 \le i \le (n-3)/2 \\ a^{t_{2n-2i+2}}b & , j = 4i \quad , 1 \le i \le (n-1)/2 \\ a^{t_{2n-2i+1}} & , j = 4i+1 \quad , 1 \le i \le (n-1)/2 \\ a^{t_{n+1}}b & , j = 2n+1 \\ a^{t_{2n-2i+1}} & , j = 4i \quad , (n+1)/2 \le i \le n-1 \\ a^{t_{2n-2i}}b & , j = 4i+1 \quad , (n+1)/2 \le i \le n-1 \\ a^{t_{2n-2i+1}}b & , j = 4i+2 \quad , (n+1)/2 \le i \le n-2 \\ a^{t_{2n-2i}} & , j = 4i+3 \quad , (n+1)/2 \le i \le n-2 \\ b & , j = 4n-2 \\ a^{t_3}b & , j = 4n-1 \\ a^{t_2} & , j = 4n \end{cases}$$

It can be shown that the associated C is a 2-sequencing of D_{2n}, but it is much more enlightening to do things another way.

We wish to build a Hamiltonian path D through D_{2n} and want the associated C to be a 2-sequencing. Since s is a symmetric sequencing of Z_{2n},

$$E = \{\{t_2, t_3\}, \ldots, \{t_{n-1}, t_n\}, \{t_{n+2}, t_{n+3}\}, \ldots, \{t_{2n-1}, t_{2n}\}\}$$

is an even starter for Z_{2n}.

Step 1. For each pair $\{t_i, t_{i+1}\}$ of E, construct two edges $\{a^{t_i}, a^{t_{i+1}}b\}$ and $\{a^{t_i}b, a^{t_{i+1}}\}$. If an edge is part of D, then "travelling the edge" corresponds to defining an element x of C. By Lemma 6

$$a^{t_i}x = a^{t_{i+1}}b \quad \text{implies} \quad x = a^{t_{i+1}-t_i}b$$

$$(a^{t_i}b)x = a^{t_{i+1}} \quad \text{implies} \quad x = a^{t_i-t_{i+1}}b.$$

The differences of the pairs of E are the exponents of a. Since it is immaterial which way one travels these edges (Lemma 6 (iii)), if they are part of D, then C will contain all a^ib except possibly $a^0b = b$ and a^nb. This step and succeeding ones are shown graphically in Figure 2.

Step 2. For each pair $\{t_i, t_{i+1}\}$ of E, construct the edge $\{a^{t_i}b, a^{t_{i+1}}b\}$. By Lemma 6

$$(a^{t_i}b)x = a^{t_{i+1}}b \text{ implies } x = a^{t_i - t_{i+1}}$$

$$(a^{t_{i+1}}b)x = a^{t_i}b \text{ implies } x = a^{t_{i+1} - t_i}.$$

If PS = $\{\{1, 2n-1\}, \{2, 2n-2\}, \ldots, \{n-1, n+1\}\}$ and the edges just defined are part of D, then any orientation of them will force the associated elements $x = a^j$ of C to have the property that the exponents will be a choice function on the pairs of PS.

Step 3. Replace $\{a^{t_2}b, a^{t_3}b\}$ by $\{b, a^{t_3}b\}$. Since s is special,

$$\pm(t_2 - t_3) = \pm(t_3 - 0)$$

and the exponents are still a choice function on the pairs of PS.

Step 4. For each pair $\{t_i, t_{i+1}\}$ of E + n, construct the edge $\{a^{t_i}, a^{t_{i+1}}\}$. Again the associated elements $x = a^j$ of C will be such that the exponents are a choice function on the pairs of PS.

Step 5. Replace $\{e = a^0, a^{t_2}\}$ by $\{b, a^{t_2}b\}$.

The elements a^n, b and a^nb still must be added to C.

Step 6. Add edges $\{e, a^{t_{2n}}\}$, $\{a^{t_{n+1}}, a^{t_{n+1}}b\}$ and $\{a^{t_{n+1}}b, a^{t_n}\}$.

It is easy to see that all edges defined give a Hamiltonian path D and that C is a 2-sequencing of D_{2n}.

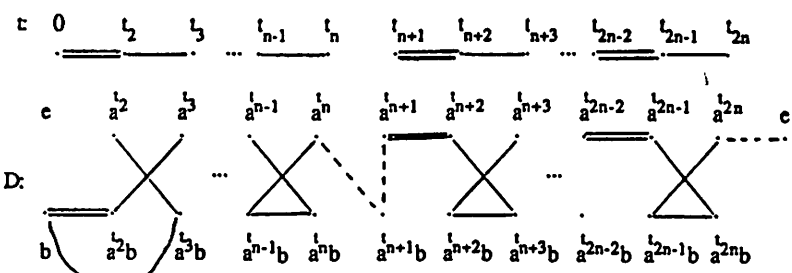

Figure 2

The solid lines in the t-row of figure 2 are the pairs of E. These go to the pairs of D_{2n} defined in steps 1, 2 and 3. The double lines in the t-row are the pairs of E + n. They go to the pairs of D_{2n} defined in steps 4 and 5. The dotted lines in the D part of figure 2 are the three edges defined in step 6.

A specific example may be constructed from the following t on Z_{10}.

$$t:\ 0, 4, 2, 3, 6, 1, 8, 7, 9, 5$$

Suppose now that $n \geq 6$ is even (it is not hard to see - use Theorem 13 - that Z_8 does not have a special symmetric sequencing) and s is a special symmetric sequencing of Z_{2n}. The construction is analogous to the odd case so the definitions can be given without explanation. Define D as follows.

$$d_j = \begin{cases} e & , j = 1 \\ a^{t_{2n-2i}} & , j = 4i + 2 \quad , 0 \leq i \leq (n-2)/2 \\ a^{t_{2n-2i-1}}b & , j = 4i + 3 \quad , 0 \leq i \leq (n-2)/2 \\ a^{t_{2n-2i+2}}b & , j = 4i \quad , 1 \leq i \leq n/2 \\ a^{t_{2n-2i+1}} & , j = 4i + 1 \quad , 1 \leq i \leq n/2 \\ a^{t_n}b & , j = 2n + 2 \\ a^{t_{2n-2i}} & , j = 4i + 3 \quad , n/2 \leq i \leq n - 2 \\ a^{t_{2n-2i+1}} & , j = 4i \quad , (n+2)/2 \leq i \leq n - 1 \\ a^{t_{2n-2i}}b & , j = 4i + 1 \quad , (n+2)/2 \leq i \leq n - 1 \\ a^{t_{2n-2i+1}}b & , j = 4i + 2 \quad , (n+2)/2 \leq i \leq n - 2 \\ b & , j = 4n - 2 \\ a^{t_3}b & , j = 4n - 1 \\ a^{t_2} & , j = 4n \end{cases}$$

As before, the proof will be outlined by considering various parts of the construction. This time

$$E = \{\{t_2, t_3\}, \ldots, \{t_{n-2}, t_{n-1}\}, \{t_{n+1}, t_{n+2}\}, \ldots, \{t_{2n-1}, t_{2n}\}\}$$

is an even starter for Z_{2n}.

Step 1. For each pair $\{t_j, t_{j+1}\}$ of E, construct two edges $\{a^{t_j}, a^{t_{j+1}}b\}$ and $\{a^{t_j}b, a^{t_{j+1}}\}$.

Step 2. For each pair $\{t_i, t_{i+1}\}$ of E, construct the edge $\{a^{t_i}b, a^{t_{i+1}}b\}$.

Step 3. Replace $\{a^{t_2}b, a^{t_3}b\}$ by $\{b, a^{t_3}b\}$. Again, this is where the special property is used.

Step 4. For each pair $\{t_i, t_{i+1}\}$ of E + n, construct the edge $\{a^{t_i}, a^{t_{i+1}}\}$.

Step 5. Replace $\{e = a^0, a^{t_2}\}$ by $\{b, a^{t_2}b\}$.

Note that, except for the fact that the collections of pairs $\{t_i, t_{i+1}\}$ are not the same in the two cases, the first five steps are identical. The last steps are not the same.

Step 6. Add edges $\{e, a^{t_{2n}}\}$, $\{a^{t_{n+1}}, a^{t_n}b\}$ and $\{a^{t_n}b, a^{t_n}\}$.

As in the odd case, the edges define a Hamiltonian path D through D_{2n} and the associated C is a 2-sequencing. Figure 3 is the analogue of figure 2 with the single lines, double lines and dotted lines playing the same roles.

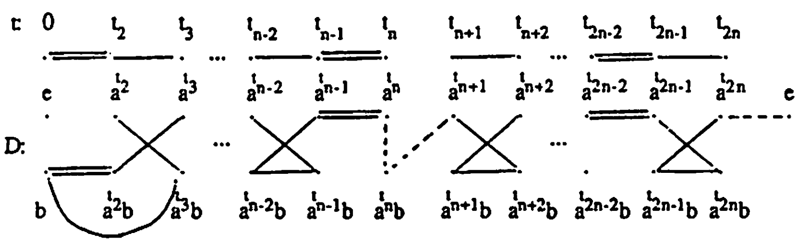

Figure 3

A specific example may be constructed from the following t on Z_{12}.

$$t:\ 0, 2, 1, 4, 11, 3, 9, 5, 10, 7, 8, 6$$

In order to apply Theorem 12, it will be necessary to gather information about special symmetric sequencings.

DEFINITION 10. Suppose G is a finite group of order 2n with identity e and unique element z of order 2. Let π: G $\rightarrow$ G/Z_2 be the natural projection and suppose

$$s: k_1, k_2, \ldots, k_n$$

is a 2-sequencing of G/Z_2. Then

$$L:\ e, l_2, l_3, \ldots, l_n$$

is a <u>lifting</u> of s iff for each i, $2 \le i \le n$, $l_i \in G$ and $\pi(l_i) = k_i$.

THEOREM 13. For $n \ge 3$, Z_n has a special 2-sequencing iff Z_{2n} has a special symmetric sequencing.

PROOF. Suppose Z_{2n} has a special symmetric sequencing s. By [3, Theorem 2],

$$\pi(s_1), \pi(s_2), \ldots, \pi(s_n)$$

is a 2-sequencing of G/Z_2. Since π is a homomorphism and s is special, the 2-sequencing is also special.

Suppose, conversely, that Z_n has a special 2-sequencing

$$\text{(8)} \qquad \begin{aligned} s&: \ 0, 2y, -y, \ldots, s_n \\ t&: \ 0, 2y, \ y, \ldots, t_n. \end{aligned}$$

Represent y as a coset of $\{0,n\}$ in Z_{2n} so that $y = \{x, x+n\}, x \in Z_{2n}$ and $2y = \{2x, 2x+n\}$. The problem is to show that one can always lift to a special symmetric sequencing of Z_{2n}. Clearly there are four possible ways to lift in the first three positions.

$$\text{(a)} \begin{aligned} s&: 0, 2x, -x \\ t&: 0, 2x, x \end{aligned}, \quad \text{(b)} \begin{aligned} s&: 0, 2x, -x+n \\ t&: 0, 2x, x+n \end{aligned}, \quad \text{(c)} \begin{aligned} s&: 0, 2x+n, -x \\ t&: 0, 2x+n, x+n \end{aligned}, \quad \text{(d)} \begin{aligned} s&: 0, 2x+n, -x+n \\ t&: 0, 2x+n, x. \end{aligned}$$

Of these, (a) and (b) have the right properties to be special but (c) and (d) do not.

There are also restrictions when lifting to a symmetric sequencing. These are listed in the proof of Theorem 4 of [3] and repeated here.

Case 1. $v \neq -v$

i) If v (-v similar) appears twice in s, then the two occurrences of $v = \{x, x+n\}$ must be lifted to x and x + n.

ii) If v and -v each appear once in s, then each lift of v forces a unique lift for -v. In particular, if v is lifted to x, then -v must be lifted to -x + n and if v is lifted to x + n, then -v must be lifted to -x.

Case 2. $v = -v, v \neq 0$

In this case v occurs only once in s and can be lifted to either x or x + n.

Now consider (8) from the standpoint of wanting to lift to a symmetric sequencing.

Case A. Either 2y or -y is a non-zero element of order 2 in Z_n (it is easy to see that both can't be of order 2). One can use either (a) or (b) and extend to a lifting of s that will generate a symmetric sequencing.

Case B. Neither $2y$ nor $-y$ has order 2 in Z_n.

i) $2y \neq -y$ and $-2y \neq -y$ in Z_n

Either (a) or (b) can be extended.

ii) $-2y = -y$

This implies $y = 0$, a contradiction.

iii) $2y = -y$ in Z_n.

If $2x = -x$ in Z_{2n}, then (b) can be extended.

If $2x = -x + n$ in Z_{2n}, then (a) can be extended.

The result follows.

COROLLARY 14. If Z_n has a special 2-sequencing and $m \geq 1$, then $D_{2^m n}$ has a 2-sequencing.

PROOF. This follows from Theorems 12 and 13 and the obvious fact that a special symmetric sequencing is also a special 2-sequencing.

COROLLARY 15. If $m \geq 2$, then D_{2^m} is 2-sequenceable.

PROOF. It is known [3] that D_4 and D_8 are 2-sequenceable. It is easy to see that Z_4 does not have a special 2-sequencing, but Z_8 does have one as follows.

s: 0, 2, 3, 1, 1, 4, 6, 3

t: 0, 2, 5, 6, 7, 3, 1, 4

The result follows by Corollary 14.

Since, as has been mentioned, it is known that for $n \geq 3$, n odd, D_n is 2-sequenceable, the entire problem could be settled affirmatively if the following conjecture is true.

CONJECTURE. If $n \geq 3$, n odd, then Z_n has a special 2-sequencing.

Evidence for this conjecture will be presented later when examples of special 2-sequencings will be given for Z_n, n odd and $3 \leq n \leq 25$. Although the conjecture has not yet been verified, there is a very general result available by means of a single construction. This result (Theorem 18 below) says that "twice" the conjecture is true.

DEFINITION 11. If $n \geq 5$ is odd, then E_{2n} is a collection of pairs in Z_{2n} as follows.

$$\{1, 2\}$$

$$\{n, 2n-1\}, \{n+1, 2n-2\}, \ldots, \{n+(n-3)/2, 2n-(n-1)/2\}$$

$$\{(n+3)/2, 2n-(n+1)/2\}$$

and if $n \geq 7$,

$$\{3, n-1\}, \{4, n-2\}, \ldots, \{(n-1)/2, (n+5)/2\}.$$

LEMMA 16. If $n \geq 5$ is odd, then E_{2n} is an even starter for Z_{2n} such that $\{1, 2\} \in E_{2n}$ and $m = (n+1)/2$.

PROOF. This is straightforward.

DEFINITION 12. If $n \geq 5$ is odd, then H_{2n} is a collection of pairs in Z_{2n} as follows.

$$\{0, 2\}, \{1, n\}$$

$$\{n+1, 2n-1\}, \{n+2, 2n-2\}, \ldots, \{n+(n-3)/2, 2n-(n-3)/2\}$$

$$\{(n+1)/2, 2n-(n-1)/2\}, \{(n+1)/2, (n+3)/2\}$$

and if $n \geq 7$,

$$\{(n+5)/2, 2n-(n+1)/2\}$$

and if $n \geq 9$

$$\{(n-1)/2, (n-1)/2+4\}, \{(n-3)/2, (n-3)/2+6\}, \ldots, \{4, n-1\}.$$

LEMMA 17. If $n \geq 5$ is odd, then H_{2n} has the following properties.

i) H_{2n} has n pairs containing $2n-1$ elements,

ii) one element, $(n+1)/2$ is in two pairs,

one element, (7 if $n = 5$, 3 if $n \geq 7$) is in no pair,

iii) all non-zero elements of Z_{2n} appear as a difference of some pair of H_{2n};

n appears twice in this way,

iv) The pairs of H_{2n} and E_{2n} together ($H_{2n} \cup E_{2n}$) form a Hamiltonian path through Z_{2n} that begins 0, 2, 1.

PROOF. The computations are straightforward.

THEOREM 18. If $n \geq 5$ is odd, then Z_{2n} has a special 2-sequencing.

PROOF. It is clear that $H_{2n} \cup E_{2n}$ is the partial sum sequence associated with a special 2-sequencing of Z_{2n}.

Note that the special 2-sequencing described in Theorem 18 is not a symmetric sequencing.

COROLLARY 19. If $n \geq 5$ is odd, and $m \geq 2$, then $D_{2^m n}$ has a 2-sequencing.

PROOF. This is an immediate consequence of Theorem 18 and Corollary 14.

LEMMA 20. If $n = 3$ and $m \geq 1$, then $D_{2^m 3}$ has a 2-sequencing.

PROOF. This also follows from Corollary 14 since

$$s:\ 0, 2, 2$$

$$t:\ 0, 2, 1$$

is a special 2-sequencing for Z_3.

The only cases still in doubt are those of type D_{2n}, n odd, $n \geq 5$. In [4] a hill-climbing algorithm is used to generate sequencings (hence 2-sequencings) of D_n, $5 \leq n \leq 50$.

The following special 2-sequencings were found by various methods. No general construction has yet been discovered for Z_n, n odd.

Z_5 s: 0, 2, 4, 3, 4
t: 0, 2, 1, 4, 3

Z_7 s: 0, 2, 6, 3, 1, 5, 3
t: 0, 2, 1, 4, 5, 3, 6

Z_9 s: 0, 2, 8, 3, 4, 4, 3, 8, 2
t: 0, 2, 1, 4, 8, 3, 6, 5, 7

Z_{11} s: 0, 2, 10, 8, 6, 6, 4, 4, 10, 2, 8
t: 0, 2, 1, 9, 4, 10, 3, 7, 6, 8, 5

Z_{13} s: 0, 2, 12, 3, 7, 7, 5, 9, 3, 11, 1, 8, 9
t: 0, 2, 1, 4, 11, 5, 10, 6, 9, 7, 8, 3, 12

Z_{15} s: 0, 2, 14, 3, 10, 4, 6, 14, 2, 12, 4, 10, 6, 8, 8
t: 0, 2, 1, 4, 14, 3, 9, 8, 10, 7, 11, 6, 12, 5, 13

Z_{17} s: 0, 2, 16, 14, 6, 12, 4, 7, 16, 2, 14, 4, 12, 6, 10, 8, 8
t: 0, 2, 1, 15, 4, 16, 3, 10, 9, 11, 8, 12, 7, 13, 6, 14, 5

Z_{19} s: 0, 2, 18, 16, 17, 11, 7, 14, 9, 12, 11, 10, 3, 13, 14, 1, 6, 15, 15
t: 0, 2, 1, 17, 15, 7, 14, 9, 18, 11, 3, 13, 16, 10, 5, 6, 12, 8, 4

Z_{21} s: 0, 2, 20, 6, 6, 13, 11, 1, 14, 10, 5, 7, 8, 16, 4, 12, 3, 17, 19, 18, 12
t: 0, 2, 1, 7, 13, 5, 16, 17, 10, 20, 4, 11, 19, 14, 18, 9, 12, 8, 6, 3, 15

Z_{23} s: 0, 2, 22, 8, 10, 9, 21, 7, 4, 3, 3, 1, 17, 12, 8, 6, 12, 4, 5, 13, 7, 18, 14
t: 0, 2, 1, 9, 19, 5, 3, 10, 14, 17, 20, 21, 15, 4, 12, 18, 7, 11, 16, 6, 13, 8, 22

Z_{25} s: 0, 2, 24, 13, 24, 2, 22, 4, 20, 6, 18, 22, 14, 10, 16, 8, 18, 6, 20, 4, 17, 13, 11, 15, 9
t: 0, 2, 1, 14, 13, 15, 12, 16, 11, 17, 10, 7, 21, 6, 22, 5, 23, 4, 24, 3, 20, 8, 19, 9, 18

4. Summary. For reasons mentioned in the introduction, it is of interest to show that if $n \geq 3$, then Q_{2n} has a symmetric sequencing. This is the main goal of the current work. The first result of this paper (Theorem 5) shows that if $n \geq 3$ is odd, then the number of symmetric sequencings of Z_{2n} is a lower bound for the number of symmetric sequencings on Q_{2n}. This information has the potential to be of use to statisticians concerned with certain randomization properties of collections of so-called quasi-complete Latin squares of a given order [7,17].

The main goal can be achieved by showing that if $n \geq 3$, then D_n has a 2-sequencing. The important idea of a starter-translate 2-sequencing is used (Theorem 9) to show that if $n \geq 3$ is odd, then D_n has a number of 2-sequencings of an especially nice kind. This idea can be modified (Theorem 12) to attack the case where $n \geq 4$ is even. The modification used is the notion of a special symmetric sequencing. Because of a doubling construction (Theorem 13) it would suffice to know when Z_n has a special 2-sequencing. Methods are given that settle this question affirmatively for all even n, $n \geq 6$, but leave the question unsettled for most odd n. Consequently, the information in this paper allows one to exhibit 2-sequencings for all D_n, $n \geq 3$, except when n is twice an odd number.

The companion paper [6] takes care of these remaining cases as follows. It generalizes the "special" idea to infinitely many possibilities that break down into eight basic types. It is necessary to find constructions for four of these types in order to reach the desired conclusion.

Considerable simplification would be achieved if a relatively straightforward proof of the conjecture that Z_{2n+1} has a 2-sequencing could be found.

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