Recent Results on Edge-Colouring Graphs, With Applications to the Total-Chromatic Number

A.J.W. HILTON

Department of Mathematics,
University of Reading,
Whiteknights,
P.O. Box 220,
Reading, RG6 2AX,
England.

Abstract. We give a brief account of some recent results on edge-colouring simple graphs and of some recent results on the total-chromatic number of simple graphs. We illustrate the kind of arguments which have been found to be successful by proving one of the simpler results on edge-colouring graphs, and by showing how to apply this to obtain one of the recent results on the total-chromatic number.

---

* This paper was originally presented at the Second Carbondale Combinatorics Conference in May, 1987.

Introduction. In this paper an account is given of recent results of the author, A.G. Chetwynd and P.D. Johnson. We are concerned here with simple graphs, that is finite graphs without loops or multiple edges. Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degrees respectively. An edge-colouring of a graph G is a map $\phi: E(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set of colours, such that no two vertices with the same colour have a common vertex. The chromatic index $\chi'(G)$ is the least number j of colours such that G can be edge-coloured with j colours. A famous theorem of Vizing [24] says that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 .$$

If $\chi'(G) = \Delta(G)$, then G is said to be Class 1, and otherwise G is Class 2. The question of deciding whether or not a graph is Class 1 was shown by Holyer [21] to be NP-hard. However for certain types of graph, the problem of classifying Class 2 graphs seems to be tractible.

If G satisfies the inequality

$$|E(G)| > \Delta(G) \left\lfloor \frac{|V(G)|}{2} \right\rfloor ,$$

then G is overfull. Clearly if G is overfull, then $|V(G)|$ is odd. An overfull graph has to be Class 2, since no colour-class of G can have more than $\lfloor \frac{1}{2}|V(G)| \rfloor$ edges. In [8], Chetwynd and Hilton made the following conjecture (now slightly modified).

Conjecture 1. *Let G be a simple graph with* $\Delta(G) > \frac{1}{3}|V(G)|$. *Then G is Class 2 if and only if G contains an overfall subgraph H with* $\Delta(G) = \Delta(H)$.

The graph G obtained from Petersen's graph by removing one vertex is Class 2, but contains no subgraph H with $\Delta(H) = \Delta(G)$; this shows that the figure $\frac{1}{3}$ in Conjecture 1 cannot be lowered.

Conjecture 1 has been proved in a number of cases. Plantholt ([22], [23]) and Chetwynd and Hilton ([6], [7], [8], [9]) have between them established the following.

Theorem 1. *Conjecture 1 is true if* $\Delta(G) \geq |V(G)| - 3$.

If G is a regular graph of even order with $d(G) \geq \frac{1}{2}|V(G)|$, then it is fairly easy to establish that G cannot contain an overfull subgraph H with $\Delta(H) = d(G)$. The following results of Chetwynd and Hilton ([4], [10]) therefore provide additional evidence for Conjecture 1.

Theorem 2. *Let* G *be a regular graph of even order. If either*

$$d(G) \geq \tfrac{6}{7}|V(G)| ,$$

or

$$d(G) \geq |V(G)| - 4 ,$$

then G *is Class* 1.

In [5], Chetwynd and Hilton proved some further results of a similar nature. We mention the following two.

Theorem 3. *Let* G *be a graph with* r *vertices of maximum degree. If* $|V(G)| = 2n$ *and*

$$\delta(G) \geq n + \tfrac{3}{2}r - 2 ,$$

then G *is Class* 1.

Theorem 4. *Let* G *be a graph with* r *vertices of maximum degree*, $|V(G)| = 2n + 1$ *and*

$$\delta(G) \geq n + \tfrac{5}{2}r - 1 .$$

Then G *is Class* 2 *if and only if* G *is overfull*.

The first part of Theorem 2 can in fact be deduced from Theorem 4.

Let G_Δ denote the subgraph of a graph G induced by the vertices of degree $\Delta(G)$ in G. Fournier [15] proved the following.

Theorem 5. *If* G_Δ *is a forest, then* G *is Class* 1.

Chetwynd and Hilton [12], together with Hoffman [13], investigated the graph G_Δ further. Chetwynd and Hilton [12] made the following conjecture.

Conjecture 2. *Let* G *satisfy*

(i) $\Delta(G) > \frac{2}{3}(|V(G)|)$, *and*

(ii) $\delta(G_\Delta) \leq 1$.

Then G *is Class* 1.

Chetwynd and Hilton showed that Conjecture 1 implies Conjecture 2, so that, in the cases when Conjecture 1 is verified, Conjecture 2 is also verified. The lower bound on $\Delta(G)$ cannot be lowered further.

In a series of papers ([18], [19], [20]) Hilton and Johnson have described a number of further interesting consequences of Conjecture 1 in various situations. Some of their results are given in [16].

Now let us turn to total-colourings. A total-colouring of a graph is a map $\psi: V(G) \cup E(G) \to \mathcal{C}$ such that no two incident or adjacent elements receive the same colour. Thus if two vertices are adjacent, then they receive different colours, if two edges have a vertex in common, then they receive different colours, and if an edge is incident with a vertex then they receive different colours. The total-chromatic number $\chi_T(G)$ of a graph G is the least number of colours needed to totally-colour G. A long-standing and fascinating conjecture made independently by Behzad [1] and Vizing [25] in 1965 is that

$$\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2 .$$

The results to date on this conjecture are rather weak. Recently Bollobás and Harris [2] showed that if $\Delta(G) \geq 3917$ then $\chi_T(G) \leq \frac{11}{6}\Delta(G)$. Also Chetwynd and Häggkvist [3] showed that, for triangle free graphs, $\chi_T(G) \leq \frac{9}{5}\Delta(G) + 2$. We shall call a graph G Type 1 if $\chi_T(G) = \Delta(G) + 1$ and Type 2 if $\chi_T(G) = \Delta(G) + 2$. If the Total-Chromatic Number Conjecture is true, then every graph is Type 1 or Type 2.

Recently the present author [1] proved the following result about $\chi_T(G)$ when G has a spanning star.

Theorem 6. *Let J be a subgraph of K_{2n}, let $e = |E(J)|$ and let j be the maximum size (i.e. number of edges) of a matching in J. Then*

$$\chi_T\left(K_{2n} \backslash E(J)\right) = 2n + 1$$

if and only if $e + j \leq n - 1$.

For regular graphs of high degree, Chetwynd and Hilton [11] have recently obtained a number of reasonably strong results. These are summarized in the following chart. In no case is the lower bound on $d(G)$ best possible.

$d(G) \geq$	Odd order	Even order
$\frac{19}{21}\lvert V(G)\rvert$	Type 1 and Type 2 characterized	-
$\frac{6}{7}\lvert V(G)\rvert$	$\chi_T(G) \leq d(G) + 2$	-
$\frac{3}{4}\lvert V(G)\rvert$	$\chi_T(G) \leq d(G) + 3$	$\chi_T(G) \leq d(G) + 2$

Chart 1. Recent results on $\chi_T(G)$ when G is regular.

The characterization when G has odd order and $d(G) \geq \frac{19}{21}|V(G)|$ is easily stated.

Theorem 7. *Let G be a regular graph of odd order with*

$$d(G) \geq \tfrac{19}{21}|V(G)| - \tfrac{3}{7} .$$

Then G is Type 1 if and only if $\bar{G}$, the complement of G, contains a subgraph $K_{i_1} \cup K_{i_2} \cup \ldots \cup K_{i_s}$, where $K_{i_1}, K_{i_2}, \ldots, K_{i_s}$ are vertex-disjoint complete graphs of orders $i_1, \ldots, i_s$ respectively, where i_j is odd and ≥ 3 $(1 \leq j \leq s)$, and $(i_1 + \ldots + i_j) - s \geq |V(G)| - d - 1$. Otherwise G is of Type 2.

For regular graphs of even order, Chetwynd and Hilton made the following conjecture.

Conjecture 3. Let G be a regular graph of even order ≥ 6 with $d(G) > \frac{1}{2}|V(G)|$. Then G is Type 1 if and only if $\bar{G}$ contains a subgraph $K_{i_1} \cup K_{i_2} \cup \ldots \cup K_{i_s}$, where $K_{i_1}, \ldots, K_{i_s}$ are vertex-disjoint complete graphs of orders $i_1, \ldots, i_s$, where i_j is even $(1 \leq j \leq s)$ and $i_1 + \ldots + i_s = 2n$.

Conjecture 3 would be a close analogue of Theorem 7; however it appears to be very much more difficult to prove.

2. Proof techniques

Our object in this section is to illustrate the rather novel proof techniques we have found by giving two of the simpler proofs, one of Theorem 3 on edge-colouring, and one of the inequality $\chi_T(G) \leq d(G) + 2$ for regular graphs of odd order satisfying $d(G) \geq \frac{6}{7}|V(G)|$. The second of these results follows from the first.

Our results are at the moment much less strong than they would be if we had a more appropriate tool to use than Dirac's theorem [14] on the existence of Hamiltonian circuits in graphs satisfying $\delta(G) \geq \frac{1}{2}|V(G)|$. However it is not clear whether, if we had such a tool, our method is good enough to give the best possible results. Only time will tell.

Proof of Theorem 3. Suppose that G has r vertices of maximum degree, has $|V(G)| = 2n$ and satisfies $\delta(G) \geq n + \frac{3}{2}r - 2$.

Let G_r be the induced subgraph of G on the r vertices of maximum degree. Partition $E(G_r)$ into r matchings, $M_1, \ldots, M_r$, such that, for $1 \leq i \leq r$, M_i is a maximal (by inclusion) matching in the graph $G_r \backslash (M_1 \cup \ldots \cup M_{i-1})$. Using Vizing's theorem, it is clear that $E(G_r)$ can be partitioned into r matchings; then edges can be transformed between these matchings, if necessary, so that the maximality condition is satisfied.

Next let $F_1, \ldots, F_{r-1}$ be $r-1$ edge-disjoint 1-factors of G such that $M_i \subseteq F_i$ $(1 \le i \le r-1)$. We now show that such 1-factors do exist. Let $1 \le j \le r-1$ and suppose that $F_1, \ldots, F_{j-1}$ exist and that $(F_1 \cup \ldots \cup F_{j-1}) \cap (M_j \cup \ldots \cup M_r) = \emptyset$; we now show that F_j exists.

Let $H_j = G \backslash (F_1 \cup \ldots \cup F_{j-1})$.

Then

$$\delta(H_j \backslash V(M_j)) \ge \delta(G) - (j-1) - |V(M_j)| .$$

By Dirac's theorem, if

$$\delta(H_j \backslash V(M_j)) \ge \tfrac{1}{2} |V(H_j \backslash V(M_j))| ,$$

then $H_j \backslash V(M_j)$ has a Hamiltonian cycle. But

$$\begin{aligned} \delta(H_j \backslash V(M_j)) &\ge \delta(G) - (j-1) - |V(M_j)| \\ &\ge \delta(G) - (r-2) - |V(M_j)| \\ &= \delta(G) - r + 2 - |V(M_j)| ; \end{aligned}$$

also $|V(H_j \backslash V(M_j))| = 2n - V(M_j)$. Therefore

$$\begin{aligned} &\delta(H_j \backslash V(M_j)) - \tfrac{1}{2} |V(H_j \backslash V(M_j))| \\ &\ge \delta - r + 2 - |V(M_j)| - n + \tfrac{1}{2} |V(M_j)| \\ &= \delta - r + 2 - n - \tfrac{1}{2} |V(M_j)| \\ &\ge \delta - r + 2 - n - \tfrac{1}{2} r \\ &= \delta - \tfrac{3r}{2} + 2 - n \\ &\ge 0 , \quad \text{since } \delta \ge n + \tfrac{3r}{2} - 2 . \end{aligned}$$

Therefore $H_j \setminus V(M_j)$ has a Hamilton cycle (which is necessarily of even length). Let F_j consist of M_j together with alternate edges of the Hamiltonian cycle. Since M_j was a maximal matching in $G_r \setminus (M_1 \cup \ldots \cup M_{j-1})$, it follows that F_j contains no edge of $M_{j+1} \cup \ldots \cup M_r$. This shows that a suitable F_j does exist.

The graph $G \setminus (\bigcup_{i=1}^{r-1} F_i)$ has exactly r vertices of maximum degree, and each of these r vertices is joined to at most one other vertex of maximum degree. Therefore by Theorem 5 (Fournier's theorem), $G \setminus (\bigcup_{i=1}^{r-1} F_i)$ is Class 1. Working back, it follows that G is also Class 1.

This proves Theorem 3.

Now we use Theorem 3 to prove the following.

<u>Theorem</u> 8. <u>Let</u> G <u>be a regular graph of odd order with</u>

$$d(G) \geq \tfrac{6}{7}|V(G)| - \tfrac{10}{7}$$

<u>Then</u>

$$\chi_T(G) \leq d(G) + 2 .$$

<u>Proof</u>. Let $|V(G)| = 2n + 1$. The theorem is true if $n = 1$ or 2, so suppose that $n \geq 3$. Let the vertices of G be $v_1, \ldots, v_{2n+1}$ and let v_j and $v_{(2n-d)+j}$ be non-adjacent for $1 \leq j \leq 2n-d$. To see that there are such vertices, note that $\bar{G}$, the complement of G, is regular of degree $2n-d$. By Vizing's theorem, $\bar{G}$ can be edge-coloured with $2n-d+1$ colours, and so $\bar{G}$ has a colour class with at least

$$\frac{(2n+1)(2n-d)}{2(2n+1-d)} \geq \frac{(2n+1)(2n-d)}{2((2n+1) - \frac{6}{7}(2n+1) + \frac{10}{7})} = \frac{7}{2(1+10/(2n+1))}(2n-d) \geq 2n-d$$

edges, since $d \geq \frac{6}{7}(2n+1) - \frac{10}{7}$ and $n \geq 2$.

From G form a graph G^* by introducing a vertex v^* and joining it to $v_{2n-d+1}, \ldots, v_{2n+1}$. Then G^* has $d+2$ vertices, $v_{2n-d+1}, \ldots, v_{2n+1}, v^*$, of degree $d+1$ and $2n-d$ vertices, $v_1, \ldots, v_{2n-d}$, of degree d.

Now let $F_1, \ldots, F_{2n-d}$ be edge-disjoint matchings of G^* such that, for $1 \leq j \leq 2n-d$, F_j misses the two vertices v_j and $v_{j+2(2n-d)}$; contains the edge $v_{j+(2n-d)}v^*$, and misses no further vertex. We show now that these matchings do exist.

We pick out these matchings one by one. For $1 \leq j \leq 2n-d$, suppose that $F_1, \ldots, F_{j-1}$ have been constructed. Let $G_j^* = G^* \backslash (F_1 \cup \ldots \cup F_{j-1})$ and let G_j^+ be the simple graph formed from G_j^* by adding in the edges $v_j v_{2(2n-d)+j}$ and $v_{(2n-d)+j} v_{2(2n-d)+j}$ if they are not already in G_j^*. If $j \leq 2n-d-1$ then $\delta(G_j^+) \geq d-(j-1) = d-j+1 \geq d-(2n-d-1)+1 = 2d-2n+2$, and if $j = 2n-d$, then $\delta(G_j^+) \geq d-(j-2) = 2d-2n+2$. Since $d \geq \frac{6}{7}(2n+1) - \frac{10}{7}$, it follows that

$$\delta(G_j^+) \geq \tfrac{12}{7}(2n+1) - \tfrac{20}{7} - 2n+2 = n+2 + \tfrac{3n}{7} - \tfrac{8}{7} \geq n+2 = \tfrac{1}{2}((2n+1) + 3) ,$$

since $n \geq 3$. Therefore, by Lemma 3, the graph G_j^+ contains a Hamiltonian circuit containing the path $v_j, v_{2(2n-d)+j}, v_{(2n-d)+j}, v^*$. Therefore G_j^* contains a matching F_j which contains the edge $v^*v_{j+(2n-d)}$, misses the vertices v_j and $v_{j+2(2n-d)}$, and misses no other vertex. Iterating this gives the required matchings $F_1, \ldots, F_{2n-d}$.

Let $G^{**} = G^* \backslash (F_1 \cup \ldots \cup F_{2n-d})$. Then G^{**} has $2n-d$ vertices $v_{(2n-d)+1}, \ldots, v_{3(2n-d)}$ of the maximum degree $d+1 - (2n-d-1) = 2d-2n+2$, and the remaining vertices have degree $2d-2n+1$. Since $d \geq \frac{6}{7}(2n+1) - \frac{10}{7}$, it follows that

$$\delta(G^{**}) = 2d - 2n+1 \geq (n+1) + \frac{3}{2}(2n-d) - 2 ,$$

and so it follows from Lemma 1 that G^{**} is Class 1.

From an edge-colouring of G^{**} with the $2d-2n+1$ colours, $c_{2n-d+1},\ldots,c_{d+2}$, we form a total-colouring of G with the $d+2$ colours $c_1,\ldots,c_{d+2}$ as follows. Each edge of G which is also an edge of G^{**} receives the same colour. For $2(2n-d) + 1 \leq j \leq 2n+1$, vertex v_j receives the colour of the edge v_jv^*. For $1 \leq j \leq 2n-d$, the two vertices v_j and v_{2n-d+j} and the edges of $F_j\backslash\{v^*v_{(2n-d)+j}\}$ all receive the colour c_j. It is easy to check that this is a total-colouring of G.

This proves Theorem 8.

References

1. M. Behzad, Graphs and their chromatic numbers, Doctoral Thesis (Michigan State University), 1965.

2. B. Bollobás and A.J. Harris, List colourings of graphs, Graphs and Combinatorics, 1 (1985), 115-127.

3. A.G. Chetwynd and R. Häggkvist, A note on list-colourings, submitted.

4. A.G. Chetwynd and A.J.W. Hilton, Regular graphs of high degree are 1-factorizable, Proc. London Math. Soc. (3), 50 (1985), 193-206.

5. A.G. Chetwynd and A.J.W. Hilton, The edge-chromatic class of graphs with large maximum degree, where the number of vertices of maximum degree is relatively small, Jour. Combinatorial Theory (B), to appear.

6. A.G. Chetwynd and A.J. W. Hilton, Partial edge-colourings of complete graphs or of graphs which are nearly complete, Graph Theory and Combinatorics, Vol. in honour of P. Erdös' 70th birthday, (1984), 81-98.

7. A.G. Chetwynd and A.J. W. Hilton, The chromatic index of graphs of even order with many edges, J. Graph Theory, 8 (1984), 463-470.

8. A.G. Chetwynd and A.J. W. Hilton, The edge-chromatic class of graphs with maximum degree at least $|V|-3$, Proceedings of the conference held in Denmark in memory of G.A. Dirac, to appear.

9. A.G. Chetwynd and A.J. W. Hilton, Star multigraphs with three vertices of maximum degree, Math. Proc. Camb. Phil. Soc., 100 (1986), 303-317.

10. A.G. Chetwynd and A.J. W. Hilton, The edge-chromatic class of regular graphs of degree 4 and their complements, Discrete Applied Math., 16 (1987), 125-134.

11. A.G. Chetwynd and A.J.W. Hilton, On the total chromatic number of regular graphs of high degree, submitted.

12. A.G. Chetwynd and A.J.W. Hilton, A Δ-subgraph condition for a graph to be Class 1, Jour. Combinatorial Theory (B), to appear.

13. A.G. Chetwynd, A.J.W. Hilton and D.G. Hoffman, On the Δ-subgraph of graphs which are critical with respect to the chromatic index, submitted.

14. G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. (3), 2 (1952), 69-81.

15. J.-C. Fournier, Colorations des arêtes d'un graphe, Cahiers de CERO, 15 (1973), 311-314.

16. A.J. W. Hilton, Recent progress on edge-colouring graphs, Discrete Math., 64 (1987), 303-307.

17. A.J.W. Hilton, A total chromatic number analogue of Plantholt's theorem, submitted.

18. A.J.W. Hilton and P.D. Johnson, Graphs which are vertex-critical with respect to the edge-chromatic number, Math. Proc. Comb. Phil. Soc., to appear.

19. A.J.W. Hilton and P.D. Johnson, Graphs which are vertex-critical with respect to the edge-chromatic class, submitted.

20. A.J.W. Hilton and P.D. Johnson, Reverse class critical multigraphs, Discrete Math., to appear.

21. I.J. Holyer, The NP-completeness of edge-colourings, SIAM J. Computing, 10 (1980), 718-720.

22. M. Plantholt, The chromatic index of graphs with a spanning star, J. Graph Theory, 5 (1981), 5-13.

23. M. Plantholt, On the chromatic index of graphs with large maximum degree, Discrete Math., 47 (1983), 91-96.

24. V.G. Vizing, On an estimate of the chromatic class of a p-graph (in Russian), Diskret. Analiz. 3 (1964), 25-30.

25. V.G. Vizing, Some unsolved problems in graph theory (in Russian), Math. Nauk., 23 (1968), 117-134; Russian Math. Surveys, 23 (1968), 125-142.