On the existence of (2,4;3,m,h)-frames for h=1,3 and 6

E.R. Lamken and S.A. Vanstone

ABSTRACT

Let V be a set of v elements. Let $G_1,G_2,...,G_m$ be a partition of V into m sets. A $\{G_1,G_2,...,G_m\}$-frame F with block size k, index λ and latinicity μ is a square array of side v which satisfies the properties listed below. We index the rows and columns of F with the elements of V. (1) Each cell is either empty or contains a k-subset of V. (2) Let F_i be the subsquare of F indexed by the elements of G_i. F_i is empty for $i=1,2,...,m$. (3) Let $j \in G_i$. Row j of F contains each element of $V-G_i$ μ times and column j of F contains each element of $V-G_i$ μ times. (4) The collection of blocks obtained from the nonempty cells of F is a GDD($v;k;G_1,G_2,...,G_m;0,\lambda$). If $|G_i|=h$ for $i=1,2,...,m$, we call F a $(\mu,\lambda;k,m,h)$-frame. Frames with $\mu=\lambda=1$ and $k=2$ were used by D.R. Stinson to establish the existence of skew Room squares and Howell designs. $(1,2;3,m,h)$-frames with $h=1,3$ and 6 have been studied and can be used to produce $KS_3(v;1,2)$s. In this paper, we prove the existence of $(2,4;3,m,h)$-frames for $h=3$ and 6 with a finite number of possible exceptions. We also show the existence of $(2,4;3,m,1)$-frames for $m \equiv 1 (mod 12)$. These frames can be used to construct $KS_3(v;2,4)$s.

1. Introduction.

Let V be a set of v elements. Let $G_1,G_2,...,G_m$ be a partition of V into m sets. A $\{G_1,G_2,...,G_m\}$-frame F with block size k, index λ and latinicity μ is a square array of side v which satisfies the properties listed below. We index the rows and columns of F by the elements of V.

(1) Each cell is either empty or contains a k-subset of V.

(2) Let F_i be the subsquare of F indexed by the elements of G_i. F_i is empty for $i=1,2,...,m$.

(3) Let $j \in G_i$. Row j of F contains each element of $V-G_i$ μ times and

column j of F contains each element of $V - G_i$ μ times.

(4) The collection of blocks obtained from the nonempty cells of F is a GDD$(v;k;G_1,G_2,...,G_m;0,\lambda)$. (See [14] for the notation for group divisible designs (GDD).)

If there is a $\{G_1,G_2,...,G_m\}$-frame H with block size k, index λ and latinicity μ such that

(1) $H_i = F_i$ for $i = 1,2,...,m$ and

(2) H can be written in the empty cells of $F - \bigcup_{i=1}^{m} F_i$,

then H is called a complement of F and denoted $\overline{F}$. The superposition of F and $\overline{F}$, $F \circ \overline{F}$, is a $\{G_1,G_2,...,G_m\}$-frame with block size k, index 2λ and latinicity 2μ. If $|G_i| = h$ for $i = 1,2,...,m$, we call F a $(\mu,\lambda;k,m,h)$-frame. If a complement of F exists, we call F a complementary $(\mu,\lambda;k,m,h)$-frame.

Existence results for $\{G_1,G_2,...,G_m\}$-frames with $k=2$, $\mu=1$ and $\lambda=1$ and for a special type of complementary frame with $k=2$, $\mu=\lambda=1$ (called a skew frame) can be found in [10]. These frames were used to establish the existence of skew Room squares and the existence of Howell designs, $H(s,n)$ (see [10]).

Frames with $k \geq 3$ have also been considered. For $k=3$, $\mu=1$, $\lambda=1$ there are few results. A (1,1;3,14,2) and a (1,1;3,20,2) are known to exist ([2]).

Frames have also been investigated for $k=3$, $\mu=1$ and $\lambda=2$. The following results appear in [3].

Theorem 1.1. *There exist* (1,2;3,m,3)*-frames for* $m \geq 5$ *except possibly for* $m \in \{6,10,14,16,18,20,22,24,26,28,30,32,34,38,39,42,43,44,46,47,51,52,59,118,123\}$.

Theorem 1.2. *There exist* (1,2;3,m,6)*-frames for* $m \geq 5$ *except possibly for* $m \in \{10,11,14,15,17,18,19,20,23,24,27,28,32, 34, 39\}$.

These frames were used to construct $KS_3(v;1,2)$s for $v \equiv 3 (mod\, 12)$ in [6]. Similarly, $KS_3(v;2,4)$s can be constructed from $(2,4;3,m,h)$-frames and complementary $(1,2;3,m,h)$-frames [7].

In this paper, we establish the existence of $(2,4;3,m,h)$-frames for $h=3$ and $h=6$ with a finite number of possible exceptions in each case. The next section contains constructions for $(2,4;3,m,h)$-frames. The existence results for $h=3$ and $h=6$ are in section 3. In the last section, we prove the existence of $(2,4;3,m,1)$-frames for $m \equiv 1 (mod\, 12)$.

2. Constructions

Before describing the main recursive construction for frames, we recall the definition of a pairwise balanced design. A pairwise balanced design, denoted PBD(v;K), is a collection B of subsets (called blocks) of a finite v-set of elements V such that every pair of distinct elements of V is contained in precisely one block of B and for each $b \in B$, $|b| \in K$, where K is some subset of positive integers. We state and prove the construction in the most general form; a special case of it was used in [3].

Theorem 2.1. *Let K be some subset of positive integers. If there exists a PBD(v;K) such that for each $\ell \in K$ there is a $(\mu,\lambda;k,\ell,h)$-frame, then there is a $(\mu,\lambda;k,v,h)$-frame. If the $(\mu,\lambda;k,\ell,h)$-frames are all complementary, then the $(\mu;\lambda;k,v,h)$-frame is also complementary.*

Proof. Let $V = \{x_1,x_2,\ldots,x_v\}$ and let $H = \{1,2,...,h\}$. Form a $vh \times vh$ array F with the rows and columns of F indexed by the elements of $V \times H$. Let D be a PBD(v;K) defined on V. Let B be a block of D of size ℓ. The rows and columns of F indexed by the elements of $B \times H$ form an $\ell h \times \ell h$ subarray. Replace this subarray by a $(\mu,\lambda;k,\ell,h)$-frame defined on $B \times H$ where the holes of the frame are indexed by $\{b\} \times H$ for $b \in B$. Do this for all of the blocks in D. The resulting array is a $(\mu,\lambda;k,v,h)$-frame. □

It will be useful to have a frame singular direct product. We generalize the product stated in [3]. For completeness, we include the proof.

Theorem 2.2. *If there is a (complementary) $(\mu,\lambda;k,m,h)$-frame containing a (complementary) $(\mu,\lambda;k,n,h)$-frame $(n \geq 0)$, a (complementary) $(\mu,\lambda;k,s,h')$-frame and three mutually orthogonal Latin squares of side $\dfrac{h(m-n)}{h'}$, then there is a (complementary) $(\mu,\lambda;k,s(m-n)+n,h)$-frame.*

Proof. Let $V = \{x_1^i,x_2^i,\ldots,x_{h'}^i \mid i=1,2,\ldots,s\}$ and let $G_i = \{x_1^i,x_2^i,\ldots,x_{h'}^i\}$ for $i=1,2,\ldots,s$. Let $H = \dfrac{h(m-n)}{h'}$. Let $W = \{1,2,\ldots,H\}$ and let $N = \{\infty_1,\infty_2,\ldots,\infty_{nh}\}$.

Let L_1,L_2 and L_3 be a set of three mutually orthogonal Latin squares of side H. L will denote the array of triples formed by the superposition of L_1,L_2 and L_3. L_{ijk} is the $H \times H$ array formed by replacing each triple (a,b,c) in L with the triple (a_i,b_j,c_k) where $a_i \in H \times \{i\}$, $b_j \in H \times \{j\}$ and $c_k \in H \times \{k\}$.

Let F be a $(\mu,\lambda;k,s,h')$-frame defined on V. F is a $\{G_1,G_2,\ldots,G_m\}$-frame. Construct an $hs(m-n) \times hs(m-n)$ array A from F by replacing each triple (x,y,z) in F with the $H \times H$ array L_{xyz} and by replacing each empty cell in F with an $H \times H$ empty array. A contains a diagonal of s $(m-n)h \times (m-n)h$ empty arrays.

Let T_i be a $(\mu,\lambda;k,m,h)$-frame defined on $(W \times G_i) \cup N$ which contains a $(\mu,\lambda;k,n,h)$-frame defined on N. Let T denote the $(\mu,\lambda;k,n,h)$-frame. T_i can be partitioned as follows.

$$T_i = \begin{array}{|c|c|} \hline T & R_i \\ \hline C_i & K_i \\ \hline \end{array} \begin{array}{l} \} \ nh \\ \} \ (m-n)h \end{array}$$

We construct a new array B from A and the T_i's. B has size $h\{s(m-n)+n\} \times h\{s(m-n)+n\}$.

$$B = \begin{array}{|ccccc|} \hline T & R_1 & R_2 & \cdots & R_s \\ C_1 & K_1 & & & \\ C_2 & & K_2 & A & \\ \vdots & & & \ddots & \\ C_s & & & & K_s \\ \hline \end{array}$$

It is straightforward to verify that B is a $(\mu,\lambda;k,s(m-n)+n,h)$-frame defined on $(W \times V) \cup N$. If each of the frames used in the construction is complementary, then the resulting array B is also a complementary frame. □

The next result uses complementary $(1,2;3,m,1)$-frames to construct $(2,4;3,m,3)$-frames.

Theorem 2.3. *If there exists a complementary $(1,2;3,m,1)$-frame and a set of three mutually orthogonal Latin squares of side m with a common transversal, then there is a $(2,4;3,m,3)$-frame.*

Proof. Let $V_i = \{1_i,2_i,\ldots,m_i\}$ for $i=1,2,3$. Let F_j be a complementary $(1,2;3,m,1)$-frame defined on V_j such that cell (i,i) is missing the element i_j. $\bar{F}_j$ will denote the complement of F_j.

Let L_1,L_2 and L_3 be a set of three mutually orthogonal Latin squares of side m with a common transversal. Suppose L_i is defined on V_i. Let h be the array of triples formed by the superposition of L_1,L_2 and L_3. We may assume that cell (i,i) of L contains the triple $\{i_1,i_2,i_3\}$ for $i=1,2,\ldots,m$. Delete the main diagonal of L and call the resulting array L'.

We construct a new array A as follows.

A =

$F_1 \circ \bar{F}_2$	F_3	L'
F_3	$F_1 \circ \bar{F}_2$	L'
L'	L'	

A is a $(2,4;3,m,3)$-frame defined on $\bigcup_{i=1}^{3} V_i$. □

The last construction in this section uses doubly resolvable $(v,3,1)$-BIBDs to construct $(2,4;3,m,1)$-frames.

Theorem 2.4. *If there exists a doubly resolvable* $(2n+1,3,1)$*-BIBD, then there is a* $(2,4;3,n,1)$*-frame.*

Proof. Let $N = \{1,2,\ldots,n\}$ and let $V = (N\times\{1,2\})\cup\{\infty\}$. Suppose D is a DR$(2n+1,3,1)$-BIBD defined on V such that the main diagonal of D contains the triples $\{\infty,i_1,i_2\}$. Delete the main diagonal of D and replace i_1 with the element i and i_2 with the element i for $i=1,2,\ldots,n$. The resulting array has index $\lambda=4$ and latinicity $\mu=2$. It is a $(2,4;3,n,1)$-frame defined on N. □

3. Existence of $(2,4;3,m,h)$-frames for $h=3$ and $h=6$

In this section, we prove the existence of $(2,4;3,m,h)$-frames for $h=3$ and $h=6$ with a finite number of possible exceptions for m. In the case $h=6$, we establish the existence of complementary $(1,2;3,m,6)$-frames and thus, from our remarks in section 1, the existence of $(2,4;3,m,6)$-frames.

In order to apply the main recursive construction, Theorem 2.1, we will need the existence of certain classes of pairwise balanced designs. The first lemma was proved in [3].

Lemma 3.1. *Let $k \in \{1,2,3,4\}$. If there is a set of $3+k$ mutually orthogonal Latin squares of side s, then there is a PBD$(5s+i_1+i_2+...+i_k;$ $\{s,i_1,...,i_k,5,5+1,...,5+k\})$ where $0 \le i_j \le s$.*

Lemma 3.2. *If there is a resolvable $(12n+4,4,1)$-BIBD, then there is a PBD$(16n+5;\{5,4n+1\})$.*

Proof. Let D be a resolvable $(12n+4,4,1)$-BIBD. Let $R_1,...,R_{4n+1}$ be the resolution classes of D. Add a new element x_i to each block in R_i for $i=1,2,...,4n+1$ and add a new block $\{x_1,x_2,...,x_{4n+1}\}$ to the resulting design. □

We will also make use of the well known existence result for $(v,5,1)$-BIBDs: a $(v,5,1)$-BIBD exists if and only if $v \equiv 1$ or $5 (mod 20)$ (Hanani [4]).

The recursive constructions for frames require the existence of some small $(2,4;3,m,h)$-frames for $h=1,3$ and 6. We construct these frames using starters and adders (see [3], [8] for definitions).

Lemma 3.3. *(i) There is a $(2,4;3,6,3)$-frame.*

(ii) There exist complementary $(1,2;3,m,3)$-frames for $m=5,7,8,9,11,13,15$ and 17.

Proof.

(i) A starter for a $(2,4;3,6,3)$-frame is $(\{11,13,14\},\{3,17,16\},$ $\{4,14,5\},\{8,9,16\},\{2,15,17\},\{2,7,9\},\{1,3,10\},\{1,4,8\},\{5,10,13\},\{15,7,11\})$. A corresponding adder is $(2,11,3,13,5,1,10,15,4,8)$.

(ii) In Table 3.1, we list starter-adder pairs for $(1,2;3,m,3)$-frames and their complements for $m=5,7,8,9$ and 11. For $m=13,15$ and 17, the starters for a $(1,2;3,m,3)$-frame and its complement are $S \cup -S$ where S is given in Table 3.2. The adders are $A_1 \cup -A_1$ and $A_2 \cup -A_2$ respectively. All of the starters for these tables were taken from [3]. □

Lemma 3.4. *There exist complementary $(1,2;3,m,6)$-frames for $m=5,6,7,8,9$ and 13.*

Proof. In Table 3.3, we list starter-adder pairs for $(1,2;3,m,6)$-frames and their complements for $m=5,6$ and 7. For $m=8,9$ and 13, the starters for a $(1,2;3,m,6)$-frame and its complement are $S \cup -S$ where S is given in Table 3.4. The adders are $A_1 \cup -A_1$ and $A_2 \cup -A_2$ respectively. These starters were also found in [3]. □

Lemma 3.5. *There exist* $(2,4;3,m,3)$*-frames for* $m=16$ *and* $m=19$.

Proof. We list starters and adders for complementary $(1,2;3,m,1)$-frames for $m=16$ and $m=19$. We then apply Theorem 2.3 to construct $(2,4;3,m,3)$-frames for $m=16$ and $m=19$.

$m=16$

Frame	S	3 4 6	13 14 7	9 11 15	2 5 10	8 12 1
	A	2	13	4	7	6
Complement	$\bar{S}$	13 12 10	3 2 9	7 5 1	14 11 6	8 4 15
	$\bar{A}$	14	3	12	9	10

$m=19$

Frame	S	14 15 3	1 2 6	9 11 17	5 7 10	13 16 4	8 12 18
	A	2	7	1	15	10	3
Complement	$\bar{S}$	5 4 16	18 17 13	10 8 2	14 12 9	6 3 15	11 7 1
	$\bar{A}$	17	12	18	4	9	16

□

We can now prove the existence of $(2,4;3,m,h)$-frames for $h=3$ and $h=6$ with a finite number of exceptions in each case. We consider three cases: $5 \le m \le 126$, $126 \le m \le 729$ and $m \ge 630$.

Lemma 3.6. *There exist* $(2,4;3,m,3)$*-frames for* $5 \le m \le 126$ *except possibly for* $m \in N = \{10,12,14,18,20,22,24,27,28,32,34,39\}$.

Proof. We consider four cases.

(i) $5 \le m \le 39$ $(m \notin N)$.

m	Construction	
5	Lemma 3.3	
6	Lemma 3.3	
7	Lemma 3.3	
8	Lemma 3.3	
9	Lemma 3.3	
11	Lemma 3.3	
13	Lemma 3.3	
15	Lemma 3.3	
16	Lemma 3.5	
17	Lemma 3.3	
19	Lemma 3.5	
21	$(21,5,1)$-BIBD	Thm. 2.1

25	(25,5,1)-BIBD	Thm. 2.1	
26	5.5+1	Thm. 2.1	(Lemma 3.1)
29	7(5-1)+1	Thm. 2.2	
30	PBD(30;{5,6})	Thm. 2.1	
31	(31,6,1)-BIBD	Thm. 2.1	
33	8(5-1)+1	Thm. 2.2	
35	5.7	Thm. 2.1	(Lemma 3.1)
36	5.7+1	Thm. 2.1	(Lemma 3.1)
37	5.7+1+1	Thm. 2.1	(Lemma 3.1)
38	5.7+1+1+1	Thm. 2.1	(Lemma 3.1)

(ii) $40 \le m \le 64$
We can write $m = 5.8 + i_1 + i_2 + i_3$ where $i_j \in \{0,1,5,6,7,8\}$. Since there are 7 mutually orthogonal Latin squares of side 8 and $(2,4;3,k,3)$-frames for $k \in \{0,1,5,6,7,8\}$, we apply Theorem 2.1 (Lemma 3.1) to construct $(2,4;3,m,3)$-frames.

(iii) $65 \le m \le 80$
We can write $m = 5.9 + i_1 + i_2 + i_3 + i_4$ where $i_j \in \{0,1,5,6,7,8,9\}$. Since there are 8 mutually orthogonal Latin squares of side 9 and $(2,4;3,k,3)$-frames for $k \in \{0,1,5,6,...,9\}$, we can construct $(2,4;3,m,3)$-frames by applying Theorem 2.1 (and Lemma 3.1).

(iv) $80 \le m \le 126$
We can write $m = 5.16 + i_1 + i_2 + i_3 + i_4$ where $i_j \in \{0,1,5,...,9,16\}$. Since there are 15 mutually orthogonal Latin squares of side 16 and $(2,4;3,m,3)$-frames for $k \in \{0,1,5,...,9,16\}$ we apply Theorem 2.1 (and Lemma 3.1) to construct $(2,4;3,m,3)$-frames. □

Lemma 3.7. *There exist complementary* $(1,2;3,m,6)$*-frames for* $5 \le m \le 126$ *except possibly* $m \in M = \{10,11,12,14,...,20,22,23,24,27,28,32,34,39,114,115,116,118,119,122,123,124\}$.

Proof.

(i) $5 \le m \le 39$ $(m \notin M)$

m	Construction	
5	Lemma 3.4	
6	Lemma 3.4	
7	Lemma 3.4	
8	Lemma 3.4	
9	Lemma 3.4	
13	Lemma 3.4	
21	(21,5,1)-BIBD	Thm. 2.1
25	(25,5,1)-BIBD	Thm. 2.1

26	5.5+1	Thm. 2.1	(Lemma 3.1)
29	7(5-1)+1	Thm. 2.2	
30	PBD(30;{5,6})	Thm. 2.1	
31	(31,6,1)-BIBD	Thm. 2.1	
33	8(5-1)+1	Thm. 2.2	
35	5.7	Thm. 2.1	(Lemma 3.1)
36	5.7+1	Thm. 2.1	(Lemma 3.1)
37	5.7+1+1	Thm. 2.1	(Lemma 3.1)
38	5.7+1+1+1	Thm. 2.1	(Lemma 3.1)

(ii) $40 \leq m \leq 80$
We apply Theorem 4.1 (and Lemma 4.4) as in Cases (ii) and (iii) of Lemma 3.6.

(iii) $81 \leq m \leq 113$
We can write $m = 5.13 + i_1 + i_2 + i_3 + i_4$ where $i_j \in \{0,1,5,\ldots,9,13\}$. Since there are 12 mutually orthogonal Latin squares of side 13 and complementary $(1,2;3,k,6)$-frames for $k \in \{0,1,5,\ldots,9,13\}$, we can construct complementary $(1,2;3,m,6)$-frames by applying Lemma 3.1 and Theorem 2.1.

(iv) $114 \leq m \leq 126$, $m \notin M$.

m	Construction		
117	16.7+5	Thm. 2.1	(Lemma 3.2)
120	PBD(120;{5,6})	Thm. 2.1	
121	(121,5,1)-BIBD	Thm. 2.1	
125	(125,5,1)-BIBD	Thm. 2.1	
126	(126,6,1)-BIBD	Thm. 2.1	

□

Lemma 3.8. *(i) There exist* $(2,4;3,m,3)$*-frames for* $125 \leq m \leq 729$.
(ii) There exist complementary $(1,2;3,m,6)$-frames for $125 \leq m \leq 729$.

Proof. We consider four cases.

(i) $125 \leq m \leq 209$.
We can write $m = 5.25 + i_1 + i_2 + i_3 + i_4$ where $i_j \in \{0,1,5,6,7,8,9,21,25\}$. Since there are 7 mutually orthogonal Latin squares of side 25 and $(2,4;3,k,3)$-frames and complementary $(1,2;3,m,6)$-frames for $k \in \{0,1,5,\ldots,9,21,25\}$ (Lemmas 3.6-7), we apply Theorem 2.1 and Lemma 3.1 to construct the appropriate frames.

(ii) $210 \leq m \leq 335$.
We can write $m = 5.41 + i_1 + i_2 + i_3 + i_4$ where $i_j \in \{0,1,5,\ldots,9,21,25,26,29,30,31,33\} = I_1$. Since there are 7 mutually

orthogonal Latin squares of side 41 and $(2,4;3,k,3)$-frames and complementary $(1,2;3,k,6)$-frames for $k \in I_1 \cup \{41\}$, we can apply Lemma 3.1 and Theorem 2.1.

(iii) $335 \leq m \leq 465$.
We can write $m = 5.67 + i_1 + i_2 + i_3 + i_4$ where $i_j \in I_1$. Since there are 7 mutually orthogonal Latin squares of side 67 and $(2,4;3,k,3)$-frames and complementary $(1,2;3,k,6)$-frames for $k \in I_1 \cup \{67\}$, we can apply Lemma 3.1 and Theorem 2.1.

(iv) $465 \leq m \leq 729$.
We can write $m = 5.81 + i_1 + i_2 + i_3 + i_4$ where $i_j \in I_1 \cup \{35,...,81\}$. Since there are 7 mutually orthogonal Latin squares of side 81 and $(2,4;3,k,3)$-frames and complementary $(1,2;3,k,6)$-frames for $k \in \{35,...,81\} \cup I_1$, we can apply Lemma 3.1 and Theorem 2.1 again. □

Finally, we combine these results to prove the following.

Theorem 3.9. *(i) There exist* $(2,4;3,m,3)$*-frames for* $m \geq 5$ *except possibly for* $m \in N = \{10,12,14,18,20,22,24,27,28,32,34,39\}$.

(ii) There exist complementary $(1,2;3,m,6)$*-frames for* $m \geq 5$ *except possibly for* $m \in M = \{10,11,12,14,...,20,22,23,24,27,28,32,34,39,114,115,116,118,119,122,123,124\}$.

Proof. From Lemma 3.6, there exist $(2,4;3,m,3)$-frames for $m \in \{5,...,126\} - N$. Lemma 3.8 provides $(2,4;3,m,3)$-frames for $125 \leq m \leq 729$. Similarly, from Lemma 3.7 there exist complementary $(1,2;3,m,6)$-frames for $m \in \{5,...,126\} - M$ and Lemma 3.8 provides complementary $(1,2;3,m,6)$-frames for $125 \leq m \leq 729$.

Let $m \geq 630$. We write $m = 5s + i$ where $s \geq 125$ and $i \in \{5,6,7,8,9\}$. Since there are 4 mutually orthogonal Latin squares of side 5 and $(2,4;3,k,3)$-frames and complementary $(1,2;3,k,6)$-frames for $k \in \{5,6,7,8,9,s\}$, Theorem 2.1 and Lemma 3.1 can be applied to construct the appropriate frames. □

4. Existence results for $(2,4;3,m,1)$-frames

A necessary condition for the existence of a $(2,4;3,m,1)$-frame is $m \equiv 1 (mod 3)$. In this section, we prove the existence of $(2,4;3,m,1)$-frames for $m \equiv 1 (mod 12)$. Since the proof uses the frame singular direct product (Theorem 2.2), we require the existence of $(2,4;3,m,1)$-frames for some small values of m. $(1,2;3,m,1)$-frames have been constructed by using starters and adders for $m = 10,16,19,25$ and 28 in [1]. Certain classes of $(1,2;3,m,1)$-frames were constructed in [13] by using algebraic techniques to find starters and adders. We use some of those constructions and Theorem 2.4 to prove the following result.

Lemma 4.1. *There exist* $(2,4;3,m,1)$-*frames for* $m=13,16,19,25,31,37,40,43$ *and* 49.

Proof. For $m=13,19,25,31$ and 40, there exist DR$(2m+1,3,1)$-BIBDs ([3], [8], [10], [11]) and we can apply Theorem 2.4. A $(2,4;3,16,1)$-frame was constructed using starters and adders in Lemma 3.5. We list starters and adders for complementary $(1,2;3,m,1)$-frames for $m=31,37$ and 43.

(i) $m=31$. We define a starter in $Z^*_{31}=Z_{31}-\{0\}$. Let $M=\{3^{10},3^{20},1\}$. We define $M3^j=\{3^{10+j},3^{20+j},3^j\}$. A starter S for a complementary $(1,2;3,31,1)$-frame is $(M,M3^3,M3^6,\ldots,M3^{27})$. An adder for S is $(1,3^3,3^6,\ldots,3^{27})$ and an adder for a complement is $(3^{20},3^{23},3^{26},3^{29},3,3^4,3^7,3^{10},3^{13},3^{16})$.

(ii) $m=37$. We define a starter in Z^*_{37}. Let $M=\{1,2,19\}$. A starter S for a complementary $(1,2;3,37,1)$-frame is $(M,M2^3,M2^6,\ldots,M2^{33})$. An adder for S is $(2^{27},2^{30},2^{33},2^{36},2^2,2^5,2^8,2^{11},2^{14},2^{17},2^{20},2^{23})$ and an adder for a complement is $(2^{35},2,2^4,2^7,2^{10},2^{13},2^{16},2^{19},2^{22},2^{25},2^{28},2^{31})$.

(iii) $m = 43$. We define a starter in $\mathbf{Z}^*_{43}$. Let $M = \{1,3^{14},3^{28}\}$. A starter S for a complementary $(1,2;3,43,1)$-frame is $(M,M3^3,M3^6,\ldots,M3^{39})$. An adder for S is $(1,3^3,3^6,\ldots,3^{39})$ and an adder for a complement is $(3^{14},3^{17},3^{20},3^{23},3^{26},3^{29},3^{32},3^{35},3^{38},3^{41},3,3^4,3^7,3^{10})$.

(iv) Consider GF(7^2) generated by $f(x) = x^2+x+3$ and let α be a primitive element such that $f(\alpha) = 0$. If $M = \{\alpha^0,\alpha^1,\alpha^{14}\}$ then $(\alpha^{3i}M{:}0\leq i\leq 15)$ is a starter and $(-\alpha^{3i}(\alpha^0+\alpha^1+\alpha^{14}){:}0\leq i\leq 15)$ is an adder. A complementary adder is $(\alpha^{3i+1}{:}0\leq i\leq 15)$. □

We can now prove our main result for $(2,4;3,m,1)$-frames.

Theorem 4.2. *For* $m\equiv 1 \pmod{12}$ *there is a* $(2,4;3,m,1)$-*frame.*

Proof. Let $m = 12n+1$. There exist $(2,4;3,12n+1,1)$-frames for $n=0,1,2$ and 3 (Lemma 4.1). If there exists a $(2,4;3,n,3)$-frame, then since there is a $(2,4;3,13,1)$-frame and three mutually orthogonal Latin squares of side 4, we can construct a $(2,4;3,12n+1,1)$-frame by applying Theorem 2.2.

Let $N_1 = \{10,12,14,18,22,24,32,34\}$, $N_2 = \{24,27,39\}$ and $N_3 = \{20,28\}$. Let $N = N_1 \cup N_2 \cup N_3$.

Since there exist $(2,4;3,n,3)$-frames for $n\geq 5$ and $n\notin N$ (Theorem 3.9 (i)), we can construct $(2,4;3,12n+1,1)$-frames for $n\geq 5$ and $n\notin N$.

Since there is a (2,4;3,25,1)-frame and three mutually orthogonal Latin squares of side 8, we apply Theorem 2.2 to construct $(2,4;3,24n+1,1)$-frames for $n \geq 5$ and $n \notin N$. This will construct $(2,4;3,12n+1,1)$-frames for $n \in N_1$. Similarly, using a (2,4;3,37,1)-frame and three mutually orthogonal Latin squares of side 12, we construct $(2,4;3,36n+1,1)$-frames for $n \geq 5$ and $n \notin N$. This will provide $(2,4;3,12n+1,1)$-frames for $n \in N_2$.

We now consider the remaining two values: $n = 20$ and $n = 28$. There exist a (2,4;3,16,3)-frame, a (2,4;3,16,1)-frame and three mutually orthogonal Latin squares of side 5. We can apply Theorem 2.2 to construct a (2,4;3,16.3.5+1,1)-frame. This is a (2,4;3,20.12+1,1)-frame. There exist three mutually orthogonal Latin squares of side 7, a (2,4;3,8,6)-frame and a (2,4;3,43,1)-frame. We apply Theorem 2.2 once more to construct a (2,4;3,8.6.7+1,1)-frame or a (2,4;3,12.28+1,1)-frame. □

Finally, we note the following asymptotic result for $(2,4;3,m,1)$-frames.

Theorem 4.3. *There exists a constant m_1 such that for all $m \geq m_1$ and $m \equiv 1$ (mod 3) there exists a $(2,4;3,m,1)$-frame.*

Proof. This follows immediately from the existence result for $DR(2m+1,3,1)$-BIBDs [8] and Theorem 2.4. □

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Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario N2L 3G1

Table 3.1

Starters and adders for complementary $(1,2;3,m,3)$-frames for $m=5,7,8.9$ and 11

$m=5$

Frame	S	1 12 13	2 9 11	3 4 7	6 8 14		
	A	1	7	4	13		
Complement	$\bar{S}$	$-S$					
	$\bar{A}$	14	8	11	2		

$m=7$

Frame	S	1 2 5	3 8 20	4 13 15	6 9 19	10 12 18	11 16 17
	A	4	12	9	10	13	1
Complement	$\bar{S}$	S					
	$\bar{A}$	11	19	16	17	20	8

$m=8$

Frame	S	1 2 3	4 7 13	5 12 23	6 17 21	9 14 19	10 20 22
	A	2	18	7	20	1	13
		11 15 18					
		3					
Complement	$\bar{S}$	$-S$					
	$\bar{A}$	$-A$					

$m=9$

Frame	S	1 2 3	4 6 10	5 15 19	7 13 24	8 16 33	11 22 25
	A	2	15	5	10	6	17
		12 17 20	14 21 26				
		23	12				
Complement	$\bar{S}$	$-S$					
	$\bar{A}$	25	12	22	17	21	10
		4	15				

$m=11$

Frame	S	1 2 3	4 6 9	5 8 15	7 19 25	10 23 30	12 18 27
	A	15	30	16	28	2	1
		13 21 29	14 24 28	16 20 32	17 26 31		
		27	13	10	12		
	$\bar{S}$	$-S$					
	$\bar{A}$	$-A$					

Table 3.2

Starters and Adders for complementary (1,2;3,m,3)-frames for m=13,15 and 17

$m=13$

Starter	S	2 3 27	6 8 17	7 10 15	16 34 38	4 14 20	9 21 28
	A_1	8	38	2	20	33	12
	A_2	3	21	10	16	17	14

$m=15$

Starter	S	1 3 4	8 13 17	10 21 27	7 14 36	6 25 33	5 19 29
	A_1	2	9	8	40	32	27
	A_2	1	11	25	6	28	4
		2 22 34					
		19					
		29					

$m=17$

Starter	S	1 2 37	3 5 24	13 20 23	12 21 25	6 11 35	9 15 29
	A_1	2	40	12	44	25	15
	A_2	1	3	43	46	41	13
		8 33 41	4 32 44				
		20	18				
		29	37				

Table 3.3

Starters and adders for complementary (1,2;3,m,6)-frames for $m=5,6,7$

$m=5$

Frame	S	1 2 3	4 8 12	6 18 29	7 21 24	9 16 22	11 14 23
	A	1	19	8	22	2	28
		13 19 27	17 26 28				
		9	21				
Complement	$\bar{S}$	S					
	$\bar{A}$	6	24	13	27	7	3
		14	26				

$m=6$

Frame	S	1 2 3	4 7 9	5 8 28	10 19 32	11 15 26	13 23 33
	A	2	13	5	19	8	14
		14 22 29	16 21 35	17 25 34	20 27 31		
		21	10	27	1		
Complement	$\bar{S}$	$-S$					
	$\bar{A}$	$-A$					

$m=7$

Frame	S	3 16 22	18 30 36	4 19 24	5 10 34	11 27 31	2 6 33
	A	31	33	37	1	3	5
		17 26 29	1 32 40	20 37 39	15 23 25	12 13 38	8 9 41
		6	20	24	25	26	27
Complement	$\bar{S}$	$-S$					
	$\bar{A}$	8	9	10	11	15	29
		30	32	34	40	2	4

Table 3.4

Starters and adders for complementary $(1,2;3,m,6)$-frames for $m=8,9,13$

$m=8$

Starter	S	2 11 17	25 29 39	4 6 41	3 15 22	1 18 21	5 10 28
	A_1	19	22	11	35	4	9
	A_2	18	34	45	47	21	2
		14 35 36					
		6					
		17					

$m=9$

Starter	S	1 2 4	3 7 13	5 10 31	6 21 40	8 25 37	11 24 35
	A_1	6	21	38	11	15	14
	A_2	4	30	25	7	49	44
		12 26 34	15 22 38				
		41	20				
		32	1				

$m=13$

Starter	S	2 16 27	4 19 28	1 23 30	8 24 29	5 22 25	17 18 36
	A_1	74	70	76	35	48	27
	A_2	1	3	11	69	20	66
		32 44 67	20 47 57	7 15 43	3 9 41	6 10 40	12 14 45
		5	28	17	59	21	24
		25	23	31	7	49	15