Cohesion Stability under Edge Destruction*

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ABSTRACT

In this paper we examine the concept of cohesion which was first introduced in [2] and further studied in [5]. Our purpose is to consider the global effects on cohesion when an edge is deleted from a given graph. The earlier paper dealt with such when an edge was added, and then in a local sense. After some preliminary discussions and definitions we move on to display graphs which are "nearly stable" under edge deletion and to further discover an infinite class of 2-connected graphs which are indeed "stable". This result is followed by some discussion of graphs which have more than one block.

Introduction

The concept of cohesion was first introduced in 1979 [2] wherein the authors discussed its relationship to alliance graphs and the stress placed on individual members of an alliance. In a later paper [5], the stability of a vertex relative to cohesion was defined when edges were added to the underlying graph. Our purpose here is to continue the study of the parameter, in particular with respect to the changes in the underlying graph when edges are deleted. We have in mind here graphs which underly communication networks so that a more global definition of cohesion and stability will be necessary.

The cohesion of a vertex x in a graph G, denoted $\mu(x)$, is the minimum number of edges whose deletion results in a subgraph of G for which x is a cutvertex. Cohesion is then a measure of the "nearness" of a vertex to being a cutvertex. Low cohesion implies that relatively minor damage or failure in the network could suddenly place a node into a position whereby

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its destruction could adversely affect the connectivity of the network. The parameter is related to other parameters where one attempts to disconnect a graph by using a combination of vertices and edges. In particular, Beineke and Harary defined "connectivity pairs" [1]. In their terminology, a vertex of minimum cohesion in a graph along with a set of edges which realizes its cohesion would be a "one, $\mu(x)$" connectivity pair.

One would suspect that cohesion is related to the "induced edge connectivity", which is the edge connectivity of the graph obtained when vertex x is deleted from G and is denoted $\lambda(x)$. It is related to $\mu(x)$ by the following two results from [2]:

Remark 1. Let G be a graph and x a vertex of G, then

a. $\mu(x) \geq \lambda(x)$

b. If $\lambda(x) < \lambda$, then $\mu(x) = \lambda(x)$, where λ is the edge connectivity of G.

In figure 1, $\lambda = 2$, $\mu(x)=3$, $\mu(z)=1$, $\lambda(x)=2$ and $\lambda(z)=1$.

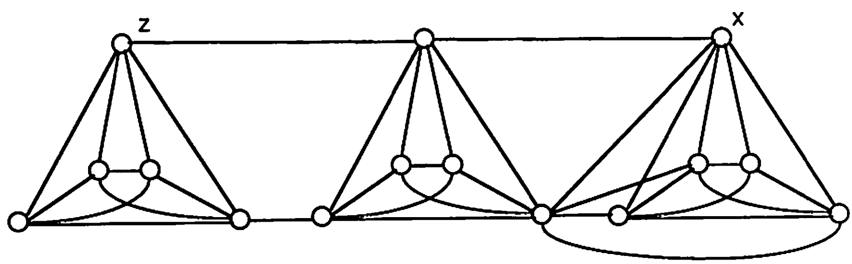


Figure 1. The cohesions of some vertices

In the 1983 paper [5] the authors were concerned with a vertex being stable under edge addition. That is, a vertex was defined to be stable if its cohesion remained unchanged whenever any edge was added to the original graph. The following result from that paper has a counterpart under edge destruction which we will soon present.

Proposition 1: Let u, v, and x be distinct vertices of a connected graph G, where e=uv and e is not an edge of G. If G'=G+e and μ' is the cohesion of a vertex of G', then

a. $\mu(x) \leq \mu'(x) \leq \mu(x) + 1$

b. $\lambda(v) \leq \mu'(v) \leq \mu(v)$.

Note the rather surprising behavior that is indicated in this result. The cohesion of a vertex *cannot* increase when edges are added incident with it, while such an action can greatly reduce the cohesion of that vertex. Said another way, one's cohesion can possibly increase only if edges are added elsewhere in the graph.

In that earlier paper a vertex is called "stable" if $\mu(x) = \mu'(x)$ where $\mu'(x)$ is the cohesion of x in the graph G + e, and e is an arbitrary edge. The graph itself is called "vertex stable" if every vertex is stable. Some theorems in that paper led to a characterization of stable vertices and to the display of some vertex stable graphs among which were the complete bipartite graphs K(m,m), $m \geq 3$.

Preliminary definitions and results.

When one is concerned with communication networks it is essential that the possibility of edge failure or destruction be considered. Such is the fundamental aim here. Proposition 1 has a counterpart for edge deletion. Throughout the remainder of this paper G' will denote the graph G with a single edge deleted and μ' will be the cohesion of a vertex in G'.

Proposition 2. Let u, v, and x be distinct vertices of G with G'=G-e, where e=uv is an edge of G. Then

a. $\mu(x)-1 \leq \mu'(x) \leq \mu(x)$

b. $\lambda'(v) \leq \mu(v) \leq \mu'(v)$.

This proposition means that when an edge is destroyed, the cohesion of its endvertices may go up arbitrarily, but a vertex away from the edge can either not change at all or *only go down* and then by at most one.

Throughout the remainder of this paper we will refer to the cohesion set of a vertex x as a set of edges of size $\mu(x)$ whose deletion leaves a graph in which x is a cutpoint. If one deletes an edge in a cohesion set for a vertex in a graph then the cohesion of that vertex has been lowered by one. The following remark makes this fact formal and thus demonstrates that there is no hope for vertex stability in a graph when considering edge deletion. Such a circumstance forces one to consider stability only in the global sense.

Remark 2. For any vertex v with positive cohesion in a graph, there exists an edge for which $\mu'(v)$ is less than $\mu(v)$.

The cohesion of a graph could be defined in at least two ways. The one which we do not consider here in light of the last remark is to define the cohesion of a graph as the minimum among the cohesions of its vertices. This parameter will be considered in later work. Rather, we take the "average cohesion" point of view in the following definition.

Definition: The cohesion of a graph G, denoted $\mu(G)$, is the sum of the cohesions of its vertices.

The cohesion of a graph can change in all possible ways when an edge is deleted from it. For example, figure 2 displays a graph so that the deletion of edge e results in a graph with higher cohesion. Each pictured vertex has cohesion three while the other vertices have cohesion seven. When e is removed each of its endpoints' cohesion increases to seven while each of the other pictured vertices has its cohesion drop by one.

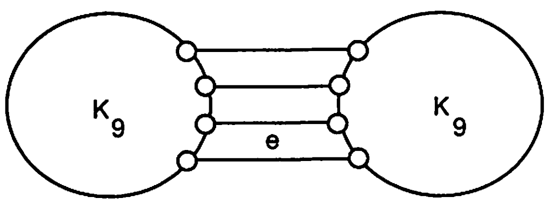


Figure 2. A Cohesion Increasing Graph

Although there are no graphs in which one can delete an arbitrary edge and have all vertex cohesions remain the same, it may be true that a graph has some subset of edges which each have that property. An edge is called a stable edge if its deletion does not change the cohesion of any vertex. The graph in figure 3 has the stable edge x. The following result is clear and shows that there are many graphs with stable edges.

Proposition 3. If G is any graph obtained by adding an edge to a graph which is vertex stable in the sense of [5], then the edge is a stable edge for G. (Under some circumstances one obtains another vertex stable graph and large numbers of stable edges can be created, one after another; see [3] and [4].)

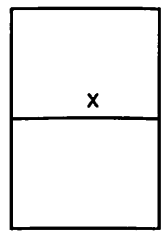


Figure 3. A stable edge

This proposition creates many graphs with large numbers of stable edges when combined with the following result from [5].

Proposition 4. If G is a regular graph of degree r, $r \geq 3$ and x is a vertex with $\lambda(x)=r-1$, then x is stable. (Which means that any regular graph where *every* vertex satisfies this property is vertex stable; persons interested in a learning more about stable edges in graphs are referred to [3] and [4].)

When dealing with the global concept of cohesion of a graph there is another more general kind of stable edge. An edge e is called s-stable if $\mu(G') = \mu(G)$, where G' is obtained by deleting e. Of course a stable edge is s-stable. An edge will be called unstable if it is not s-stable.

A graph G is called a stable graph if each of its edges is s-stable. The next and main part of this paper is concerned with the search for stable graphs. There is one kind of cohesion set which seems to play an important role in such a pursuit. A cohesion set for a vertex x is called a neighborhood cohesion set if it consists exactly of the edges incident with a neighbor of x, excluding the edge incident with x. (Such a set makes x a cutpoint by creating a vertex of degree one adjacent to x.)

Toward a Stable Graph.

It is not readily apparent that there should be graphs which are stable in this "average" cohesion sense. Before we display an infinite class of such graphs we first examine another related class which has some properties which are nicer than our stable graphs and in which every member is "almost" stable. The results in the last section may be used to create these graphs which may be called "asymptotically stable". That is, the ratio of s-stable edges to all edges as the number of vertices in the given class goes to infinity is one.

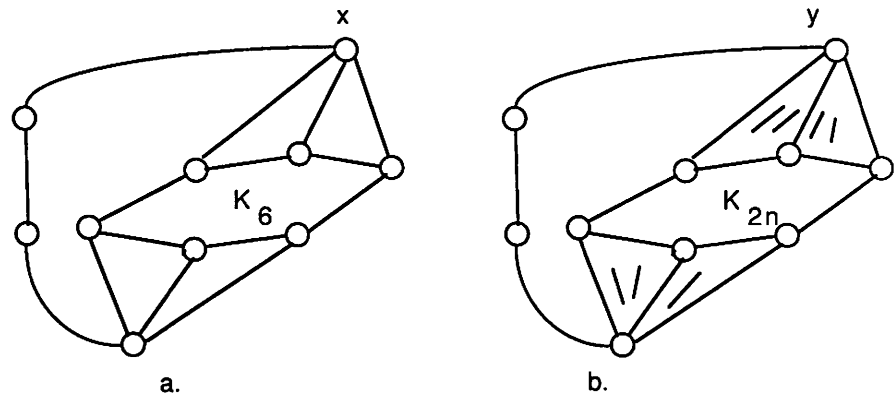


Figure 4. Some nearly stable graphs

The graph in figure 4a has 24 edges, exactly three of which are not s-stable. In fact the edges within the K_6 are stable, the edges incident with the degree two vertices are unstable, and the other edges are s-stable. Notice that if one of these latter edges incident with x is deleted, vertex x has its cohesion remain unchanged, while the other endpoint has its cohesion go from three (neighborhood cohesion set at x) to five (neighborhood cohesion set within K_6). On the other hand, the other K_6 vertices adjacent to x each has its cohesion drop by one. This graph is very close to stable indeed!

In figure 4b we notice that the graph in 4a generalizes when we replace K_6 by K_{2n} and construct the other vertices and edges as shown. The reader can notice that the same three edges remain unstable and that the edges within K_{2n} behave just as before. The removal of an edge incident with y leaves the cohesion of y unchanged while the cohesion of its other endpoint goes from n to 2n-1. However each of the n - l vertices of K_{2n} incident with y has its cohesion drop by 1. Hence if q is the number of edges for an arbitrary graph in this class, then it has q-3 s-stable edges and is thus "asymptotically stable". Note also that each graph in this class has diameter three and both edge and vertex connectivity two.

The search for an infinite class of stable graphs has not been an easy one. At this time there is no theory which will generate such graphs, although a result in [3] allows one to do so in the case where one is willing to have cutpoints in the graphs. Even there, however, the graphs are dependent on the existence of some "available" stable graphs. The following theorem does answer the question as to existence of an infinite class of 2-connected stable graphs, however.

Theorem 1. For each positive integer k, there is a 2-connected stable graph with 16k vertices.

Proof: For k=1 consider the graph in Figure 5. Notice that as in the "almost stable" graph considered earlier, the edges within each copy of K_6 are stable.

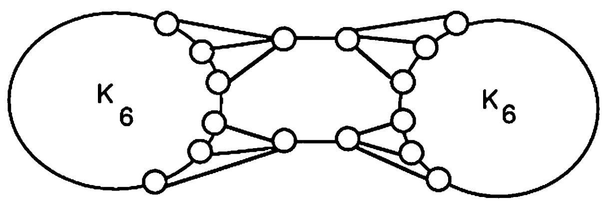


Figure 5. A Stable Graph

Other edges incident with vertices of the K_6's behave just as did their counterparts in that earlier example. The two edges which comprise a 2-edge cutset are the most interesting. If either such edge is deleted, each of its endpoints has its cohesion go from 1 to 5 while the six vertices adjacent to the endpoints and in the two K_6's each have a drop of one in cohesion, as do the endpoints of the other member of the edge cutset. Thus the deletion results in a gain of eight in the cohesion of the graph and an equal loss, creating a stable graph with 16 vertices.

Suppose now that k > 1 and examine the graph in Figure 6.

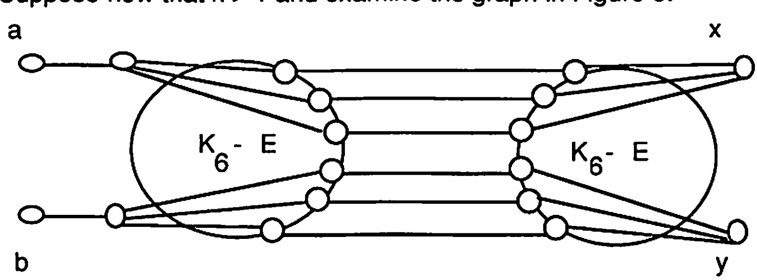
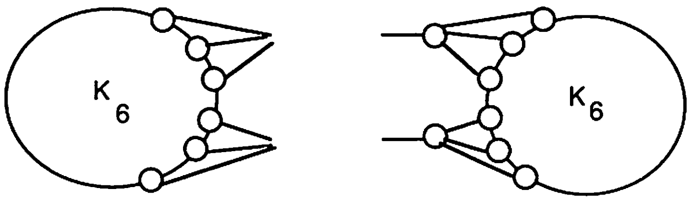


Figure 6. Toward a stable graph.

Each of the large circles in this figure represents a copy of K_6 from which a 1-factor, E, has been deleted. One forms a graph with 16k vertices by first attaching k- l copies of this graph, each to the left of the other by identifying a and b in the right most copy with x and y in its successor, respectively. One completes the construction by attaching the structure in

Figure 7b to the right of the previously created graph at vertices x and y and the structure in Figure 7a to the left at vertices a and b.



a. b.

Figure 7. Finishing a Stable Graph

To see that the graph so created is stable one first notices that the edges arising from the six "parallel" edges in the drawing for the graph of Figure 6 are stable. The cohesions of each of their endpoints is 3 and are therefore not dependent on these edges, which also are in no cohesion sets for other vertices. Furthermore, all other edges behave just as the similar edges did in the case k=1. (One sees here the necessity for removing the 1-factor, since otherwise the "loss and gain" would not work out properly.) Since the graphs are clearly 2-connected we have created the desired stable graphs.

From a communications network point of view the graphs of theorem 1 are not ideal. Although stable graphs would be of great value, one would prefer such graphs with high connectivity and small diameter. These graphs have both edge and vertex connectivity equal to two, which is acceptable. However the diameter of each graph increases with k, which is hardly ideal. We have found no stable graphs which have both high connectivity and low diameter. When one begins to examine graphs of connectivity one, some interesting results can be obtained.

The first result is clear since the cohesion of a cutvertex is zero in any graph. Theorem 2 says that a graph doesn't necessarily obtain stability from stable blocks and, in fact, a stable graph as an endblock must have exactly the opposite effect.

Proposition 5. If G is a graph with blocks B_i, i=1,...,k, then $\mu(G) = \Sigma\, \mu^*(B_i)$, i=1,...,k, where $\mu^*(B_i)$ is the sum of the cohesions of all the vertices of B_i which are not cut vertices of G.

Theorem 2. If a nonblock graph G has an endblock which is stable when considered as a graph, then G is not stable.

Proof. Let G have endblock B which is stable as a graph. Let the one cutvertex of G which is in B be v. We refer to the "graph" B as BG.

Since v is not a cutpoint of BG it has positive cohesion and so let e be an edge of BG which is in a cohesion set of v. In BG'=BG-e let μ' denote the cohesions of the vertices. Then $\mu'(v)=\mu(v)-1$. We write the cohesion of BG' as $\mu(BG') = \mu'(v) + S'$, where S' is the sum of the cohesions of the other vertices of BG'. Hence $\mu(BG') = \mu(v)-1 + S' = \mu(BG)$. If we let S be the sum of the cohesions of the vertices of BG which are not v then we have that $S' = S+1$.

By proposition 5, $\mu(G) = S + T$, where T is the sum of the starred cohesions of the other blocks of G. Suppose the edge e is removed from the graph G. Then $\mu(G\text{-}e) = S' + T$ since the removal of e can affect the cohesion of vertices in no other blocks of G. Then $\mu(G\text{-}e) = S' + T = S+1+ T = \mu(G)+1$ and thus G is not stable.

We finish this section by noting that there are stable graphs with cutvertices. If one replaces any of the edge cutsets of size 2 in the graphs of theorem 1 by the structure given in Figure 8, a graph with a cutpoint is obtained and this new graph remains stable. We also note that in [3] and [4] more results are obtained concerning creating stable graphs with cutpoints from other such graphs.

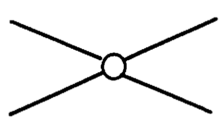


Figure 8. An Insert

The reader is referred to [3] for further results on stable graphs, stable edges, and the examination of these concepts relative to other important properties for communication networks.

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