THE MINIMUM NUMBER OF CYCLES IN GRAPHS WITH GIVEN CYCLE RANK AND SMALL CONNECTIVITY

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ABSTRACT

The cycle rank ,$r(G)$, of a graph $G = (V, E)$ is given by $r(G) = |E| - |V| + 1$. Let $f(k,r)$ be the minimum number of cycles possible in a k-connected graph with cycle rank r. We show $f(1,r) = r$, $f(2,r) = \binom{r+1}{2}$, $f(3,r) = r^2 - r + 1$ and characterize the extremal graphs. Bounds are obtained for $f(k,r)$, $k \geq 4$; the upper bound is polynomial in r.

0. INTRODUCTION

The *cycle rank*, $r(G)$, of the graph $G = (V, E)$ is defined by $r(G) = |E| - |V| + 1$. We wish to determine the minimal number, $f(k,r)$, of cycles possible in a k-connected graph with cycle rank r.

We denote by $\Psi(G)$ the total number of cycles in the graph G and by $\Psi(G; P)$ the number of cycles of G containing the path P of G (P may be just an edge or a vertex). Consequently, $f(k,r) = \min \Psi(G)$ where the minimum is taken over all k-connected graphs G with cycle rank r. We denote the connectivity of G by $\kappa(G)$.

Our graphs will all be finite and simple, i.e., no loops or multiple edges are allowed. Consequently, if $\kappa(G) \geq k$ we have $|V| \geq k+1$ and $|E| \geq \frac{1}{2}k|V|$ so that $r(G) \geq \frac{1}{2}(k^2 - k)$.

We will determine $f(k,r)$ for $k = 1, 2, 3$ and $r \geq \frac{1}{2}(k^2 - k)$ and give bounds when $k \geq 4$.

All undefined notation and terminology can be found in Bondy and Murty [4]; we will, however, denote the complement of a graph G by $\bar{G}$. Since vertices and edges of our graphs are labeled only for purposes of discussion, we need not distinguish between equal and isomorphic graphs and will use equality in both situations.

Beginning in Section 3, the wheel W_{r-1} will be important. We note that W_{r-1} has a unique center for $r \geq 4$ and call the edges incident with this center the *spokes* of W_{r-1}. All other edges are called *rim edges.*

We preface the next section by noting that if G has s components and cycle rank r, then by the addition of $s-1$ edges, a connected graph H with cycle rank $r+s-1$ is formed. By Theorem 1 of the next section, we have $\Psi(G) = \Psi(H) \geq r+s-1$. Thus, graphs with given cycle rank $r \geq 0$ but with no connectivity restriction must be connected if they are to have the minimum number of cycles possible.

1. CONNECTED GRAPHS

Theorem 1. $f(1,r) = r$, $r \geq 0$. If $\kappa(G) \geq 1$ and $r(G) = r$, then $\Psi(G) = r$ iff $e \in E(G)$ implies $\Psi(G;e) \leq 1$.

Proof. Consider the graph H consisting of the path $v_0v_1 \cdots v_{2r}$ with the additional edges $v_{2i-2}v_{2i}$, $1 \leq i \leq r$, (if $r=0$, $H = K_2$). Clearly, $\kappa(H) \geq 1$ and $r(H) = r$ so that $f(1,r) \leq r$.

If $\kappa(G) \geq 1$ and $r(G) = r$, then choose a spanning tree T of G and label the edges of G not in T as e_i, $1 \leq i \leq r$. Since $\Psi(T + e_i) = 1$ and those cycles are distinct, we have $\Psi(G) \geq r$ so that $f(1,r) = r$.

If $e \in E(G)$ implies $\Psi(G;e) \leq 1$, then no e_i and e_j, $i \neq j$, lie in the same cycle of G and $\Psi(G) = r$. On the other hand, if $\Psi(G;e) > 1$ for some $e \in E(G)$, then let T' be a spanning tree of $G - e$ and observe that $\Psi(G) = \Psi(G-e) + \Psi(G;e) > r$. Thus $\Psi(G) = r$ iff $e \in E(G)$ implies $\Psi(G,e) \in 1$. ■

The extremal graphs characterized in Theorem 1 are known variously as cacti and Husimi trees.

2. 2-CONNECTED GRAPHS

Throughout this section, we use many of the commonly known properties of 2-connected graphs without further reference. Statements and proofs of these properties may be found in [3], [4], [5], [8], and [11].

A *major vertex* of a graph G is any vertex with degree at least 3. A path $P = (x \dots y)$ of G in which x and y are major vertices of G and all internal vertices of P have degree 2 in G will be called a ***suspended path*** (SP) of G if $x \neq y$. By $G \ominus P$, we mean the graph G' obtained from G be deleting all internal vertices of P. We also will refer to G as being obtained from G' by *adding a path*. We note that if P and Q are SP's of a graph G, then they are edge disjoint. An SP P of a graph G is called a *non-essential suspended path* (NSP) if $\kappa(G \ominus P) \geq 2$. A 2-connected graph G is said to be *minimally 2-connected* if $e \in E(G)$ implies $\kappa(G - e) < 2$.

We will develop and use several properties of 2-connected graphs for determining $f(2, r)$ and characterizing the related extremal graphs. Of particular importance is the contraction operation. An edge $e = uv$ of a graph G is *contracted* to a vertex x by adding to $G - \{u, v\}$ a vertex x and all edges xw where w is adjacent to u or v in G. We denote this operation by $G \cdot e$ or $G \cdot uv$ and note that $G \cdot e$ is simple. If e does not lie in a C_3 of G, then $r(G \cdot e) = r(G)$; in any case, $r(G \cdot e) \leq r(G)$. We note, further, that $\Psi(G \cdot uv) \leq \Psi(G)$ and the equality is strict iff G has a cycle that contains both u and v but not the edge uv. Finally, we say that the graph G *contracts* to the graph H if H can be obtained from G by recursively contracting edges.

Fact A. If $\delta(G) \geq k$ and $\kappa(G \cdot e) \geq k$ for some $e = uv \in E(G)$, then $\kappa(G) \geq k$.

Proof. Suppose, to the contrary, that G has a vertex cut S with $|S| < k$. If $\{u, v\} \subseteq S$ or $\{u, v\} \subseteq V(G) \setminus S$, then $\kappa(G \cdot e) \leq |S| < k$. Hence, we may assume $u \in S$ and v is in some component C of $G - S$. If $V(C) \setminus \{v\} \neq \emptyset$, then $\kappa(G \cdot e) \leq |S| < k$. Thus, $V(C) = \{v\}$ and $\delta(G) \leq |S| < k$. ■

Fact B. If H is a 2-connected proper subgraph of the 2-connected graph G, then G contains a NSP P for which $E(P) \cap E(H) = \emptyset$.

Proof. (by induction on $|V(G)|$). We clearly have $|V(G)| \geq 4$ and the theorem is obvious if $|V(G)| = 4$. We proceed to the inductive step. Thomassen [12, p. 46] has remarked that, given $e \in E(G)$, it is an easy exercise to show that at least one of $G - e$ and $G \cdot e$ is 2-connected. Thus, if we choose $e = uv \in E(G) \setminus E(H)$ we have either $\kappa(G - e) \geq 2$ and we are done or $\kappa(G - e) = 1$ (so that $\{u, v\} \not\subseteq V(H)$) and $\kappa(G \cdot e) \geq 2$. If $V(G) \setminus V(H) = v$, say, then $N(v) = \{u, w\}$ for some w in $V(H)$ so that we may take $P = uvw$ and we are done. Consequently, we may assume that H is a proper subgraph of $G \cdot e$. It follows from the inductive hypothesis that $G \cdot e$ has an NSP P' for which $E(P') \cap E(H) = \emptyset$. Letting x be the vertex to which uv was contracted, we take P to be P' or P' with an edge subdivided in case x is an internal vertex of P'. Certainly $E(P) \cap E(H) = \emptyset$ and it follows from Fact A that $\kappa(G \cdot e - P') \geq 2$ implies $\kappa(G - P) \geq 2$, i.e., P is an NSP of G. ■

Whitney [14] proved and later Hedetniemi [8] gave a different proof that every minimally 2-connected graph can be constructed from C_3 by recursively subdividing edges and adding paths between distinct vertices. Thomassen [12, p. 45] states that an easy exercise shows that any 2-connected graph can be obtained from a cycle by recursively adding paths between distinct vertices. For later use, we require the following variant of these results.

Fact C. If $\kappa(G) \geq 2$, $r(G) = r \geq 1$ and P is an SP of G, then there is a sequence $G_1, \ldots, G_r$ of 2-connected subgraphs of G such that G_1 is a cycle containing P, $G_r = G$ and, for $2 \leq i \leq r$, $G_{i-1} = G_i \ominus P_i$ for some NSP P_i of G_i.

Proof. (by induction on r). The result is vacuously true for $r = 1$ since G must then be a cycle. We proceed to the induction step. Let G_1 be a cycle of G containing P. By Fact B, there is an NSP P_r for which $E(P_r) \cap E(G_1) = \emptyset$. We apply the inductive hypothesis to $G \ominus P_r$ to obtain a sequence $G_1, \ldots, G_{r-1}$ which, along with $G_r = G$, has the requisite properties. ■

It should be noted that a path P_i of Fact C, although an NSP of G_i, need not be an NSP of G_{i+1}. We do, however, have the following result.

Fact D. If $\kappa(G) \geq 2$ and $r(G) = r \geq 2$, then G has at least $r+1$ NSP's.

Proof. (by induction on r). Using either Fact B or Fact C, we conclude that G has an NSP $P = u \ldots v$. If $r = 2$, then $G - P$ is a cycle and G obviously has 3 NSP's. Hence, we may assume $r - 1 \geq 2$ so that, by the induction hypothesis, $G - P$ has a set R of r NSP's. If none of these NSP's contains u or v as an internal vertex then $R \cup \{P\}$ is the required set. Otherwise, there are two cases to consider.

Case 1. One or both of u and v are internal vertices of NSP's of $G - P$ and, if both, the two NSP's are distinct. We replace each of these NSP's of $G - P$ by an NSP of G that is not an NSP of $G - P$ as follows.

Let $P' = (x \cdots y)$ be an NSP of $G - P$ that contains u as an internal vertex. We replace P' in R by the subpath P'' of P' with end vertices x and u. Since $\kappa(G \ominus P \ominus P') \geq 2$, we see that the graph $G \ominus P'$ with the path $y \cdots u \cdots v$ added has connectivity at least 2. Thus, since u and x are major vertices of G, P'' is an NSP of G but not of $G - P$. Applying similar arguments at v if necessary, we obtain a set of $r+1$ NSP's of G.

Case 2. Both u and v are internal vertices of the same NSP P' of $G - P$. We replace P' in R by the subpath P'' of P' that has end vertices u and v. Since P'' is clearly an NSP of G, we again obtain a set of $r+1$ NSP's of G. ■

In the following result, the subscripts i in C_i are for indexing purposes only and do not necessarily indicate the length of a cycle.

Fact E. If $\kappa(G) \geq 2$, $r(G) = r \geq 1$ and P is an SP of G then $\Psi(G; P) \geq r$.

Proof. Let $G_1, \ldots, G_r$ denote the sequence described in Fact C. Let $C_1 = G_1$ and, for $2 \leq i \leq r$, let C_i be a cycle containing P and P_i (such a cycle exists since $\kappa(G_i) \geq 2$). Since $C_i \neq C_j$ for $i \neq j$, the proof is complete. ■

Our characterization of the extremal graphs in this section will be in terms of forbidden subgraphs one of which is K_4 and the other is a graph we denote by F. The graph F consists of the cycle $v_1v_2v_3v_4v_5v_6$ together with the triangle $v_1v_3v_5$.

A contraction of a graph G to a graph H is said to be *cycle rank preserving* (CRP) iff $r(H) = r(G)$. We note that cycle rank is preserved iff none of the edges contracted to obtain H lies in a triangle. Equivalently, if the edge uv is contracted to the vertex x, then $d(x) = d(u) + d(v) - 2$. We see, in particular, that a graph G has a CRP contraction to K_4 iff G is a subdivision of K_4 and G has a CRP contraction to F iff G consists of a cycle $v_1 \ldots v_n$ together with the added internally vertex disjoint paths $v_nu_1 \ldots u_rv_i$, $v_jw_1 \ldots w_sv_k$ and $v_\ell x_1 \ldots x_tv_m$ where $1 < i \leq j < k \leq \ell < m \leq n$.

Fact F. If $\kappa(G) \geq 2$, P is an SP of G and G contains a subgraph H that has a CRP contraction to K_4 or F then G contains a subgraph H' that contains P and has a CRP contraction to K_4 or F.

Proof. (outline) Either $E(P) \subseteq E(H)$ and we are done or we may assume $E(P) \cap E(H) = \emptyset$. In the latter case, we choose any $e \in E(H)$ and observe that $\kappa(G) \geq 2$ implies e and $P = (u \ldots v)$ lie in a cycle. Thus, G contains vertex disjoint paths $Q = (u \ldots x)$ and $R = (v \ldots y)$ with $V(H) \cap V(Q) = \{x\}$ and $V(H) \cap V(R) = \{y\}$. It can be verified, by examination of the various cases determined by the relative positions of x and y in H, that if H has a CRP contraction to K_4 then a subgraph H' that contains P and has a CRP contraction to K_4 exists and if H has a CRP contraction to F then a subgraph H' of G that contains P and has a CRP contraction to K_4 or F exists. ■

Theorem 2. $f(2,r) = \binom{r+1}{2}$, $r \geq 1$. If $\kappa(G) \geq 2$, $r(G) = r \geq 1$ and $e \in E(G)$ then $\Psi(G;e) \geq r$, and $\Psi(G) = \binom{r+1}{2}$ iff no subgraph of G has a CRP contraction to K_4 or F.

Proof. The theorem is obvious when $r = 1$ so we may assume $r \geq 2$. But then every edge e of G lies in an SP so that, by Fact E, $\Psi(G;e) \geq r$.

Let H be the graph obtained by appending to the path P_{r-1} of length r a vertex v adjacent to all vertices of P_{r-1}. It is easily verified that $\Psi(H) = \binom{r-1}{2}$ and $\Psi(H; e) = r$ for all edges e incident with v. Thus, $f(r,2) \leq \binom{r-1}{2}$ and the inequality $\Psi(G; e) \geq r$ is sharp for $r \geq 1$.

We next prove, by induction on r, that $f(r,2) \geq \binom{r+1}{2}$. Since $r \geq 2$, Fact B implies that G contains an NSP P. By the inductive hypothesis $\Psi(G \ominus P) \geq \binom{r}{2}$ and so, by Fact E, $\Psi(G) = \Psi(G \ominus P) + \Psi(G; P) \geq \binom{r}{2} + r = \binom{r+1}{2}$. We conclude $f(r,2) = \binom{r+1}{2}$, $r \geq 1$ and next show that $\Psi(G) > \binom{r+1}{2}$ if G contains a subgraph H that has a CRP contraction to K_4 or F.

We may assume $\kappa(H) \geq 2$. If $H = G$, it immediately follows that $\Psi(G) > \binom{r+1}{2}$ so we may assume H is a proper subgraph of G and, by Fact B, choose an NSP P satisfying $E(P) \cap E(H) = \emptyset$. By the induction hypothesis $\Psi(G \ominus P) > \binom{r}{2}$ so that $\Psi(G) \geq \Psi(G \ominus P) + \Psi(G; P) > \binom{r}{2} + r > \binom{r+1}{2}$.

It remains to show that if G contains no subgraph that has a CRP contraction to K_4 or F, then $\Psi(G) = \binom{r+1}{2}$. We may assume $\kappa(G) = 2$ since, as is easily shown, every 3-connected graph contains a subdivision of K_4. Furthermore, it is an exercise [4, p. 124] to show that G has a vertex v with $d(v) = 2$. Our proof will be by induction on $|E(G)|$, it being obvious for $|E(G)| = 3$, 4 and 5. Let u and w be the two vertices of G adjacent to v. We have two cases to consider.

Case 1. $uw \notin E(G)$. The graph $G - v + uw$ is 2-connected, has cycle rank r and contains no subgraph that has a CRP contraction to K_4 or F. Thus, by the induction hypothesis, $\Psi(G) = \Psi(G - v + uw) = \binom{r+1}{2}$.

Case 2. $uw \in E(G)$. We have two subcases to consider.

Subcase 2.1. $\kappa(G - v - uw) \geq 2$. Since neither $G - v$ nor $G - v - uw$ contains a subgraph that has a CRP contraction to K_4 or F, we have, by the induction hypothesis,

$$\begin{aligned}\Psi(G) &= 1 + 2\Psi(G - v; uw) + \Psi(G - v - uw) = 1 + 2[\Psi(G - v) - \Psi(G - v - uw)] \\ &\quad + \Psi(G - v - uw) = 1 + 2\binom{r}{2} - \binom{r-1}{2} = \binom{r+1}{2}.\end{aligned}$$

Subcase 2.2. $\kappa(G-v-uw)=1$. Let x be a cut vertex of $G-v-uw$. We observe that $u \neq x \neq w$ and that u and w lie in different components of $G-v-uw-x$. Let G'_u be the component of $G-v-uw-x$ containing u and let G'_w be the remainder of $G-v-uw-x$. Now set $G_u = G-v-uw-V(G'_w)$ and $G_w = G-v-uw-V(G'_u)$ so that $V(G_u)\cap V(G_w) = x$. Since G contains no subgraph that has a CRP contraction to F, it cannot simultaneously be that G_u contains distinct u-x paths and that G_w contains distinct w-x paths. Thus, we may assume the u-x path P in G_u is unique. Now it is easily argued that $G_u = P$. Let e be an edge of P; the induction hypothesis implies $\Psi(G) = \Psi(G\cdot e) = \binom{r+1}{2}$. ∎

Two observations are now in order. First, it follows from Theorem 2 that if $\kappa(G)\geq 2$, $r(G)=r$, $\Psi(G)=\binom{r+1}{2}$ and P is an NSP of G then $\Psi(G;P)=r$.

Secondly, if $\kappa(G)\geq 2$, $r(G)=r$ and u and v are distinct vertices of G then G contains at least $r+1$ u-v paths. If $uv\in E(G)$, this follows immediately from the fact, Theorem 2, that $\Psi(G;uv)\geq r$. If $uv\notin E(G)$ then, again from Theorem 2, $\Psi(G+uv;uv)\geq r+1$ so that G contains at least $r+1$ u-v paths.

3. 3-CONNECTED GRAPHS

Theorem 3. $f(3,r)=r^2-r+1$, $r\geq 3$. If $\kappa(G)\geq 3$, $r(G)=r\geq 3$ and $e\in E(G)$ then $\Psi(G;e)\geq 2r-2$, and $\Psi(G)=r^2-r+1$ iff $G=W_{r+1}$.

Proof. We first prove, by induction on $|E(G)|$, that $\Psi(G;e)\geq 2r-2$. This is obvious for $|E(G)|=6$ so we proceed to the inductive step. Choose $e'\in E(G)$ so that $e\neq e'$ and e' is not adjacent to e. Thomassen [12, p. 46] has shown that either $G-e'$ is a subdivision of a 3-connected graph or $\kappa(G\cdot e')\geq 3$. We consider these two cases.

Case 1. $G-e'$ is a subdivision of a 3-connected graph G'. Since $r(G')=r(G-e')=r-1$, the induction hypothesis implies $\Psi(G-e';e)\geq 2(r-1)-2$. Since, as is easily proven by induction on k, any two edges e and e' of a k-connected graph, $k\geq 1$, lie in at least $k-1$ common cycles, we have $\Psi(G;e)=\Psi(G-e';e)+2\geq 2r-2$.

Case 2. $\kappa(G \cdot e') \geq 3$. If e' does not lie in a triangle then, by the inductive hypothesis, $\Psi(G;e) \geq \Psi(G \cdot e';e) \geq 2r-2$. Suppose, finally, that $e' = uv$ lies in $t \geq 1$ triangles uw_iv, $1 \leq i \leq t$. Then $r(G \cdot e') = r-t$ so that $\Psi(G \cdot e';e) \geq 2(r-t)-2$. Now let x be the vertex of $G \cdot e'$ to which uv was contracted. Each cycle of $G \cdot e'$ containing e and w_ix for some i (there are at least two such cycles) corresponds in a natural way to a pair of cycles of G; one containing e and w_iu but not v (or w_iv but not u) and the other containing e and w_ivu (w_iuv, respectively). Thus, $\Psi(G;e) \geq \Psi(G \cdot e';e) + 2t \geq 2r-2$ in this case also.

We now show that this bound for $\Psi(G;e)$ is sharp. It is easily verified that $r(W_{r+1}) = r$, $\Psi(W_{r+1}) = r^2 - r + 1$ and $\Psi(W_{r+1};e) = 2r-2$ for every edge e incident with the center of W_{r+1}, $r \geq 3$. Consequently $f(3,r) \leq r^2 - r + 1$ and we next show, by induction on $|E(G)|$, that $\Psi(G) \geq r^2 - r + 1$.

This last inequality is obvious if $|E(G)| = 6$ and we proceed to the inductive step. Barnette and Grünbaum [2, or see 12, p. 46] have shown that G contains an edge e such that $G - e$ is a subdivision of a 3-connected graph G'. Since $r(G') = r(G-e) = r-1$, the inductive hypotheses and our earlier result give $\Psi(G) = \Psi(G-e) + \Psi(G;e) = \Psi(G') + \Psi(G;e) \geq (r-1)^2 - (r-1) + 1 + 2r - 2 = r^2 - r + 1$. Thus, $f(3,r) = r^2 - r + 1$ and we now show, by induction on $|E(G)|$, that this bound is achieved only by W_{r+1}.

Suppose $\kappa(G) \geq 3$, $r(G) = r \geq 3$, $\Psi(G) = r^2 - r + 1$ and $|E(G)| \geq 6$. If $r = 3$, then $G = K_4 = W_4$. If $r = 4$, then $G = W_5$, $K_{3,3}$ or $K_2 \times K_3$. But $\Psi(W_5) = 13 < 14 = \Psi(K_2 \times K_3) < 15 = \Psi(K_{3,3})$. Thus we may assume $r \geq 5$. There are two cases to consider.

Case 1. There is an $e \in E(G)$ for which $\kappa(G-e) \geq 3$. Since $f(3,r-1) \leq \Psi(G-e) = \Psi(G) - \Psi(G;e) \leq f(3,r) - (2r-2) = f(3,r-1)$ it follows that $\Psi(G;e) = 2r-2$ and $\Psi(G-e) = f(3,r-1)$. By the induction hypothesis, $G - e = W_r$. Since e is not incident with the center of W_r, easy calculations show that $\Psi(G;e) \geq \frac{1}{2}(r^2 - r - 2) > 2r - 2 = \Psi(G;e)$. Consequently, this case cannot happen.

Case 2. G is minimally 3-connected. If $G = W_{r+1}$, we are done. If not, we use a result of Tutte [13, or see 3, p. 46] stating that G contains an edge e, not in a K_3, for which $\kappa(G \cdot e) \geq 3$. Thus, $r(G \cdot e) = r$ and $f(3,r) \leq \Psi(G \cdot e) \leq \Psi(G) = f(3,r)$ so that, by the inductive hypothesis, $G \cdot e = W_{r+1}$.

Since e does not lie in a triangle of G, e contracted to the center of W_{r+1}. Thus, G consists of a cycle of length r, C_r, and two adjacent vertices u and v, not in C_r, with u adjacent to some number a of the vertices of C_r and v adjacent to the remaining $r - a$ vertices of C_r. But then G has a cycle containing neither u nor v, $2\binom{a}{2}$ cycles containing u but not v, $2\binom{r-a}{2}$ cycles containing v but not u, $2a(r-a)$ cycles using edge e and at least one cycle containing u and v but not e. We conclude that $\Psi(G) \geq r^2 - r + 2 > f(3,r)$. Since this is impossible, $G = W_{r+1}$. ∎

We remark, in closing this section, that although the wheels W_{m+1} are not the 3-connected planar graphs on $2m$ edges with the least number of spanning trees as conjectured by Tutte [4, p. 248], they are the 3-connected graphs with cycle rank r that have the least number of cycles.

4. BOUNDS FOR $f(k,r)$, $k > 3$

C. A. Barefoot has calculated the numbers of cycles in the set of eighteen 4-connected graphs with cycle rank 6, 7, 8 or 9 with the following results. Let e_1, e_2 and e_3 be pairwise independent edges of K_6. Then $f(4,6) = \Psi(K_5) = 37$, $f(4,7) = \Psi(K_6 - e_1 - e_2 - e_3) = 63$, $f(4,8) = \Psi(K_6 - e_1 - e_2) = 91$ and $f(4,9) = \Psi(K_6 - e_1) = 133$. In all cases, the extremal graphs are unique. No other exact values of $f(k,r)$ are known to us. We do, however, have the following bounds for $f(k,r)$, $k \geq 4$.

Clearly $\kappa(K_{k,m}) = k$ and $r(K_{k,m}) = (k-1)(m-1)$ so that $f(k,r) \leq \Psi(K_{k,m})$ when $r = (k-1)(m-1)$. Easy calculations show that, for $4 \leq k \leq m$,

$$\Psi(K_{k,m}) = \sum_{i=2}^{k} \frac{k!m!}{2i(k-i)!(m-i)!} \leq \frac{ek!m!}{4(m-k)!} = \frac{e}{4} \prod_{j=1}^{k} j(m-j+1) .$$

Certainly $j(m-j+1) \le (k-1)(m-1)$ for $2 \le j \le k-3$ and, if $4 \le k \le m$, then $1 \le \left(1 - \frac{1}{m}\right)(k-1)$ so that $m \le (k-1)(m-1)$. Furthermore, if $4 \le k \le m$ then

$$\prod_{j=k-2}^{k} j(m-j+1) \le (k-1)^3(m-1)^3$$

so that for $r = (k-1)(m-1)$ we have

$$f(k,r) \le \Psi(K_{k,m}) \le \frac{e}{4}(k-1)^k(m-1)^k = \frac{e}{4}r^k .$$

Mader [9] has shown that if $p \ge 4$, $t(p) = 3 \cdot 2^{p-3} - 3$, $|V(G)| \ge t(p)$ and

$$|E(G)| \ge t(p)|V(G)| - \binom{t(p)+1}{2} + 1$$

then G contains a subdivision of K_p. We use this result to give a lower bound for the number of cycles in dense graphs. Suppose $\kappa(G) \ge k$, $|V(G)| = n$, $|E(G)| = m$ and $r(G) = r$ where $k \ge 2n^\alpha$ for $\alpha \in (0,1)$. For sufficiently large n, choose an integer p satisfying

$$\frac{\log(m/n)}{\log 4} \le p \le \frac{\log(m/n)}{\log 2} \text{ (so } 2^p \le m/n \le n) .$$

By the result of Mader, G contains a subdivision of K_{p+1} and hence, for sufficiently large n,

$$\begin{aligned}\Psi(G) &\ge \Psi(K_{p+1}) \ge p! \ge p^{\frac{p}{2}} \ge m^{p\log p/(2\log m)} \\ &\ge m^{p\log p/(4\log n)} \ge m^{2\beta\log(8\beta\log n)} \ge m^{\beta\log\log n} \ge r^{\beta\log\log n}\end{aligned}$$

where $\beta = \alpha/(8\log 4)$ and the sixth inequality holds since $m \ge n^{\alpha+1}$.

5. CONCLUDING REMARKS

We expect the lower bound given in Section 4 to be very weak. The upper bound, however, may be the correct order of magnitude. Without attaching much significance to it, we note that the graphs $K_{2,r+1}$ are extremal graphs for $f(2,r)$.

The authors, along with C. A. Barefoot [1] had earlier considered a similar extremal problem where the restraint on cycle rank was replaced by the requirement that the graphs be cubic. There also we were able to give exact results only for connectivity less than 4.

The second author and P. J. Slater [6] have studied the related extremal problem of determining the maximum number of cycles possible in graphs with given cycle rank. There it was shown that the extremal graphs were cubic. We conjectured that the cubic graphs of given order with the maximum number of cycles were precisely those with maximum girth. This conjecture has been disproved by Guichard [7] who discovered, by computer, a cubic graph of order 16 that has one more cycle than the unique cubic graph of order 16 that has girth 6.

Finally, we mention that Perl [10] has studied digraphs with the maximum number of cycles.

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