# CERTAIN IMPLICATIONS OF THE MULTIPLIER CONJECTURE

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Abstract. Some interesting implications of the multiplier conjecture are pointed out in this paper. We show the nonexistence of seven unknown difference sets, assuming the multiplier conjecture. If any of those difference sets is found by other means, it would, therefore, disprove the multiplier conjecture. These difference sets correspond to seven missing entries in Lander's table.

## 1. Introduction.

We refer the reader to [6] and [11] for the basic facts about difference sets and their multipliers. We state Hall's multiplier theorem.

*Hall's multiplier theorem* [9]. Let $D$ be a $(v, k, \lambda)$ difference set in an abelian group $G$. Let $p$ be a prime divisor of $k - \lambda$ such that $(p, v) = 1$. If $p > \lambda$, then $p$ is a multiplier of $D$.

There are several generalizations of the above theorem, all of which require a hypothesis similar to "$p > \lambda$" of the above theorem. All known examples seem to suggest that this condition "$p > \lambda$" (or its variations thereof) is unnecessary. However, all the known proofs require this condition very heavily.

*The multiplier conjecture.* The condition "$p > \lambda$" is superflous in Hall's multiplier theorem.

In this paper, we gather some recent results which seem to be interesting implications of the multiplier conjecture. We also show the nonexistence of $(189, 48, 12)$ difference sets in $Z_3 \times Z_3 \times Z_3 \times Z_7$, $(176, 50, 14)$ difference sets in $Z_4 \times Z_4 \times Z_{11}$, $Z_2 \times Z_2 \times Z_4 \times Z_{11}$ and $Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_{11}$ and $(208, 46, 10)$ difference sets in $Z_4 \times Z_4 \times Z_{13}$, $Z_2 \times Z_2 \times Z_4 \times Z_{13}$ and $Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_{13}$, *assuming* the multiplier conjecture. Consequently, the multiplier conjecture would be disproved if any of these seven difference sets could be found by other methods. Incidentally, these seven difference sets are listed as undecided cases in Lander's table [11].

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## 2. Some recent results.

The following recent results assume the multiplier conjecture:

**Theorem 1.** *Assume that the multiplier conjecture is true. Let $D$ be a $(v, k, \lambda)$ difference set in an abelian group $G$.*

- (i) *Arasu [1]. If $k - \lambda \equiv 2 \pmod 4$, then either $k = 2\lambda$ or $2\lambda + 1$, according as $\lambda$ is even or odd. Consequently, the corresponding design is Hadamard (up to complementation).*
- (ii) *Pott [13]. Assume that $n = k - \lambda$ is a nonsquare and $(v, n) = 1$. Let $H$ be a normal subgroup $G$ of order $h$ ($G$ need not be abelian) and suppose $G/H$ is abelian. If $p$ is a prime divisor of $n$, then $vh(-1)^{(\frac{v}{h}-1)\frac{1}{2}}$ is a square modulo $p$.*

*In [1], Arasu showed that the multiplier conjecture implies a conjecture of Hall and one of Ryser, both for the case $k - \lambda \equiv 2 \pmod 4$.*

In Theorem 1, (i) (resp. (ii)) holds, whenever 2 is a multiplier of $D$ (resp. $p$ is a $G/H$ multiplier of $D$).

## 3. Lander's table.

In [11], Lander concludes by giving all parameter triples $(v, k, \lambda)$ satisfying the basic equation $k(k-1) = \lambda(v-1)$ and which do not contradict the conclusions of Schutzenberger's theorem and the Bruck-Ryser-Chowla theorem, $k \leq 50$ and $k \leq v/2$. For a fixed such $v$, Lander considers all possible abelian groups of order $v$ and obtains 268 triples. Of these, 65 correspond to known difference sets, 178 are shown not to exist and 25 are undecided. Arasu ([2] and [3]) filled the entries 34, 48, 49, 147, 180-183 with answer "no", Bozikov [7] independently knocked off entries 34, 48 and 49, and Jungnickel and Pott [10] independently supplied a short but elegant proof to knock off entries 180-183. In an unpublished work, Turyn [15] and Arasu and Reis [5] filled entries 113 and 115 respectively, both with answer 'yes'. Recently, Davis [8], in his Ph.D. thesis, has constructed an infinite family of difference sets the parameters of which include those discovered by Turyn, Arasu and Reis. Thus, only 15 more entries in Lander's table now need to be filled. In the rest of this paper we will show: assuming the multiplier conjecture, we can fill 7 more entries in Lander's table with a negative response.

## 4. Background material.

In this section, we lay some ground work.

Let $D$ be a $(v, k, \lambda)$ difference set in an abelian group $G$. Let $H$ be a subgroup of $G$ of index $m$. Let $G/H = \{H_0, H_1, H_2, \ldots, H_{m-1}\}$. Define $s_i = |D \cap H_i|$ for $i = 0, 1, \ldots, m-1$. The following is well-known (for instance, see [4] or [11]).

**Proposition 4.1.**

$$\text{(i)} \quad \sum_{i=0}^{m-1} s_i = k$$

$$\text{(ii)} \quad \sum_{i=0}^{m-1} s_i^2 = k - \lambda + \lambda|H|.$$

**Theorem 4.2.** *(McFarland and Rice [12]). There exists a translate of D fixed by all the numerical multipliers of D.*

**Proposition 4.3.** *With notations as above, if* $-1$ *is a* $G/H$*-multiplier of* $D$*, then all but one of the* $s_i$ *are* $\equiv \lambda|H| \pmod 2$ *and the remaining* $s_i$ *is* $\equiv k - \lambda + \lambda|H| \pmod 2$.

Proof: See Corollary 6.1 in [4].

## 5. (189, 48, 12) case.

**Proposition 5.1.** *(189, 48, 12) difference sets do not exist in* $Z_3 \times Z_3 \times Z_3 \times Z_7$*, admitting 2 as a multiplier. Thus, the validity of the multiplier conjecture establishes their nonexistence.*

Proof: Let $D$ be a hypothetical (189, 48, 12) difference set in $G = Z_3 \times Z_3 \times Z_3 \times Z_7$ with multiplier 2. Let $H = (0) \times (0) \times (0) \times Z_7$. Since 2 is a multiplier of $D$, it follows that 2 is also $G/H$-multiplier of $D$. But $G/H \cong Z_3 \times Z_3 \times Z_3$ and $2 \equiv -1$ (mod exponent of $G/H$). Thus $-1$ is a $G/H$ multiplier of $D$. With $s_i$ ($i = 1, \ldots, 26$) as in Section 4, it follows from Proposition 4.3 that each $s_i$ is even. Since $2^2 = 4$ is also a multiplier of $D$, we may assume $D$ is a union of orbits of $G$ under $\langle x \to 4x \rangle$ (using Theorem 4.2). We note that 4 fixes each coset $H_i$ of $H$ setwise and forms the orbits $\{0\}$, $\{1,2,4\}$ and $\{3,5,6\}$ on $Z_7$. Hence, each $s_i = 0$ or 4. Thus, $D$ picks up 12 such orbits of size 4. But each of these orbits forms a (7, 4, 2) difference set in $Z_7$, thereby producing 24 differences for each element of $H$, contradicting $\lambda = 12$. Hence, $D$ cannot exist.

## 6. (176, 50, 14) case.

**Proposition 6.1.** *The unknown (176, 50, 14) difference set in* $G = Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_{11}$ *cannot admit 3 as a multiplier. Hence, if the multiplier conjecture is true, then such a difference set cannot exist.*

Proof: Let $D$ be a (176, 50, 14) difference set in $G$ having 3 as a multiplier. Assume that $D$ is fixed by the multiplier 3. Let $H = (0) \times (0) \times Z_2 \times Z_2 \times Z_{11}$. Define $H_i, s_i$ ($i = 0, 1, 2, 3$) be as in Section 4. Then by Proposition 4.1,

$$s_0 + s_1 + s_2 + s_3 = 50$$
$$s_0^2 + s_1^2 + s_2^2 + s_3^2 = 652.$$

Solving these equations, we find 4 sets of solutions for the $s_i$, viz,

(i) (14, 14, 14, 8)
(ii) (17, 11, 11, 11)
(iii) (16, 14, 10, 10)
(iv) (15, 15, 11, 9).

Each $H_i$ is fixed setwise by $\langle x \to 3x \rangle$. The orbits of $G$ under $\langle x \to 3x \rangle$ are of the form: $(i_1, i_2, i_3, i_4, 0)$, $(i_1, i_2, i_3, i_4) \times S$, where $S$ = set of nonzero squares in $Z_{11}$ or set of nonsquares of $Z_{11}$. [Note: there are 16 singleton orbits, 4 in each $H_i$ and 32 orbits of size 5, 8 in each $H_i$). In any case, $S$ forms a (11, 5, 2) difference set in $Z_{11}$.

If the solutions of $s_i$ are (ii), (iii) or (iv), $D$ must pick up 9 orbits of size 5, in which case differences within the same orbit yield $(0, 0, 0, 0, x)$, $x \in Z_{11}^*$, twice, for a total of 18 times as $d - d'$, $(d, d' \in D)$, contradicting $\lambda = 14$.

It remains to consider the case $(s_i) = (14, 14, 14, 8)$. In this case, $D$ picks up 7 orbits of size 5 and 15 singleton orbits. The singleton orbits form a (16, 15, 14) difference set in $Z_2 \times Z_2 \times Z_2 \times Z_2 \times (0)$. But then it can easily be seen that some nonzero element of $Z_2 \times Z_2 \times Z_2 \times Z_2 \times (0)$ occurs as a difference $d - d'$, $(d, d' \in D)$ at least once more, again contradicting $\lambda = 14$. Thus, $D$ does not exist, completing the proof of Proposition 6.1.

Remark: Proof of Proposition 6.1 uses ideas of [14].

**Proposition 6.2.** *The unknown (176, 50, 14) difference sets in $Z_4 \times Z_4 \times Z_{11}$ and $Z_4 \times Z_2 \times Z_2 \times Z_{11}$ cannot admit 3 as a multiplier. Thus, the validity of multiplier conjecture would establish their nonexistence.*

Proof: Identical to the proof of Proposition 6.2, considering more cases, however.

## 7. (208, 46, 10) case.

**Proposition 7.1.** *The unknown (208, 46, 10) difference sets in $Z_4 \times Z_4 \times Z_{13}$, $Z_2 \times Z_2 \times Z_4 \times Z_{13}$ and $Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_{13}$ do not admit 3 as a multiplier. Hence, the validity of multiplier conjecture would prove their nonexistence.*

Proof: Along the lines of Proposition 6.1, noting that the equations

$$s_0 + s_1 + s_2 + s_3 = 46 \text{ and}$$
$$s_0^2 + s_1^2 + s_2^2 + s_3^2 = 556$$

have 4 sets of solutions, viz,

(i) (13, 13, 13, 7)
(ii) (15, 13, 9, 9)
(iii) (14, 14, 10, 8) and
(iv) (16, 10, 10, 10).

Even though the nonsquares of $Z_{13}$ do not form a difference set in $Z_{13}^*$ (unlike in $Z_{11}^*$) arguments similar to those in the proof of 6.1 (with minor changes) work to finish the proof. We omit the details.

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