A SURVEY OF EXTREMAL COVERINGS OF PAIRS AND TRIPLES*

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Abstract. Suppose that one is given v elements, and wishes to form a design that covers all t-sets from these elements exactly once. The design is to obey the further restriction that the longest block in the design has k elements in it; furthermore, we wish the design to contain as few blocks as possible.

The number of blocks in such a minimal design is denoted by the symbol $g^{(k)}(1,t;v)$; determination of this number is closely connected with the behaviour of Steiner Systems. Recently, much progress has been made in two important cases, namely, when t = 2 (pairwise balanced designs) and t = 3 (designs with balance on triples). This survey paper presents the subject from its inception up to current results.

1. Introduction.

Supppose that we start with a variety set comprising v elements denoted by the integers 1,2,3, ... , v; let t be an integer less than v. We define a perfect covering of t-sets to be a selection of subsets formed from the variety set such that each subset is proper, and such that each t-set occurs exactly λ times. For example, if v = 7, t= 2, and λ = 3, then the following family of subsets is a perfect covering:

1236, 2347, 3451, 4562, 5673, 6714, 7125, together with twenty-one pairs 12,13,14, ... , 56,57,67.

Another perfect covering would be provided by taking the first 7 sets in the above family (the quadruples) and repeating them to give a covering in 14 sets.

Not a great deal of work has been done on coverings with $\lambda > 1$, and we shall only be considering the case $\lambda = 1$ in this paper. With $\lambda = 1$, there are two trivial solutions. We shall exclude the trivial case when all of the elements are placed in a single set {1,2,3, ... ,v}. However, there is also always the trivial (but important) solution where we take a covering family made up of all possible t-sets. However, in order to have interesting covering families to study, we need further restrictions. Three particular kinds of coverings have been studied in considerable detail.

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(1) If we impose the restriction that all sets in the covering family have the same cardinality k, then we are dealing with Steiner Systems S(t,k,v); see, for example, [25]. Steiner Systems are particularly symmetric and fascinating, but only a dozen or so are known for which t is greater than 3. For t = 2, the Steiner Systems S(2,k,v) are just the well known Balanced Incomplete Block Designs, about which there is a very large literature. However, even in the case of Balanced Incomplete Block Designs, it frequently happens that an appropriate design does not exist. For example, if we take v = 11, then there are 55 pairs from an 11-set; if we are going to cover these pairs by blocks of equal size k (with k greater than 2), then k(k-1)/2 must divide 55. But 55 is not divisible by 3, or 6, or 10, or 15, or 21, or 28. This illustration emphasizes the fact that, if we are going to have a perfect covering family, then some of the block sizes will generally have to be different from one another.

(2) We might restrict the covering family by permitting two distinct block sizes in the covering family. As an example, let v = 12 and t = 3. Then a perfect covering of triples can be displayed as follows; take A = {1,2,3,4,5,6} and B = A+6 as two disjoint sextuples. Define a 1-factorization of the set A (see [20], for example) as follows:

$F_1 = \{(1,2),(3,4),(5,6)\}$, $F_2 = \{(1,3),(2,5),(4,6)\}$, $F_3 = \{(1,4),(3,5),(2,6)\}$,
$F_4 = \{(1,5),(2,4),(3,6)\}$, $F_5 = \{(1,6),(2,3),(4,5)\}$.

Also, define a 1-factorization of B by taking the sets $G_i = F_i + 6$. Then it is easy to see that we can construct a perfect covering family by taking the two sets A and B, together with the 45 quadruples formed by taking every pair from F_i with every pair from G_i (this gives 9 quadruples for any fixed i, and i may take on the values 1,2,3,4,5).

(3) Another restriction on the covering family that leads to problems of extreme interest is to demand that the family have minimal cardinality. This leads to the introduction of the concept of a g-covering. A g-covering is a covering family such that no other covering family has smaller cardinality; we denote this minimal cardinality by g(1,t;v). For example, it can be shown that the family just constructed in the previous paragraph is minimal for v = 12 and t = 3; thus g(1,3;12) = 45. An example of a non-minimal family for v = 12 and t = 3 would be provided by taking the set C = {1,2,3,4,5,6,7,8,9,10,11} and the 55 triples of the form {c,d,12}, where c and d are elements of C; this would be a perfect covering in 56 sets.

We shall discuss g-coverings of pairs (t=2) and of triples (t=3). Only a few results are known for t > 3, largely because progress depends heavily on a knowledge of Steiner Systems.

2. Coverings of Pairs; the Numbers g(1,2;v).

We first mention that **perfect pair coverings**, that is, families that cover all pairs exactly once, also appear in the literature under the name of **finite linear spaces** or under the name of **pairwise balanced designs.** We prefer the first and third of these three terms. Pairwise balanced designs also subsume the well known Balanced Incomplete Block Designs (BIBDs); these are just pairwise balanced designs for which only a single block size is involved.

One of the first results on perfect pair-coverings was given in 1948 by de Bruijn and Erdos **[1]**. They proved that the cardinality of the minimal covering family was given by g(1,2;v) = v. Furthermore, they showed that this minimum could always be achieved by taking one block C containing all elements of the v-set except v, and then taking all pairs {c,v}, where c ranges over the v-1 elements of C (this particular configuration is often called a near-pencil). In the special case when $v = k^2-k+1$ and a finite geometry exists (that is, k-1 is a prime power), then the finite geometry provides a second minimal covering in v sets, and those v sets all contain precisely k elements (this second type of minimum, of course, can only occur if v is an integer selected from the set { 7, 13, 21, 31, 57, 73, 91, ...}; it is just a Balanced Incomplete Block Design with parameters (v,b,r,k,λ), where $v = b = k^2-k+1$, $r=k$, $\lambda=1$).

Actually, it is not too illuminating to consider the number g. In **[14]**, the quantity $g^{(k)}(1,2;v)$ was introduced; this is the minimum size of a covering family that contains at least one block of length k but no block of larger size. Henceforth, in this section, we shall write g(v) for g(1,2;v), and we shall write $g^{(k)}(v)$ for $g^{(k)}(1,2;v)$. We thus see that the quantity g(v) is rather accidental; it is just the global minimum for the quantity $g^{(k)}(v)$, the minimum being taken over all values of k. We shall now provide a summary of developments in the theory, phrasing the results in terms of $g^{(k)}(v)$, since results appear more simple and direct in this context.

The first important step forward was taken by Woodall **[26]** in 1968. Woodall proved a general result for any t; in the special case when t = 2, his result specializes to

$$g^{(k)}(v) \geq W = 1 + (v-k)(3k-v+1)/2.$$

We call W the Woodall bound. Then Stanton and Kalbfleisch showed **[21]** (again we specialize their result to the case t = 2) that

$$g^{(k)}(v) \geq 1 + k^2(v-k)/(v-1).$$

We call this quantity SK. If we graph the quantity SK, we obtain a particularly simple proof of the Erdos-de Bruijn result (see [18] or [19]). Of course, we should also mention the combinatorial bound C; for small values of k, it is extremely useful to note that

$$g^{(k)}(v) \geq C = v(v-1)/k(k-1).$$

The combinatorial bound obviously gives the exact result for $k = 2$.

There are many open questions for particular values of k and v. However, a great deal of progress has now been made. In [14], it was shown that the Woodall Bound gave the exact value for $g^{(k)}(v)$ as long as $k \geq (v-1)/2$ and v does not have the form 4m+1 (the latter case was handled in [6]). This basically means that the combinatorial bound C holds (approximately) from k equal to 2 until k reaches a value in the neighbourhood of $v^{1/2}$; the SK bound then holds, approximately, from k in the neighbourhood of $v^{1/2}$ to k in the neighbourhood of v/2; then the bound W holds exactly when k exceeds v/2 (special consideration is needed at the transitional points where we change from one bound to another). It thus becomes of importance to see just how close $g^{(k)}(v)$ is to the bound SK.

In [7], it was pointed out that the bound SK is exact when all other blocks meet the block of length k and when they form a resolvable balanced incomplete block design. A very important extension of this result was given by Stinson in [24]; Stinson defined s to be the greatest integer in the quantity (v-1)/k, and showed that

$$g^{(k)}(v) \geq 1 + (v-k)(2sk-v+k+1)/s(s+1).$$

We call this bound S (the Stinson bound). It is an improvement over the SK bound, except when SK is exact. Furthermore, if S is integral, then S is exact if and only if all blocks meet the base block of length k and form a resolvable pairwise balanced design with block sizes s and s+1. It is possible to give a particularly straightforward account of the behaviour of the Stinson bound, and we do this in the next section.

3. The Stinson Bound for t = 2.

Suppose that we consider all the blocks of the covering with the exception of the base block of length k. We say that there are b_i blocks of length i (in a block of length i, there may be i points not on the base block, or there may be one point on the base block and i-1 points not on the base block). Then, we may write:

(1) $g^{(k)}(v) - 1 = g - 1$ (for short) $= b_2 + b_3 + b_4 + b_5 + \ldots$

We now count all pairs not in the base block and find:

(2) $v(v-1)/2 - k(k-1)/2 = b_2 + 3b_3 + 6b_4 + 10b_5 + \ldots$

Now let the points in the base block be called j (j = 1,2,3, ..., k), and let b_{ij} denote the number of blocks of length i through point j. Clearly,

$$\Sigma_i (i-1)\, b_{ij} = v\text{-}k, \text{ for all i;}$$

thus we have $\Sigma_{i,j} (i - 1)\, b_{ij} = k\,(v\text{-}k)$. However, we must not forget the blocks that do not meet the base block; suppose that b_{i0} denotes the number of blocks of length i that do not meet the base block; then $\Sigma_i (i - 1)b_{i0} = \varepsilon$, where ε is a non-negative integer. Adding all of these expressions together gives us our third equation:

(3) $k(v\text{-}k) + \varepsilon = b_2 + 2b_3 + 3b_4 + 4b_5 + \ldots$

We now multiply these equations by the quantities s(s+1)/2, 1, and -(s+1), respectively, and add the three equations. This has the effect of eliminating the terms in b_{s+1} and b_{s+2} to leave the result:

(4) $s(s+1)(g\text{-}1) + (v^2\text{-}v+k^2\text{-}k) = 2(s+1)k(v\text{-}k) + 2\varepsilon(s+1) + 2P,$

where P is the non-negative integer

$$b_s+b_{s+3}+3(b_{s-1}+b_{s+4})+6(b_{s-2}+b_{s+5})+\ldots$$

Then we find

(5) $g\text{-}1 = (v\text{-}k)(2sk\text{-}v+k+1)/s(s+1) + 2\varepsilon/s + 2P/s(s+1).$

Equation (5) gives the Stinson Bound when we ignore the two last terms, which are certainly non-negative.

It is easy to see that the optimal value for s, in Equation (5), is the greatest integer in (v-1)/k; suppose that we assign that particular value to s.

Now consider $s = 1$ (that is, k lies between v/2 and v). The Stinson bound then becomes the Woodall Bound W and we have

(6) $g-1 = (v-k)(3k-v+1)/2 + 2\varepsilon + b_4 + 3b_5 + 6b_6 + \ldots = (W-1) + 2\varepsilon + P.$

Equation (6) immediately gives us the

Theorem. The Woodall Bound can only be achieved if all blocks meet the long block of length k (that is, $\varepsilon = 0$) and if all the other blocks have lengths 2 and 3 (thus the other blocks fall into resolution classes with blocks of lengths 1 and 2 hanging on to the points of the base block).

That the Woodall bound is actually achieved in this region can be shown by a straightforward construction (cf. [14]).

We now turn our attention to the case when $s = 2$ in Equation (5); this is when k lies between v/3 and v/2. Equation (5) can then be written in the form

(7) $g = 1 + (v-k)(5k-v+1)/6 + \varepsilon + P/3 = S + \varepsilon + (b_2+b_5)/3 + b_6+\ldots$

Now the Stinson Bound may not be integral, but we see that it can not be achieved (in the nearest-integer sense) unless $\varepsilon = 0$ (recall that ε is an integer). Thus, we have the

Theorem. When $s = 2$ (that is, k lies between v/3 and v/2), the Stinson Bound is only attained if all blocks meet the base block.

However, we can go further; the numerator $(v-k)(5k-v+1)$ is an even integer, and so the quantity $P/3 = (b_2+b_5)/3$ can only assume the values 0/3, 1/3, or 2/3. Thus we have the

Theorem. If the Stinson Bound is met with $s = 2$, then all blocks must have lengths 3 and 4, except that there may be one or two exceptional blocks with lengths of 2 or 5.

It is relatively easy to specify when these rogue blocks appear. We let $k = 6t+a$, and let $v = 2k+6u+b$; then one can carry out the requisite algebra and find

(8) $b_2+ b_5 = 3\psi(X/6) - X/2,$

where $X = (a+b)(3a+1-b)$ and ψ denotes the ceiling function. Of course, in any particular case, it is probably easier to carry out the specific elimination that led to Equation (5).

For example, let us consider $v = 24$ and $k = 8$; then $a = 2$, $b = 2$, and $b_2 + b_5 = 2$. In general, we find that, for $X = 0, 2, 4 \pmod 6$, respectively, then $b_2 + b_5 = 0, 2, 1$, respectively.

We should add that the procedure used in obtaining Equation (5) is equally useful for other values of s. If $s = 3$ (that is, k lies between v/4 and v/3), then we get

(9) $g-1 = (v-k)(7k-v+1)/12 + 2\varepsilon/3 + P/6.$

From this equation, we can deduce easily that $\varepsilon = 0$ if the Stinson Bound is met; furthermore, one can get a quantitative limitation on the number of rogue blocks in this case. However, this result is only a special case of a much more general theorem. Let us return to Equation (5), with s having its optimal value. If the Stinson bound is to be met, it is clear that $2\varepsilon/s$ can not exceed unity; hence the maximum length of any block disjoint from the base block is s/2 when s is even and (s+1)/2 when s is odd. This shows that the disjoint blocks are relatively "short", in order to keep down the value of ε. On the other hand, let us look at the quantity P and let us suppose that there is a block of length (s+1)-z disjoint from the base block. It will contribute an amount (z+1)z/2 to P and an amount (s-z) to ε. The total contribution from this one disjoint block will thus be

$$(z+1)z/s(s+1) + 2(s-z)/s = \{z^2 - z(2s + 1) + 2s(s + 1)\}/s(s + 1).$$

This quadratic function starts at the value 2 when $z = 0$ and decreases to the value $(s^2+s+2)/(s^2+s)$, which is always greater than 1, for $z = s-1$, that is, for block length 2. We thus see that any block disjoint from the base block must contribute more than one unit to Equation (5), and thence we obtain the following result.

Theorem. If the Stinson Bound is to be met, in the nearest-integer sense, then all blocks must meet the base block of length k.

Recently, a further strengthening of the Stinson bound has been achieved. Rolf Rees [4], in his doctoral dissertation, was able to obtain a bound R that is, in some cases, stronger than S; if the bound R is exact, then all blocks must meet the base block of length k and they must have block sizes equal to s, s+1, or s+2. The exact properties of the R bound are rather complicated, but they are described in detail by Rees and Stinson [5]; we simply note that, if τ is defined to be the residue of (v-k), modulo s, then

$$R = 1 + \{(v-k)(2k(s-1+\tau/s)-v+k+1) + 2k\tau(1-\tau/s)\}/\{s^2-s+2\tau).$$

Tables have now been produced that give the values of $g^{(k)}(v)$ for most small values of v and k; in particular, [11] and [12] give the results for all $v \leq 22$ except in the cases v = 17,18, and 19, with k = 4. The case k = 4 is discussed, for all other values of v, in [22]; the value for $g^{(4)}(17)$ is quoted in [23], and will be established in forthcoming papers by methods that may possibly also work for v = 18 and v = 19. A few initial results for the case k = 5 are given in [9]. Buskens, Rees, Stanton, and Stinson, have extended the census of $g^{(k)}(v)$ up to v = 31 (with a number of blanks). However, the next natural range of values, $32 \leq v \leq 57$, presents many opportunities for the discovery of exotic designs.

Rees [4] has given constructions for the cases v = 2k+2, 2k+3, and 2k+4 (see also [2]). The case v = 2k+1 was already given in [14] and [6]. A special instance of the case v = 2k+7 appears in [2].

4. Coverings of Triples; the Numbers g(1,3;v).

In this section, we deal with the case t = 3, that is, we are looking at the coverings of triples. We shall now write g(v) to mean g(1,3;v); similarly, we shall use $g^{(k)}(v)$ to designate $g^{(k)}(1,3;v)$, that is, the minimal cardinality of a covering family that does includes a block of length k but possesses no block of larger size.

For t = 3, the combinatorial bound is given by

$$C = v(v-1)(v-2)/k(k-1)(k-2).$$

The SK bound is given by the expression

$$SK = 1 + k(k-1)^2(v-k)/2(v-2),$$

and the Woodall bound W is given by

$$W = 1 + k(v-k)(3k-v-1)/4.$$

Also, if s denotes the greatest integer in the quantity (v-2)/(k-1), then the Stinson bound is given by

$$S = 1 + k(v-k)(2s(k-1)-v+k+1)/2s(s+1).$$

The over-all behaviour of $g^{(k)}(v)$ is described in [10], where a table of values is given for v up to 26; the C bound predominates for small values of k, and then the SK or S bound takes over. Finally, the W bound is exact (after being increased by a small perturbation factor) in the range between v/2 and v (the "long block" case).

A survey up to 1985 of progress in the $g^{(k)}(v)$ problem is found in [8]; the values of $g^{(k)}(v)$, for most of the values in the range $v \leq 26$, appear in [21], [13], [16], and [18]. These papers also introduced the hypothesis, established in some instances, that minimal families were to be found by puncturing Steiner systems, especially inversive planes. (A punctured Steiner System is merely one from which one or more points have been deleted.) This hypothesis was established in [3], where Hartman, Mullin, and Stinson proved that g(v) is basically a step function. If we have $v = q^2+1$, where q is a prime power, then the inversive planes $S(3, q+1, q^2+1)$ actually give the value of g; furthermore, this value of $q(q^2+1)$ is also the minimum when $v = q^2+1-\alpha$, where α is small relative to q (for an exact formulation of the permissible size of α, we must refer to [3]).

The results for v = 11 and 13 are particularly difficult to obtain; for v = 11, see [15]. For v = 13, see [17] and a sequel that is still to appear. There remains some work to be done for k-values lying in the region with $v \leq 26$. However, the greatest opportunity for discovery lies in the range $27 \leq v \leq 50$; just as we are handicapped, in the case of pair coverings, by the non-existence of the projective plane on 43 points, so the problem of covering by triples is made difficult in this last range by the non-existence of an inversive plane on 37 points.

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