Partitioning sets of quadruples into designs II

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ABSTRACT

It is shown that the collection of all $\binom{9}{4}$ distinct quadruples chosen from a set of nine points can be partitioned into nine mutually disjoint 3-(8,4,1) designs in just two non-isomorphic ways. Two proofs of this result are given: one by direct construction, the other by extending sets of eight mutually disjoint 2-(7,3,1) designs based on a set of eight points.

Introduction.

A *t-design* based on a v-set, X, is a collection of k-subsets (blocks) chosen from X in such a way that each unordered t-subset of X occurs in precisely λ of the blocks. Such a design has parameters t-(v,k,λ). None of the designs considered in this paper has repeated blocks.

Two t-(v,k,λ) designs are said to be *disjoint* if and only if they have no block in common. If the set of all the $\binom{v}{k}$ k-sets chosen from X can be partitioned into mutually disjoint t-(v,k,λ) designs, then these designs are said to form a *large set*. For example, Kirkman [5] in 1850 and Bays[1] in 1917 showed that the 84 triples chosen from a 9-set can be partitioned into a large set of 2-(9,3,1) designs in just two non-isomorphic ways; see also Kramer and Mesner [6] and Mathon, Phelps and Rosa [8]. Another example is provided by Kreher and Radziszowski [7] who showed that the 3432 7-subsets of a 14-set can be partitioned into a large set of two 6-(14,7,4) designs.

A necessary condition for the existence of a large set of t-(v,k,λ) designs, each with b blocks, is that b divides $\binom{v}{k}$. However, there are examples where this condition is satisfied but a large set does not exist. For example, Cayley [3] showed that from the $\binom{7}{3}$ triples on seven points at most two disjoint 2-(7,3,1) designs can be formed; the remaining 21 triples form a 2-(7,3,3) design which cannot be decomposed into smaller designs. Another example is provided by Kramer and Mesner [6] who showed that on a 12-set, there are at most two disjoint 5-(12,6,1) designs.

Whether or not a large set exists, it may be possible to pack the designs neatly by enlarging the set of points on which they are based, sometimes by adjoining just one extra point. Thus, if the set of all the $\binom{v}{k}$ k-sets chosen from X can be

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partitioned into v mutually disjoint t-$(v-1,k,\lambda)$ designs, each missing a different point of X, then we shall say that these designs form an *overlarge set*. We shall take $X = \{1,2,...,9\}$. and label the designs of an overlarge set by their missing elements.

For example, from a $(t+1)$-$(v,k+1,1)$ design, D, we can form an overlarge set of t-$(v-1,k,1)$ designs by choosing, for each $i = 1,...,v$, all the blocks of D containing i, and deleting i from each of them. These k-sets form design i, and this overlarge set is said to be *derived* from D. Such a construction was used by Rosa [**10**], for example. But an overlarge set of designs need not be derived in this way; Sharry and Street [**11**] deal with the case $v = 8, k = 3$, where only one of the 11 possible overlarge sets (partition A) is derived from a 3-$(8,4,1)$ design.

Overlarge sets of 3-$(8,4,1)$ designs.

It is well-known that a 2-$(7,3,1)$ design has a unique extension to a 3-$(8,4,1)$ design. This extension is by complementation; that is to say, the same new point is added to each of the seven blocks of the 2-$(7,3,1)$ design, and then seven further blocks are formed by taking complements with respect to the new point set. The process is reversible; any point x, and the blocks not containing x, can be deleted from a 3-$(8,4,1)$ design to yield a 2-$(7,3,1)$ design called the *restriction* on x.

There is no large set of 3-$(8,4,1)$ designs, for if there were then restrictions on the same point throughout would give a large set of 2-$(7,3,1)$ designs, and we have already seen that no such large set exists. However overlarge sets of 3-$(8,4,1)$ designs, based on a 9-set, do exist. There are two non-isomorphic cases. We shall demonstrate this in two ways: by direct construction, and by extending overlarge sets of 2-$(7,3,1)$ designs.

In an overlarge set of 3-$(8,4,1)$ designs based on a 9-set, design i omits the point i, for $i = 1,2,...,9$. In design i, the blocks fall into seven disjoint pairs, and any two disjoint blocks cover eight distinct points between them. The omitted ninth point i is called a *missing link* for either block of a disjoint pair.

LEMMA 1. *In an overlarge set of* 3-$(8,4,1)$ *designs, if the block* $123x$ *has missing link* y, *then the block* $123y$ *cannot have missing link* x.

PROOF: In design y, the unique block disjoint from $123x$ is the set

$$S = \{1,2,...,9\}\backslash\{1,2,3,x,y\}.$$

The same set, S, is the block disjoint from $123y$ in design x. But since each quadruple appears only once in the overlarge set, this is a contradiction. □

Consider the six blocks containing a given triple, say 123. They can be arranged in (cyclic) chains so that block $123x$ with missing link y is succeeded by block $123y$ with missing link z. Since each triple occurs just once in each design, the six missing links associated with 123 form a well-defined *chain of missing links*.

LEMMA 2. *In an overlarge set of* 3-$(8,4,1)$ *designs, chains of missing links have either one 6-cycle or two disjoint 3-cycles.*

PROOF: By Lemma 1, there can be no 2-cycles. □

We are now ready to establish our main result.

THEOREM. *Suppose there exists an overlarge set of* 3-(8, 4, 1) *designs, and consider its chains of missing links.*

(1) *If at least one chain of missing links is a 6-cycle, then precisely 12 chains have two disjoint 3-cycles each, and the overlarge set is uniquely determined up to isomorphism. It is associated with a* 2-(9, 3, 1) *design, and has automorphism group* $ASL(2,3)$, *of order 216.*

(2) *If every chain of missing links has two disjoint 3-cycles, then the overlarge set is uniquely determined up to isomorphism. It is associated with a set of 28 resolution classes, no two orthogonal, and has automorphism group* $P\Gamma L(2,8)$, *of order 1512.*

Construction and properties of the first overlarge set.

Assume that the overlarge set has a missing link chain with a 6-cycle. Then by a suitable labelling of points, the six blocks containing the triple 123, namely

1234, 1235, 1236, 1237, 1238, 1239,

can be assigned to designs 9, 4, 5, 6, 7, 8, respectively. Since the blocks of a 3-(8, 4, 1) design occur in disjoint pairs, this means that the blocks

5678, 6789, 7894, 8945, 9456, 4567,

must also be assigned to designs 9, 4, 5, 6, 7, 8, respectively. Now six of the 15 blocks not containing 1, 2, or 3 have been assigned in such a way that the remaining nine of these blocks must appear in designs 1, 2, and 3, three per design, placed so as to avoid repetition of a triple in two blocks of the same design. Without loss of generality, we place them as follows:

4579, 4678, 5689, in design 1;
4568, 4679, 5789, in design 2;
4578, 4689, 5679, in design 3.

Within each of designs 1, 2, 3, the complements of these blocks are now determined.

Thus design 1 now contains the blocks

2368, 4579, 2359, 4678, 2347, 5689,

and can be completed with either of the following sets of blocks:

(a) 2456, 3789, 2489, 3567, 2578, 3469, 2679, 3458;
(b) 2458, 3679, 2469, 3578, 2567, 3489, 2789, 3456.

Selecting either completion of design 1 determines the rest of the overlarge set; L_1, arising from completion (a), is shown in Table 1, where row i contains the 14 blocks of design i, for $i = 1, ..., 9$.

i :	X_1	X_2	Y_1	Y_2	Z_1	Z_2	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4
1 :	2368	4579	2456	3789	2679	3458	2347	3567	3469	4678	5689	2489	2578	2359
2 :	1567	3489	1689	3457	1346	5789	4568	1459	1358	1478	1379	3678	4679	3569
3 :	1589	2467	1248	5679	1678	2459	2568	2789	4689	1269	1479	1456	1257	4578
4 :	1235	6789	1578	2369	1569	2378	3568	1367	2567	3579	1279	2589	1389	1268
5 :	1468	2379	1349	2678	1247	3689	1679	3467	1378	1236	2348	1289	2469	4789
6 :	1249	3578	1237	4589	1258	3479	1789	2389	2579	2478	2345	1457	1348	1359
7 :	1369	2458	1259	3468	1489	2356	2349	2689	1238	3589	1568	1345	4569	1246
8 :	1347	2569	1356	2479	1239	4567	1579	1245	1469	3459	2346	3679	2357	1267
9 :	1278	3456	1467	2358	1357	2468	1234	1458	2457	1256	5678	2367	1368	3478

Table 1: the overlarge set, L_1, of 3-$(8,4,1)$ designs.

The overlarge set arising from completion (b) is isomorphic to L_1 under the permutation (47)(58)(69).

Each of the $\binom{9}{3} = 84$ triples induces a chain of missing links. Of these chains, 12 have two 3-cycles, and these lead to corresponding partitions of the point set into three triples; that is, to a resolution class. Each partition occurs three times, induced by each of its triples, and just 12 distinct triples appear. These triples are the blocks of the following 2-$(9,3,1)$ design:

126, 359, 478; 138, 257, 469; 145, 289, 367; 179, 234, 568.

This 2-$(9,3,1)$ design has automorphism group, G, of order 432, which acts 2-transitively on its points. Since $Aut(L_1)$, the automorphism group of the overlarge set L_1, must preserve this 2-$(9,3,1)$ design, it must be a subgroup of G. Elements of $Aut(L_1)$ include

(12)(389754), (1825)(3974), (13)(267954), (15)(268397).

Therefore $Aut(L_1)$ is 2-transitive on points. Elements of $Aut(L_1)$ which fix both 1 and 2 must also fix 6, and there are three such elements:

(359)(847), (395)(874) , and the identity.

Thus $|Aut(L_1)| = 9 \times 8 \times 3 = 216$. This is the group $ASL(2,3)$, or $3^2 : 2A_4$ in the notation of Conway *et al* [4].

Note that $Aut(L_1)$ is not transitive on the blocks of L_1, but partitions them into two orbits. The first orbit consists of the 54 blocks listed in columns $X_1, ..., Z_2$

of Table 1. None of these blocks contains any of the triples of the associated 2-$(9,3,1)$ design, and each pair of columns $\{X_1, X_2\}, \{Y_1, Y_2\}, \{Z_1, Z_2\}$ contains the 18 blocks of a 2-$(9,4,3)$ design, of type I in the classification of Breach [**2**]. The second orbit consists of the 72 blocks listed in the remaining columns of Table 1, which can be partitioned into 2-$(9,4,6)$ designs in various ways. For instance, the blocks of columns $A_1, ..., A_4$ form one such design, and those of columns $B_1, ..., B_4$ form another. Also, for each i, $i = 1,2,3,4$, the blocks of columns A_i and B_i cover the same pairs, so among these eight columns, those with the same subscript may be traded without upsetting the balance of the 2-$(9,4,6)$ designs.

The automorphism groups of these subdesigns have orders 144 and 54 respectively. For example, the automorphism group of the design consisting of the 18 blocks of columns $\{X_1, X_2\}$ is the group of order 144 with generators

$$(23)(68)(79),\ (28)(36)(45),\ (26)(47)(59),\ (1253)(6798),\ (1463)(5978).$$

Similarly, the design consisting of the 36 blocks of columns $\{A_1, A_2, A_3, A_4\}$ has automorphism group of order 54, with generators

$$(15)(37)(89),\ (126)(359)(487)$$

and acts transitively on the points, but partitions the blocks into two orbits, one consisting of the nine blocks of A_3, and the other of the remaining blocks.

Construction and properties of the second overlarge set.

It is now assumed that the overlarge set has no 6-cycles among its chains of missing links, and we again consider the blocks containing the triple 123. The blocks

1234, 1235, 1236, and 1237, 1238, 1239,

are assigned to designs 5, 6, 4, and to designs 8, 9, 7, respectively, giving two 3-cycles. Their disjoint mates

6789, 4789, 5789, 9456, 7456, 8456,

are then assigned to designs 5, 6, 4, 8, 9, 7, respectively. The other nine blocks not containing any of 1, 2, 3 include 4578, 4579, 4589. To avoid repetitions of a triple within a 3-$(8,4,1)$ design, these three blocks must be assigned to different designs, and none of them can belong to any of designs $4, ..., 9$. Hence without loss of generality, we assign 4578 to design 1, 4579 to design 2, and 4589 to design 3.

Somewhat surprisingly, from here on the construction is forced. The restriction to only 3-cycles in the chains of missing links is very strong. For example, consider the blocks containing the triple 458. So far, we have 4586 in design 7, and 4587 in design 1, so we must put 4581 in design 6. We also have 4589 in design 3, so 4583 cannot be in design 9, and hence 4583 must be in design 2, and 4582 in design 9. Similar arguments apply to the blocks containing 457 or 459. Every time a block is assigned to a design, a disjoint mate is also assigned. Combining this fact with the location of suitably chosen triples uniquely determines the whole overlarge set, L_2. It is given in Table 2, where again row i of the table shows the 14 blocks of design i, for $i = 1, ..., 9$.

Each triple, together with its two 3-cycles of missing links, forms a resolution class of the nine points of the large set. Since each triple is induced by each of its two associated triples, we obtain altogether 28 resolution classes, no two orthogonal, as follows:

123, 456, 789;	124, 358, 679;	125, 369, 478;	126, 347, 589;
127, 368, 459;	128, 349, 567;	129, 357, 468;	134, 259, 678;
135, 267, 489;	136, 248, 579;	137, 258, 469;	138, 269, 457;
139, 247, 568;	145, 268, 379;	146, 257, 389;	147, 289, 356;
148, 237, 569;	149, 236, 578;	156, 249, 378;	157, 234, 689;
158, 279, 346;	159, 238, 467;	167, 239, 458;	168, 235, 479;
169, 278, 345;	178, 246, 359;	179, 256, 348;	189, 245, 367.

The permutation (195274386) is an automorphism of L_2, so $Aut(L_2)$ is transitive on points. The stabilizer of the point 1 includes the permutations

$$\alpha = (456)(789), \quad \beta = (358)(679), \quad \gamma = (259)(687),$$

and thus $Aut(L_2)$ is 2-transitive. The stabilizer of both 1 and 2 includes α and β, so $Aut(L_2)$ is 3-transitive. The only elements of $Aut(L_2)$ which fix 1, 2, and 3 are α, α^2 and the identity. Hence $|Aut(L_2)| = 9 \times 8 \times 7 \times 3 = 1512$. In fact, $Aut(L_2)$ is the group $P\Gamma L(2,8)$, or $L_2(8):3$ in the notation of [4].

i :	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV
1 :	2369	4578	2347	5689	2358	4679	2459	3678	2468	3579	2567	3489	2789	3456
2 :	1349	5678	1357	4689	1368	4579	1478	3569	1589	3467	1456	3789	1679	3458
3 :	1248	5679	1259	4678	1267	4589	1457	2689	1469	2578	1568	2479	1789	2456
4 :	1236	5789	1257	3689	1289	3567	1358	2679	1379	2568	1569	2378	1678	2359
5 :	1234	6789	1268	3479	1279	3468	1369	2478	1378	2469	1467	2389	1489	2367
6 :	1235	4789	1249	3578	1278	3459	1347	2589	1389	2457	1458	2379	1579	2348
7 :	1239	4568	1246	3589	1258	3469	1348	2569	1356	2489	1459	2368	1689	2345
8 :	1237	4569	1245	3679	1269	3457	1346	2579	1359	2467	1479	2356	1567	2349
9 :	1238	4567	1247	3568	1256	3478	1345	2678	1367	2458	1468	2375	1578	2346

Table 2: the overlarge set, L_2, of 3-(8,4,1) designs.

Construction of overlarge sets by extending smaller designs.

A 2-$(7,3,1)$ design can be extended to a 3-$(8,4,1)$ design in only one way, and that is by complementation. Conversely, any restriction on a point of a 3-$(8,4,1)$ design produces a 2-$(7,3,1)$ design. Therefore, from an overlarge set of nine 3-$(8,4,1)$ designs, we can produce an overlarge set of eight 2-$(7,3,1)$ designs by deletion of all the blocks that do not contain a given point x, and the deletion of x from all the remaining blocks.

The reverse process is not always possible. For suppose that in an overlarge set of 2-$(7,3,1)$ designs the block abx belongs to design y and the block aby to design x. If we attempt to extend the overlarge set by introducing a new point z, then the block $abxz$ will belong to design y in the extended set, and the block $abyz$ to design x. In other words, we will have a 2-cycle in the chain of missing links, which is impossible. There are 11 possible non-isomorphic overlarge sets of 2-$(7,3,1)$ designs, given in [**11**] as partitions $A,...,K$. In partitions A,C,E,J,K, we have 124 in design 7 and 127 in design 4. In partitions D,H,I, we have 138 in design 2 and 132 in design 8. In partition G, we have 172 in design 3 and 173 in design 2. Thus partitions B and F are the only candidates for extension, and indeed both can be extended. The extension of F produces an isomorph of overlarge set L_1, and that of B an isomorph of L_2.

Note that the stabilizer of the point 1, say, in $Aut(L_1)$ is the group, $SL(2,3)$, of order 24 generated by the permutations

$$(2465)(3987),\ (2368)(4759),\ (275)(469)$$

and is isomorphic to the automorphism group of partition F. Similarly the stabilizer of the point 1 in $Aut(L_2)$ is isomorphic to the automorphism group, of order 168, of partition B; that is, the group with a normal subgroup of order 56, which is itself the normalizer of the (elementary abelian) Sylow 2-subgroup.

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