EMBEDDING CYCLE SYSTEMS OF EVEN LENGTH

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Abstract. We prove that if m is even then a partial m-cycle system on n vertices can be embedded in an m-cycle system on $2mn+1$ vertices.

1. Introduction and notation.

Let $V(G)$ and $E(G)$ denote the vertex and edge sets of a graph G, respectively. Let $Z_n = \{0, 1, \ldots, n-1\}$. Let K_n and $K_{x,y}$ be the complete graph and the complete bipartite graph, respectively. An *m-cycle* is a simple graph with m vertices, say $u_0, \ldots, u_{m-1}$ in which the only edges are $u_0 u_{m-1}$ and the edges joining u_i to u_{i+1} (for $0 \le i \le m-2$). We represent this cycle by $(u_0, \ldots, u_{m-1})$ or $(u_0, u_{m-1}, u_{m-2}, \ldots, u_1)$ or any cyclic shift of these. A *(partial) m-cycle system* is an order pair $(V, C(m))$ where $C(m)$ is a set of edge-disjoint m-cycles which partition (a subset of) the edge set of the complete graph with vertex set V.

A partial m-cycle system $(Z_n, C_1(m))$ is *embedded* in an m-cycle system $(Z_v, C_2(m))$ if $C_1(m) \subseteq C_2(m)$. A natural problem then is to find as small a value of v as possible so that every partial m-cycle on n vertices can be embedded in an m-cycle system on v vertices. The best result to date is Wilson's theorem [8] which shows that all partial m-cycle systems can be finitely embedded, but the size v of the m-cycle system is an exponential function of n. (Of course, Wilson's result is proved for the embedding of partial graph decompositions in general, not just for m-cycle systems.) The only other results related to this problem deal with the particular case when m is odd. A 3-cycle system is more commonly known as a *Steiner triple system*. Originally, a finite embedding of a partial Steiner triple system on n vertices in a Steiner triple system on v vertices was found by Treash [7], but v is an exponential function of n. Gradually over the years, several results [1, 4] have culminated in reducing

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v to at most $4n+1$. Most recently it has been shown [5] that if m is odd then a partial m-cycle system on n vertices can be embedded in an m-cycle system on at most $m((m-2)n(n-1)+2n+1)$ vertices. There it is also shown that a partial weak Steiner m-cycle system on n vertices can be embedded in an m-cycle system on $m(2n+1)$ vertices when m is odd.

Here we consider embedding partial m-cycle systems when m is even. This seems to be an easier problem than when m is odd. We show that a partial m-cycle system on n vertices can be embedded in an m-cycle system on at most $2mn+1$ vertices.

Let λK_n denote the graph on n vertices in which each pair of vertices is joined by exactly λ edges. Some results have been obtained on the generalized embedding problem when K_n is replaced by λK_n. However, using the technique described in [3], it can be shown that if any partial m-cycle system of K_n can be embedded in an m-cycle system of $K_{f(n)}$ then any partial m-cycle system of λK_n can be embedded in an m-cycle system of λK_v, where $v \le f(m(m-2)\lambda n^2)$. So, clearly, the case when $\lambda = 1$ is of most interest.

All graphs in this paper are simple. See [2] for any graph theoretical terms that are not defined. Throughout the rest of this paper we shall assume that m is even and at least 4; we shall write $m = 2k$. Let $[x]$ denote the greatest integer less than or equal to x.

2. Preliminary results.

In this paper, we make extensive use of the following result. We say that a graph G can be *decomposed* into m-cycles if there exists a set of m-cycles $C(m)$, the edges of which partition $E(G)$. Recall that we write $m = 2k$.

Lemma 2.1. *[6] $K_{x,y}$ can be decomposed into m-cycles if and only if $x \ge k$, $y \ge k$, m divides xy and x and y are even.*

By Lemma 2.1, $K_{m,m}$ can be decomposed into m-cycles; we denote such a set of m-cycles on the vertex set $\{i,j\} \times \{0,1,\ldots,m-1\}$ by $C(i,j;m)$.

Two partial m-cycle systems (Z_n, C_1) and (Z_n, C_2) are *mutually balanced* if for all $ij \in E(K_n)$, ij is in a cycle in C_1 if and only if ij is in a cycle in C_2. The following mutually balanced m-cycle systems, both defined on the vertex set $Z_m \times Z_m$, are of vital importance in proving our result. Let

$$A_1(m) = \bigcup_{i=0}^{m-1} C(i, i+1; m),$$

where all vertices are reduced modulo m, and

$$A_2(m) = \{ ((0,x),(1,x+y),(2,x+2y),\ldots,(m-1,x+(m-1)y)) \mid 0 \le x \le m-1,\ 0 \le y \le m-1 \}.$$

Lemma 2.2. *$(Z_m \times Z_m, A_1(m))$ and $(Z_m \times Z_m, A_2(m))$ are mutually balanced partial m-cycle systems.*

To construct m-cycle systems we need the following result. Again recall that $m = 2k$.

Lemma 2.3. *There exists an m-cycle system on $2m + 1$ vertices.*

Proof: In each of 2 cases, we define an m-cycle $a(m) = (a_1, a_2, \ldots, a_m)$ as follows.

Let $m = 4x$. For $1 \le i \le m/2$, define

$$a_i = (-1)^i i, \text{ and}$$
$$a_{(m/2)+i} = \begin{cases} (m/2) - 2 + i & \text{if } i \text{ is odd,} \\ (3m/2) - 1 - i & \text{if } i \text{ is even,} \end{cases}$$

where everything is reduced modulo $2m + 1$.

Let $m = 4x + 2$. Define

$$\begin{aligned} a_i &= (-1)^{i-1}(i-1) && \text{for } 1 \le i \le m/2 - 2, \\ a_{k-1} &= 1 - k, a_k = k - 2, \\ a_{k+1} &= -k, a_{k+2} = k + 1, \\ a_{m-2i} &= m - 1 + 2i && \text{for } 0 \le i \le (m-6)/4, \text{ and} \\ a_{m-2i+1} &= m - 1 - 2i && \text{for } 1 \le i \le (m-6)/4, \end{aligned}$$

where everything is reduced modulo $2m + 1$.

Now define $a(m) + i$ to be the m-cycle formed by adding i modulo $2m + 1$ to each vertex in $a(m)$. Then $C(m) = \{a(m) + i \mid 0 \le i \le 2m\}$ is the required set of m-cycles.

Example 2.4: $a(8) = (16, 2, 14, 4, 3, 9, 5, 7)$ and $a(10) = (0, 20, 2, 17, 3, 16, 6, 11, 7, 9)$.

Finally, we conclude this section with a construction of some m-cycle systems. Let $(\{\infty\} \cup (\{i, j\} \times Z_m), B(i, j; m))$ be an m-cycle system (which exists by Lemma 2.3). Let $E = \{(i, i + n) \mid 0 \le i \le n - 1\}$.

Theorem 2.5. *For any $n \ge 1$, $(\{\infty\} \cup (Z_{2n} \times Z_m), D(n, m))$ is an m-cycle system, where we define*

$$D(n, m) = \left(\bigcup_{i=0}^{n-1} B(i, i + n; m) \right) \cup \left(\bigcup_{\substack{0 \le i < j \le 2n-1 \\ (i,j) \notin E}} C(i, j; m) \right).$$

3. Embedding partial m-cycle systems.

Theorem 3.1. *A partial m-cycle system on n vertices can be embedded in an m-cycle system on $2mn+1$ vertices.*

Proof: Let $(Z_n, C_1(m))$ be a partial m-cycle system. Let $(\{\infty\} \cup (Z_{2n} \times Z_m), D(n,m))$ be the m-cycle system constructed in Theorem 2.5. For each m-cycle $u = (u_0, u_1, \ldots, u_{m-1}) \in C_1(m)$ let $A_2(u;m)$ be the m-cycles formed from those in $A_2(m)$ by replacing each vertex i with u_i. Then by Lemma 2.2, $\bigcup_{i=0}^{m-1} C(u_i, u_{i+1}; m)$ (of course, reducing the subscripts of u_i modulo m) and $A_2(u;m)$ form mutually balanced partial m-cycle systems. For each $u \in C_1(m)$, remove the m-cycles in $\bigcup_{u=0}^{m-1} C(u_i, u_{i+1}; m)$ from $D(n,m)$ and replace them with the m-cycles in $A_2(u;m)$, thus forming another m-cycle system $(\{\infty\} \cup (Z_{2n} \times Z_m), C_2(m))$. Then since $((u_0,0),(u_1,0),\ldots,(u_{m-1},0)) \in A_2(u;m)$, $(\{\infty\} \cup (Z_{2n} \times Z_m), C_2(m))$ is the required embedding of $(Z_n, C_1(m))$.

Remark: Clearly, Theorem 3.1 can be strengthened in several ways. For example, a partial m-cycle system on n vertices can be embedded in an m-cycle system on $2mt+1$ vertices for any $t \geq n$. Also, if ij is not an edge in any cycle in $C_1(m)$ then the vertices in $\{n+i, n+j\} \times Z_m$ need not be introduced in the embedding process, as $B(i,j;m)$ can be used instead of the two m-cycle systems $B(i,i+n;m)$ and $B(j,j+n;m)$ when constructing $D(n,m)$. This observation shows that if G is the graph consisting of the cycles in $C_1(m)$ and α' is the maximum number of independent edges in the complement of G then $(Z_n, C_1(m))$ can be embedded in an m-cycle system on $1+2m(n-\alpha')$ vertices.

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