# CONDITIONS FOR DEGREE SEQUENCES TO BE REALISABLE BY 3-UNIFORM HYPERGRAPHS

David Billington

ABSTRACT. A degree sequence which is realisable by a 3-uniform hypergraph is called 3-graphic. Seven necessary conditions, one sufficient condition, and one equivalent condition for degree sequences to be 3-graphic are derived. Moreover, four special classes of degree sequences are examined. For each class an equivalent condition for the sequences in this class to be 3-graphic is derived. Using these conditions all the 3-graphic degree sequences of length at most seven have been determined.

## Section 1. Definitions and introduction

A *hypergraph* is a set of subsets of a finite set of *points*. The number of points in the finite set of points is denoted by p. The subsets are called *blocks*. Notice that hypergraphs do not contain repeated blocks. If all the blocks of a hypergraph contain the same number of points then the hypergraph is said to be *uniform*. An s-*uniform hypergraph*, or s-*graph*, is a uniform hypergraph in which all the blocks have exactly s elements. So a 2-uniform hypergraph is just a (simple) graph. The letter s always denotes the size (number of elements) of the blocks in a uniform hypergraph. Therefore throughout this paper we shall assume that s and p denote positive integers such that $s \leq p$.

The number of blocks which contain a given point is called the degree of that point. If v is a point of the hypergraph M then the degree of v in M is denoted by deg(v,M). It is clear that a degree is a non-negative integer. When the sequence of all the degrees of a hypergraph is arranged in decreasing order it is called the *degree sequence* of the hypergraph. (A sequence of integers is *decreasing* if and only if for any adjacent pair of integers the one on the right is less than or equal to the one on the left.) The length of a degree sequence is thus p. A hypergraph *realises* (or is a *realisation* of) a sequence of integers if and only if the sequence is the degree sequence of the hypergraph. A sequence of integers is called s-*graphic* if and only if there is an s-graph which realises the sequence.

The general problem with which this paper is concerned is to find an algorithm to determine whether a given sequence is s-graphic or not. The history of this problem began in 1955 when Havel [7] solved the problem for s = 2. Other key contributions are [2], [3], [4], and [5]. A detailed history of this problem can be found in Section 1 of [1]. The case when s = 3 is the smallest for which no satisfactory algorithm, or proof of NP-completeness, has yet been found. Indeed an anonymous referee quoted Paul Erdös as saying that, even when s = 3, the problem was "probably impossibly hard". For the case when s = 3 an algorithm for solving the problem was given in [1]. Unfortunately this algorithm was not polynomial. So in this paper we restrict ourselves to polynomial algorithms and see how good they are. These polynomial algorithms are presented in the form of seven necessary conditions, one sufficient condition and five equivalent conditions for degree sequences to be 3-graphic. These conditions are good enough to determine all the 3-graphic sequences of length at most seven.

It will be convenient to have in mind some standard set of points. The following definition makes this precise.

*Definition of standard realisation.*

Let $D = (d_1, d_2, \ldots, d_p)$ be a degree sequence. Then the hypergraph M is a *standard realisation* of D if and only if

(1) D is the degree sequence of M;

(2) the set of points of M is $\{1,2,\ldots,p\}$; and

(3) for all i in M, $\deg(i,M) = d_i$.

A *standard hypergraph* is a hypergraph which is a standard realisation of its own degree sequence.

Two hypergraphs are *isomorphic* if and only if there is a bijection from the set of points of one onto the set of points of the other such that the image of a block in one is a block in the other. It is easy to see that any hypergraph is isomorphic to a standard realisation of its degree sequence. Hence there is no loss of generality in restricting our attention to standard hypergraphs.

We conclude this section with the following notational conventions. Let $P = \{1,2,\ldots,p\}$ be the set of all positive integers which are at most p. Define $C = \{\{i,j,k\} \subseteq P : i < j < k\}$ to be the set of all three element subsets of P. C is called the *complete 3-graph* on P. For non-negative integers n and r we define $\binom{n}{r}$ as follows. If $n \geq r \geq 0$, then $\binom{n}{r} = \frac{n!}{(n-r)!r!}$; otherwise $0 \leq n < r$ and then $\binom{n}{r} = 0$. If A is any finite set of integers then minA and maxA denote respectively the minimum and maximum elements of A.

## Section 2. Partial Sum Conditions

Our first result is one of the simpler partial generalisations of the Erdös-Gallai result, which can be found in [4] or on page 59 of [6]. If x and y are non-negative integers then x monus y, denoted by $x \dot{-} y$, is defined by the equation

$$x \dot{-} y = \max\{x-y,0\}.$$

THEOREM 2.1. Let $D = d_1,d_2,\dots,d_p)$ be a 3-graphic degree sequence, and let $P = \{1,2,\dots,p\}$. Then for all $n \in P$,

$$\Sigma(d_i : 1 \le i \le n) \le 3\left[\binom{n}{3} - \left(\binom{n-1}{2} \dot{-} d_n\right)\right] + 2\Sigma(\min\{d_i, \binom{n}{2}\} : n+1 \le i \le p) + \Sigma(\min\{d_i, n(i-n-1)\} : n+1 \le i \le p).$$

PROOF: Suppose that M is any standard realisation of D and for each $n \in P$ let $N(n) = \{1,2,\dots,n\}$. The right side of the inequality consists of three terms. If we can show that each term is i times an upper bound for the number of blocks of M which intersect N(n) in i points, then the result will follow.

Consider the first term. Clearly $\binom{n}{3}$ is an upper bound for the number of blocks intersecting N(n) in 3 points. If there are $\binom{n}{3}$ such blocks then each point in N(n) is in $\binom{n-1}{2}$ such blocks. So if $d_n < \binom{n-1}{2}$ then $\binom{n}{3}$ may be reduced by $\binom{n-1}{2} - d_n$. Hence the stated upper bound.

Consider the second term. Each point i in $P \setminus N(n)$ can be placed in at most $\binom{n}{2}$ blocks which intersect N(n) in 2 points. Since each point i is in at most $d_i$ blocks, we see that each point i in $P \setminus N(n)$ can be in at most $\min\{d_i, \binom{n}{2}\}$ blocks which intersect N(n) in 2 points. Hence the stated upper bound.

Now consider the third term and let i and j be points in $P \setminus N(n)$. There are at most $n(i-n-1)$ blocks of the form $\{i,j,k\}$ where $n+1 \le j < i$ and $k \in N(n)$. Since the number of blocks containing i is $d_i$, there are at most $\min\{d_i,n(i-n-1)\}$ blocks of the above form. Hence an upper bound on the number of blocks which intersect N(n) in 1 point is $\Sigma(\min\{d_i, n(i-n-1)\} : n+1 \le i \le p)$. ■

The above result can be generalised to s-graphs. Also the upper bounds of the theorem can be slightly improved at the expense of greatly increasing the complexity of the expressions. These improvements seem to have little chance of yielding sufficient conditions because no account is taken of the interactions between the blocks which intersect N(n) in 1, 2, or 3 points. Rather than pursuing this unpromising line of reasoning we shall develop a set of simpler necessary partial sum conditions.

Let $D = (d_1, d_2, \ldots, d_p)$ be a 3-graphic degree sequence. Let **M** be any standard 3-graph which realises D, and let q be the number of blocks in **M**. It will be convenient to define $P_0 = \{0,1,\ldots,p\}$. For each ordered pair $(i,j) \in P_0 \times P_0$ we define $\mathbf{M}(i,j)$ by

$$\mathbf{M}(i,j) = \{\{i,j,k\} \in \mathbf{M} : i<j<k \text{ and } k \in P\}.$$

The cardinality of $\mathbf{M}(i,j)$ is denoted by $|\mathbf{M}(i,j)|$. For each ordered pair $(m,n) \in P_0 \times P_0$ we define $s(m,n)$ to be the following sum.

$$s(m,n) = \Sigma(|\mathbf{M}(i,j)| : 0 \le i \le m, \text{ and } 0 \le j \le n)$$

LEMMA 2.2.

(1) If $i \ge j$, or $i = 0$, or $j = 0$ then $\mathbf{M}(i,j)$ is empty.

(2) For each ordered pair $(m,n) \in P_0 \times P_0$, $s(m,n) \le q$.

(3) For each $n \in P$, $s(n-1,n) \le p\binom{n}{2} - 2\binom{n+1}{3}$.

(4) For each $n \in P$, $s(n,p-1) \le p\binom{n}{2} - 2\binom{n+1}{3} + n\binom{p-n}{2}$.

(5) For each $n \in P$, $d_1 + d_2 + \ldots + d_n \le s(n-1,n) + s(n,p-1) + \binom{n}{3}$.

PROOF: Part (1) follows immediately from the definitions.

Part (2) follows from the observation that the $\mathbf{M}(i,j)$ partition the blocks r of **M**.

Before we prove parts (3) and (4) we need to observe that since $\mathbf{M}(i,j) \subseteq \{\{i,j,j+1\},\{i,j,j+2\},\ldots,\{i,j,p\}\}$ we have $0 \le |\mathbf{M}(i,j)| \le p-j$.

Part (3). For any $n \in P$,

$$\begin{aligned}
s(n-1,n) &= \Sigma(|M(i,j)| : 0 \le i \le n-1, \text{ and } 0 \le j \le n) \\
&= \Sigma(|M(i,j)| : 1 \le i \le n-1, 2 \le j \le n, \text{ and } i < j) \\
&\le 1(p-2) + 2(p-3) + \dots + (n-1)(p-n) \\
&= p(1 + 2 + \dots + (n-1)) - (1*2 + 2*3 + \dots + (n-1)(n)) \\
&= p\binom{n}{2} - 2\binom{n+1}{3}.
\end{aligned}$$

Part (4) is proved similarly. For any $n \in P$,

$$\begin{aligned}
s(n,p-1) &= \Sigma(|M(i,j)| : 0 \le i \le n, \text{ and } 0 \le j \le p-1) \\
&= \Sigma(|M(i,j)| : 1 \le i \le n, 2 \le j \le p-1, \text{ and } i < j) \\
&\le [1(p-2) + 2(p-3) + \dots + (n-1)(p-n)] + \\
&\qquad [n(p-n-1) + n(p-n-2) + \dots + n(1)] \\
&= p\binom{n}{2} - 2\binom{n+1}{3} + n(1+2+ \dots + p-n-1) \\
&= p\binom{n}{2} - 2\binom{n+1}{3} + n\binom{p-n}{2}.
\end{aligned}$$

Part(5). From the definition of $M(i,j)$, for each $k \in P$,

(a) $k$ is a member of every block in $M(i,k)$, where $1 \le i \le k-1$;

(b) $k$ is a member of every block in $M(k,j)$, where $k+1 \le j \le p-1$;

(c) $k$ is a member of at most one block in $M(i,j)$, where $1 \le i \le k-2$ and $2 \le j \le k-1$ and $i < j$.

Now the number of $M(i,j)$ which satisfy the conditions in (c) on i and j is

$$1 + 2 + \dots + (k-2) = \binom{k-1}{2}.$$

Thus for all $k \in P$,

$$d_k \ge \Sigma(|M(i,k)| : 1 \le i \le k-1) + \Sigma(|M(k,j)| : k+1 \le j \le p-1), \text{ and}$$

$$d_k \le \Sigma(|M(i,k)| : 1 \le i \le k-1) + \Sigma(|M(k,j)| : k+1 \le j \le p-1) + \binom{k-1}{2}.$$

Using the second of these inequalities, for each $n \in P$ we have
$d_1 + ... + d_k + ... + d_n$

$$
\begin{aligned}
&\le \Sigma(|M(i,1)| : 1 \le i \le 0) + \Sigma(|M(1,j)| : 2 \le j \le p-1) + \binom{0}{2} + \\
&\quad \cdots\cdots + \cdots\cdots + \cdots + \\
&\quad \Sigma(|M(i,k)| : 1 \le i \le k-1) + \Sigma(|M(k,j)| : k+1 \le j \le p-1) + \binom{k-1}{2} + \\
&\quad \cdots\cdots + \cdots\cdots + \cdots + \\
&\quad \Sigma(|M(i,n)| : 1 \le i \le n-1) + \Sigma(|M(n,j)| : n+1 \le j \le p-1) + \binom{n-1}{2} \\
&= \Sigma(|M(i,j)| : 1 \le i \le j-1 \text{ and } 1 \le j \le n) + \\
&\quad \Sigma(|M(i,j)| : 1 \le i \le n \text{ and } i+1 \le j \le p-1) + \Sigma(\binom{i}{2} : 0 \le i \le n-1) \\
&= \Sigma(|M(i,j)| : 1 \le i \le n-1,\ 1 \le j \le n \text{ and } i < j) + \\
&\quad \Sigma(|M(i,j)| : 1 \le i \le n,\ 1 \le j \le p-1 \text{ and } i < j) + \Sigma(\binom{i}{2} : 0 \le i \le n-1) \\
&= \Sigma(|M(i,j)| : 0 \le i \le n-1 \text{ and } 0 \le j \le n) + \\
&\quad \Sigma(|M(i,j)| : 0 \le i \le n \text{ and } 0 \le j \le p-1) + \Sigma(\binom{i}{2} : 0 \le i \le n-1) \\
&= s(n-1,n) + s(n,p-1) + \binom{n}{3}. \ \blacksquare
\end{aligned}
$$

THEOREM 2.3. Let $D = (d_1, d_2, ..., d_p)$ be a 3-graphic degree sequence, and let $P = \{1,2,...,p\}$. Then the following five conditions all hold.

(1) $d_1 + d_2 + ... + d_p = 3q$, for some integer q. Moreover, if M is any 3-graph which realises D then the number of blocks in M is q.

(2) $d_1 \le \binom{p-1}{2}$.

(3) $d_1 - d_p \le \binom{p-2}{2}$.

(4) For each $n \in P$, $d_1 + d_2 + ... + d_n \le 2q + \binom{n}{3}$.

(5) For each $n \in P$, $d_1 + d_2 + ... + d_n \le q + (p-1)\binom{n}{2} - \binom{n+1}{3}$.

PROOF: Part (1) follows from the fact that each block of any 3-graphic realisation of D contributes 3 to the sum of the degrees.

Part (2) follows from the observation that the maximum number of blocks which can contain a given point is at most $\binom{p-1}{2}$. However, if the inequalities in parts (3) and (4) of Lemma 2.2 are substituted into the inequality in part (5) of Lemma 2.2 we get

$$d_1 + d_2 + \dots + d_n \le n\binom{p-1}{2}, \quad \text{for each } n \in P.$$

Since D is decreasing, the only useful instance of this result is for n = 1.

Part (3) follows from the observation that

$$\begin{aligned} d_1 - d_p &= \text{degree of } 1 - \text{degree of } p \\ &= \text{number of blocks containing 1 but not p} \\ &\quad + \text{number of blocks containing both 1 and p} \\ &\quad - \text{number of blocks containing p but not 1} \\ &\quad - \text{number of blocks containing both p and 1} \\ &= \text{number of blocks containing 1 but not p} \\ &\quad - \text{number of blocks containing p but not 1} \\ &\le \text{number of blocks containing 1 but not p} \\ &= \binom{p-2}{2}. \end{aligned}$$

Part (4) can be derived by twice substituting the inequality in part (2) of Lemma 2.2 into the inequality in part (5) of Lemma 2.2.

Part (5) can be derived by substituting the inequalities in parts (2) and (3) of Lemma 2.2 into the inequality in part (5) of Lemma 2.2, and noting that $\binom{n}{3} = \binom{n+1}{3} - \binom{n}{2}$. ■

## Section 3. Weight Conditions

The *weight* of a sequence $D = (d_1,d_2,\dots,d_p)$ is denoted by wt(D) and is defined by the equation

$$wt(D) = \Sigma(id_i : 1 \le i \le p).$$

If $B = \{i,j,k\}$ is any block of a 3-graph then the sum of B, $\Sigma B$, is defined by $\Sigma B = i+j+k$. Let $M = \{B_1,B_2,\dots,B_q\}$ be any standard 3-graph then the sum of M, $\Sigma M$, is defined by $\Sigma M = \Sigma(\Sigma B_i : 1 \le i \le q)$. These definitions are related by the following lemma.

LEMMA 3.1. If M is any standard 3-graphic realisation of the degree sequence D then $wt(D) = \Sigma M$.

PROOF: The proof follows immediately from the definitions. ■

There are $\binom{p}{3}$ blocks in **C**, the complete 3-graph on P = {1,2,...,p}. Calculate the sum of each of these blocks, and arrange these sums in increasing order. Now define w(p,q) to be the sum of the first q terms of this ordered list. The number of terms in a sequence D is denoted by length(D) and the sum of all these terms is denoted by $\sum D$.

LEMMA 3.2. $w(p,q) = \min\{wt(D) : D \text{ is a 3-graphic degree sequence, } length(D) = p, \text{ and } \sum D = 3q\}$.

PROOF: Let **M** consist of the q blocks of **C** whose sums form the first q terms of the ordered list used to define w(p,q). Then $\sum \mathbf{M} = w(p,q)$. Let $D_M$ be the degree sequence of **M**. If we let

$$m = \min\{wt(D) : D \text{ is a 3-graphic degree sequence, } length(D) = p, \text{ and } \sum D = 3q\},$$

then $m \leq wt(D_M) = \sum \mathbf{M} = w(p,q)$, by Lemma 3.1. Conversely, let D be any 3-graphic degree sequence of length p whose sum is 3q. Let **H** be a standard 3-graphic realisation of D. By Lemma 3.1 $\sum \mathbf{H} = wt(D)$. But by construction $\sum \mathbf{M} \leq \sum \mathbf{H}$. Thus $\sum \mathbf{M} \leq \sum \mathbf{H} = wt(D)$. But D was arbitrary, so $\sum \mathbf{M} \leq m$. Hence $w(p,q) = \sum \mathbf{M} \leq m$. ■

Before we can prove our main result we need the following terminology and result from [1].

*Definition of flatter, and steeper.*

Let D and D' be finite decreasing sequences of non-negative integers which are the same length.

D' is *flatter* than D, and D is *steeper* than D', if and only if D' can be obtained from D by a non-empty sequence of elementary flattenings.

D' is an *elementary flattening* of D if and only if D' can be obtained from D by

(1) finding two integers of D which differ by at least 2; and then
(2) transferring 1 from the larger to the smaller; that is, taking 1 from the larger and adding 1 to the smaller; and then
(3) re-ordering the resulting sequence so that it is decreasing.

It should be clear that, since D is decreasing, it is possible to choose two integers as in (1) so that no re-ordering is necessary. The integers are chosen so that they will not only produce D' but so that they are as close together in the sequence as possible.

LEMMA 3.3. If D is s-graphic then all sequences which are flatter than D are also s-graphic.

PROOF: See Lemma 2.3 of [1]. ■

If two sequences are in the flatter/steeper relationship then their weights are related as detailed in the following lemma.

LEMMA 3.4. Let D and D' be finite decreasing sequences of non-negative integers which are the same length. If D is steeper than D' then

$$wt(D) \le wt(D').$$

PROOF: It suffices to show that an elementary flattening of D will increase the weight. Consequently suppose $D = (d_1, d_2, \ldots, d_p)$ and $d_i \ge d_j+2$. So $i < j$. Let $D' = (d_1', d_2', \ldots, d_p')$ where $d_i' = d_i - 1$, $d_j' = d_j + 1$ and $d_k' = d_k$ for all $k \in P\backslash\{i,j\}$. Then

$$\begin{aligned} wt(D') - wt(D) &= id_i' + jd_j' - id_i - jd_j \\ &= i(d_i - 1) + j(d_j + 1) - id_i - jd_j \\ &= -i + j \\ &> 0. \end{aligned}$$

Therefore flattening increases weight and steepening decreases weight. ■

LEMMA 3.5. Suppose p and q are non-negative integers, and suppose that $3q = mp+r$ where $0 \le r < p$. Then $m\binom{p+1}{2} + \binom{r+1}{2} = \max\{wt(D) : D$ is a 3-graphic degree sequence, $\text{length}(D) = p$, and $\sum D = 3q\}$.

PROOF: Let D' be the sequence whose first r terms are all m+1 and whose remaining p-r terms are all m. Then D' is flatter than any decreasing sequence of non-negative integers of length p and sum 3q. So by Lemma 3.4 the weight of D' is greater than the weight of any other such sequence. Moreover D' is 3-graphic by Lemma 3.3. Since

$$\begin{aligned} wt(D') &= 1(m+1) + 2(m+1) + \ldots + r(m+1) + (r+1)m + (r+2)m + \ldots + p \\ &= 1m + 2m + \ldots + pm \;+\; 1 + 2 + \ldots + r \\ &= m(1 + 2 + \ldots + p) \;+\; \binom{r+1}{2} \\ &= m\binom{p+1}{2} + \binom{r+1}{2}, \end{aligned}$$

the lemma is proved. ■

The main result of this section now follows easily.

THEOREM 3.6. Let D be a 3-graphic degree sequence of length p and sum 3q. Moreover suppose that $3q = mp + r$ where $0 \leq r < p$. Then

$$w(p,q) \leq wt(D) \leq m\binom{p+1}{2} + \binom{r+1}{2}.$$

PROOF: The first inequality follows immediately from Lemma 3.2, while the second inequality follows immediately from Lemma 3.5. ■

## Section 4. A Sufficient Condition

In this section we give a heuristic for constructing a 3-graph from its degree sequence. Unfortunately this heuristic does not always work, however it is a polynomial algorithm. An indication of how well it works can be seen from the numerical results in Section 6.

The basic idea of the algorithm is as follows.

Given a decreasing sequence of non-negative integers, say D, we try to construct a 3-graphic realisation of D in a straightforward manner. This attempt yields a sequence of non-negative integers called the residual sequence of D and denoted by Residual(D). If Residual(D) contains only zeros then the construction attempt succeeded. Otherwise the attempt failed and Residual(D) is used to obtain another sequence, denoted by Child(D), which is steeper than D.

The process outlined in the above paragraph is now applied to Child(D). In this way we keep producing steeper and steeper sequences until either the minimum weight bound of Theorem 3.6 is violated or a 3-graphic realisation is constructed. Lemma 3.4 guarantees that unless a 3-graph is constructed the minimum weight bound will be violated. If a 3-graph is eventually constructed then Lemma 3.3 assures us that D was 3-graphic. Moreover the proof of Lemma 3.3 indicates how to obtain a realisation of D from the constructed realisation of the sequence which was steeper than D. We shall now give the details of the above algorithm.

Let $D = (d_1,d_2,...,d_p)$ be a decreasing sequence of non-negative integers. Suppose that $C = \{B_1,B_2,...,B_b\}$, where $b = \binom{p}{3}$, is such that the $B_i$ are lexicographically ordered. That is, $i < j$ if and only if $B_i < B_j$, where $B_i < B_j$ is defined as follows. If $B \in C$ let B(1), B(2), and B(3) denote the smallest, middle, and largest elements of B respectively. Now $B_i < B_j$ means that one of the following three conditions holds.

(1) $B_i(1) < B_j(1)$.

(2) $B_i(1) = B_j(1)$ and $B_i(2) < B_j(2)$.

(3) $B_i(1) = B_j(1)$ and $B_i(2) = B_j(2)$ and $B_i(3) < B_j(3)$.

The straightforward attempt to realise D can now be described. The blocks of C are considered in order. A block is added to the 3-graph we are constructing if and only if its addition will not make the degree of any point, i say, greater than $d_i$. When all the blocks of C have been considered the constructed 3-graph, denoted by M(D), will have $(a_1,a_2,...,a_p)$ say, as its degree sequence. The residual sequence of D, after the above attempt to construct a realisation of D, is defined by

$$\text{Residual}(D) = (d_1-a_1,d_2-a_2,...,d_p-a_p).$$

It is clear that Residual(D) is a (not necessarily ordered) sequence of p non-negative integers. Also Residual(D) consists of p zeros if and only if M(D) is in fact a 3-graphic realisation of D.

Suppose that $\text{Residual}(D) = (r_1,r_2,...,r_p)$. If Residual(D) has a positive element then a steeper sequence than D is obtained from D as follows. Let Position be a function which accepts a residual sequence and returns a member of $P\backslash\{1\}$. (We define Position in the next paragraph.) If Position(Residual(D)) = m then define the sequence $D' = (d'_1,d'_2,...,d'_p)$ by $d'_m = d_m-1$, $d'_{m-1} = d_{m-1}+1$ and for all $i \in P\backslash\{m-1, m\}$, $d'_i = d_i$. Since D' may not be decreasing we define Child(D) to be D' re-arranged into decreasing order.

The definition of Position(Residual(D)) is as follows. Let max be the maximum of $\{r_2,...,r_p\}$, and define

$$\text{Posns1} = \{i \in P\backslash\{1\} : r_i = \max\}.$$

Let mindiff be the minimum of $\{d_{i-1} - d_i \ : i \in \text{Posns1}\}$, and define

$$\text{Posns2} = \{i \in \text{Posns1} : d_{i-1} - d_i = \text{mindiff}\}.$$

Let min be the minimum of $\{r_{i-1} : i \in \text{Posns2}\}$, and define

$$\text{Posns3} = \{i \in \text{Posns2} : r_{i-1} = \min\}.$$

Finally we define Position(Residual(D)) to be the maximum of Posns3.

We can now define the function Graf which takes a decreasing sequence of non-negative integers and returns either TRUE or FALSE. Let D be a decreasing sequence of p non-negative integers such that $\sum D = 3q$. Then Graf(D) is defined recursively by the following three statements.

(1) If $wt(D) < w(p,q)$ then $Graf(D) = FALSE$.
(2) If $wt(D) \geq w(p,q)$ and every term of Residual(D) is zero then $Graf(D) = TRUE$.
(3) If $wt(D) \geq w(p,q)$ and Residual(D) has at least one non-zero term then $Graf(D) = Graf(Child(D))$.

LEMMA 4.1. If Graf(D) is TRUE then D is 3-graphic.

PROOF: The result is clear from the above discussion and Lemma 3.3. ■

## Section 5. Equivalent Conditions

Recall that C is the complete 3-graph on $P = \{1,2,...,p\}$. If M is any 3-graph on P then the *complement* of M, denoted by $M^c$, is defined by $M^c = C \setminus M$. If $D = (d_1,d_2,...,d_p)$ is a 3-graphic degree sequence then the *complement* of D, denoted by $D^c$, is defined by

$$D^c = \left( \binom{p-1}{2} - d_p, \binom{p-1}{2} - d_{p-1}, ..., \binom{p-1}{2} - d_1 \right).$$

The following theorem gives the main results about complements.

THEOREM 5.1. Let M be a 3-graph whose degree sequence is D. Then the following three statements are true.

(1) $M^c$ is a 3-graph whose degree sequence is $D^c$.

(2) $(M^c)^c = M$, and $(D^c)^c = D$.

(3) If M has q blocks then $M^c$ has $\binom{p}{3} - q$ blocks.

PROOF: (1) That $M^c$ is a 3-graph is clear from the definition. Since each point of C is in $\binom{p-1}{2}$ blocks it follows that the degree sequence of $M^c$ is $D^c$.

Parts (2) and (3) are immediate from the definitions. ■

COROLLARY 5.2. D is 3-graphic if and only if $D^c$ is 3-graphic.

PROOF: This result follows easily from Theorem 5.1 parts (1) and (2). ■

We now consider four special classes of sequences and derive necessary and sufficient conditions for these classes of sequences to be 3-graphic.

If **H** is a hypergraph then it will be convenient to define

$$\mathbf{H}(1) = \{B \in \mathbf{H}: 1 \in B\},$$

the set of all blocks of **H** containing the point 1.

THEOREM 5.3. Suppose that $p \geq 4$. Then $(\binom{p-1}{2}, d_2, ..., d_p)$ is 3-graphic if and only if $(d_2-(p-2), d_3-(p-2), ..., d_p-(p-2))$ is 3-graphic.

PROOF: Let $D = (\binom{p-1}{2}, d_2, ..., d_p)$ and $D' = (d_2-(p-2), d_3-(p-2), ..., d_p-(p-2))$.

Suppose that D' is 3-graphic. Let **M'** be a 3-graph whose set of points is $\{2,3,...,p\}$ and such that $\deg(i,\mathbf{M}') = d_i-(p-2)$, where $2 \leq i \leq p$. Then $\mathbf{M}' \cup C(1)$ is a 3-graphic realisation of D.

Conversely suppose that D is 3-graphic and let **M** be a standard 3-graphic realisation of D. Then $\deg(1,\mathbf{M}) = \binom{p-1}{2}$, so $C(1) \subseteq \mathbf{M}$. Thus $\mathbf{M} \backslash C(1)$ is a 3-graphic realisation of D'. ■

LEMMA 5.4. Let **H** be a hypergraph. If $\deg(i,\mathbf{H}) > \deg(j,\mathbf{H})$ then there are $\deg(i,\mathbf{H}) - \deg(j,\mathbf{H})$ blocks, B, such that

(a) $B \in \mathbf{H}$,

(b) $i \in B$,

(c) $j \notin B$, and

(d) $(B \backslash \{i\}) \cup \{j\} \notin \mathbf{H}$.

PROOF: Let $B_1, ..., B_m$ be all the blocks in **H** which contain i but not j. Let $A_1, ..., A_n$ be the blocks in **H** which contain j but not i. Since $\deg(i,\mathbf{H})$ is greater than $\deg(j,\mathbf{H})$, then $m-n = \deg(i,\mathbf{H})-\deg(j,\mathbf{H})$. For each $k \in \{1,...,m\}$, if the i in $B_k$ is replaced by j then the resulting block is either in $\{A_1,...,A_n\}$ or not in **H**. Hence there are at least m-n blocks which satisfy the required four properties. ■

Let **M** be a standard 3-graph. Define A(**M**), the absent set of **M**, by $A(\mathbf{M}) = \{a \in P \backslash \{1\} : \text{there is a } B \in C(1) \backslash \mathbf{M} \text{ such that } a \in B\}$.

The cardinality of A(**M**) is denoted by $|A(\mathbf{M})|$. For any $k \in \{0,1,...,p\}$, we define $W(k) = \{p-k+1, p-k+2, ..., p\}$.

The following Lemma shows that we may assume that the absent set of a 3-graph consists of the points with the smallest degrees.

LEMMA 5.5. Let M be a standard 3-graphic realisation of D. Let $|A(M)| = k$. Then there is a standard 3-graphic realisation, M*, of D such that $A(M^*) = W(k)$.

PROOF: Let M be a standard 3-graphic realisation of D and let $|A(M)| = k$. If $A(M) \neq W(k)$ then there is an $a \in A(M)$ and a $w \in W(k)$ such that $a \notin W(k)$ and $w \notin A(M)$. The result follows if we can construct a standard 3-graphic realisation, M', of D such that $A(M') = (A(M)\backslash\{a\}) \cup \{w\}$.

Since M is standard and $a < w$, we have $deg(a,M) \geq deg(w,M)$. Now let $n = deg(w,M(1)) - deg(a,M(1))$. Then $n > 0$. So if $H = M\backslash M(1)$ then $deg(a,H) - deg(w,H) \geq n$. By Lemma 5.4, there are n blocks, $B_1,...,B_n$, in H such that for all $i \in \{1,...,n\}$, $a \in B_i$, $w \notin B_i$ and if $B_i' = (B_i\backslash\{a\}) \cup \{w\}$ then $B_i' \notin H$. Moreover, since each $B_i$ contains no 1's, $B_i'$ can not contain any 1's either. Hence $B_i' \notin M(1)$ and so $B_i' \notin M$.

Let $A = C(1)\backslash M(1)$ be the set of blocks which contain 1 but are absent from M. Then for each $x \in P\backslash\{1\}$,

$$deg(x,A) + deg(x,M(1)) = deg(x,C(1)) = p-2.$$

So $deg(a,A) - deg(w,A) = n$. But $deg(w,A) = 0$ because $w \notin A(M)$, and so $deg(a,A) = n$. Let the n absent blocks which contain a be $\{1,a,x_1\}, ..., \{1,a,x_n\}$. Since $w \notin A(M)$, the blocks $\{1,w,x_1\}, ..., \{1,w,x_n\}$ are in M(1) and hence in M. Define M' to be M with $B_1,...,B_n$ and $\{1,w,x_1\}, ..., \{1,w,x_n\}$ replaced by $B_1',...,B_n'$ and $\{1,a,x_1\}, ..., \{1,a,x_n\}$.

Now M' is a standard 3-graphic realisation of D such that $A(M') = (A(M)\backslash\{a\}) \cup \{w\}$. ■

Before we state the next three theorems we shall need the following definition. Suppose that $p \geq 4$ and that $D = (d_1, d_2, ..., d_p)$ is a sequence of integers. Let $(t_1, t_2, ..., t_n)$ be a sequence of at least two positive integers. Define $D(t_1, ..., t_n)$ to be the sequence of p-1 terms as follows.

If $n \geq p$ then $D(t_1,...,t_n)$ is not defined.

If $n = p-1$ then $D(t_1,...,t_n)$

$$:= (d_2-(p-2) + t_1, d_3-(p-2) + t_2, ..., d_p-(p-2) + t_n).$$

If $2 \leq n \leq p-2$ then $D(t_1,...,t_n)$

$$= (d_2-(p-2), ..., d_{p-n}-(p-2), d_{p-n+1}-(p-2) + t_1, ..., d_p-(p-2) + t_n).$$

That is $D(t_1,...,t_n)$ is formed from D by deleting the first term, subtracting p-2 from the remaining terms, and then adding $t_1,...,t_n$ to the last n terms.

THEOREM 5.6. Suppose that $p \geq 4$. Let $D = (\binom{p-1}{2}-1, d_2,...,d_p)$ and let D' be D(1,1) arranged in decreasing order. Then D is 3-graphic if and only if D' is 3-graphic.

PROOF: Suppose that D' is 3-graphic. Let M' be a 3-graph on the set of points $\{2,3,...,p\}$ such that $\deg(i,M') = d_i - (p-2)$, where $2 \leq i \leq p-2$, and $\deg(p-1,M') = d_{p-1}-(p-2) + 1$ and $\deg(p,M') = d_p-(p-2) + 1$. If $H = C(1)\backslash\{\{1,p-1,p\}\}$ then $M' \cup H$ is a 3-graphic realisation of D.

Conversely suppose that D is 3-graphic. In any 3-graphic realisation, M, of D there will be exactly one block in $C(1)\backslash M$, and so $|A(M)| = 2$. By Lemma 5.5, there is a standard 3-graphic realisation, M, of D such that $A(M) = W(2) = \{p-1,p\}$. Then $M\backslash M(1)$ is a 3-graphic realisation of D'. ■

The following lemma says that the points with the lowest degrees may be assumed to occur more often in the absent blocks.

LEMMA 5.7. Let M be a standard 3-graphic realisation of D. Then there is a standard 3-graphic realisation, M*, of D such that $A(M^*) = A(M)$ and if $a,b \in A(M)$ and $a < b$ then $\deg(a,M^*(1)) \geq \deg(b,M^*(1))$, and so $\deg(a,C(1)\backslash M^*) \leq \deg(b,C(1)\backslash M^*)$.

PROOF: Let M be a standard 3-graphic realisation of D. Take any a, $b \in A(M)$ and suppose that $a < b$. If $\deg(a,M(1)) \geq \deg(b,M(1))$ then there is nothing to prove. So suppose that $\deg(a,M(1)) < \deg(b,M(1))$. The result will follow if we can construct a standard 3-graphic realisation, M', of D such that $A(M') = A(M)$, $\deg(a,M'(1)) > \deg(b,M'(1))$ and for all $x \in P\backslash\{a,b\}$ $\deg(x,M'(1)) = \deg(x,M(1))$. We shall now construct such an M'.

Since M is standard $\deg(a,M) \geq \deg(b,M)$. Let $\deg(b,M(1)) - \deg(a,M(1)) = n$. So if $H = M\backslash M(1)$ then $\deg(a,H) - \deg(b,H) \geq n$. By Lemma 5.4, there are n blocks, $B_1,...,B_n$, in H such that for all $i \in \{1,...,n\}$, $a \in B_i$, $b \notin B_i$ and if $B_i' = (B_i\backslash\{a\}) \cup \{b\}$ then $B_i' \notin H$. Moreover, each such $B_i'$ is not in M.

Let $A = C(1)\backslash M(1)$ be the set of blocks which contain 1 but are absent from M. Then $\deg(x,A) + \deg(x,M(1)) = \deg(x,C(1)) = p-2$ for each $x \in P\backslash\{1\}$. So $\deg(a,A) - \deg(b,A) = n$. By Lemma 5.4, there are n blocks, $\{1,a,x_1\},\{1,a,x_2\},...,\{1,a,x_n\}$, in A such that, for all $i \in \{1,...,n\}$, $b \neq x_i$ and $\{1,b,x_i\} \notin A$. So, for all $i \in \{1,...,n\}$,

$\{1,b,x_i\} \in M$. Define M' to be M with $B_1,\ldots,B_n$ and $\{1,b,x_1\},\ldots,\{1,b,x_n\}$ replaced by $B_1',\ldots,B_n'$ and $\{1,a,x_1\},\ldots,\{1,a,x_n\}$.

Now M' is a standard 3-graphic realisation of D such that $\deg(a,M'(1)) - \deg(b,M'(1)) = n$, and $\deg(x,M'(1)) = \deg(x,M(1))$ for all $x \in P\backslash\{a,b\}$. Moreover, since $a,b \in A(M)$ and $n > 0$, we have $a,b \in A(M')$ and so $A(M') = A(M)$. ■

THEOREM 5.8. Suppose that $p \geq 4$. Let $D = (\binom{p-1}{2}-2, d_2,\ldots,d_p)$, D' be D(1,1,1,1) arranged in decreasing order, and D" be D(1,1,2) arranged in decreasing order. Then D is 3-graphic if and only if either D' or D" is 3-graphic.

PROOF: If D' is 3-graphic then there is a 3-graph M' on the set of points $\{2,3,\ldots,p\}$ such that $\deg(i,M') = d_i-(p-2)$, where $2 \leq i \leq p-4$, and $\deg(j,M') = d_j-(p-2) + 1$, where $p-3 \leq j \leq p$. If $H = C(1)\backslash\{\{1,p-3,p-2\}, \{1,p-1,p\}\}$ then $M' \cup H$ is a 3-graphic realisation of D.

If D" is 3-graphic then there is a 3-graph M" on the set of points $\{2,3,\ldots,p\}$ such that $\deg(p,M") = d_p-(p-2) + 2$, $\deg(p-1,M") = d_{p-1} - (p-2) + 1$, $\deg(p-2,M") = d_{p-2}-(p-2) + 1$, and $\deg(i,M") = d_i-(p-2)$, where $2 \leq i \leq p-3$. If $H = C(1)\backslash\{\{1,p-2,p\}, \{1,p-1,p\}\}$ then $M" \cup H$ is a 3-graphic realisation of D.

Conversely suppose that D is 3-graphic. In any 3-graphic realisation, M, of D there will be exactly two blocks in $C(1)\backslash M$. These two absent blocks have one of the following two forms.

(1) $\{1,a_1,a_2\}$ and $\{1,a_3,a_4\}$.
(2) $\{1,a_1,a_2\}$ and $\{1,a_1,a_3\}$.

We consider these two cases separately.

CASE 1: The absent blocks are $\{1,a_1,a_2\}$ and $\{1,a_3,a_4\}$, where $a_i = a_j$ if and only if $i = j$. By Lemma 5.5, there is a standard 3-graphic realisation, M, of D such that $A(M) = W(4) = \{p-3,p-2,p-1,p\}$. Then $M\backslash M(1)$ is a 3-graphic realisation of D'.

CASE 2: The absent blocks are $\{1,a_1,a_2\}$ and $\{1,a_1,a_3\}$, where $a_i = a_j$ if and only if $i = j$. By Lemma 5.5, there is a standard 3-graphic realisation, M, of D such that $A(M) = W(3) = \{p-2, p-1, p\}$. By Lemma 5.7, there is a standard 3-graphic realisation, M*, of D such that $C(1)\backslash M^* = \{\{1,p-1,p\}, \{1,p-2,p\}\}$. Then $M^*\backslash M^*(1)$ is a 3-graphic realisation of D". ■

THEOREM 5.9. Suppose that $p \geq 4$. Let $D = (\binom{p-1}{2}-3, d_2, \ldots, d_p)$, D1 be D(1,1,1,1,1,1) arranged in decreasing order, D2 be D(1,1,1,1,2) arranged in decreasing order, D3 be D(1,1,2,2) arranged in decreasing order, D4 be D(2,2,2) arranged in decreasing order, and D5 be D(1,1,1,3) arranged in decreasing order. Then D is 3-graphic if and only if one of D1, D2, D3, D4, or D5 is 3-graphic.

PROOF: If D1 is 3-graphic then there is a 3-graph **M1** on the set of points $\{2,3,\ldots,p\}$ such that $\deg(i,\mathbf{M1}) = d_i-(p-2)$, where $2 \leq i \leq p-6$, and $\deg(j,\mathbf{M1}) = d_j - (p-2) + 1$, where $p-5 \leq j \leq p$. $\mathbf{M1} \cup \mathbf{H}$ is a 3-graphic realisation of D if $\mathbf{H} = \mathbf{C}(1)\backslash\{\{1,p-5,p-4\}, \{1,p-3,p-2\}, \{1,p-1,p\}\}$.

If D2 is 3-graphic then there is a 3-graph **M2** on the set of points $\{2,3,\ldots,p\}$ such that $\deg(i,\mathbf{M2}) = d_i-(p-2)$, where $2 \leq i \leq p-5$, $\deg(j,\mathbf{M2}) = d_j-(p-2) + 1$, where $p-4 \leq j \leq p-1$, and $\deg(p,\mathbf{M2}) = d_p-(p-2) + 2$. If $\mathbf{H} = \mathbf{C}(1)\backslash\{\{1,p-4,p-3\},\{1,p-2,p\},\{1,p-1,p\}\}$ then $\mathbf{M2} \cup \mathbf{H}$ is a 3-graphic realisation of D.

If D3 is 3-graphic then there is a 3-graph **M3** on the set of points $\{2,3,\ldots,p\}$ such that $\deg(i,\mathbf{M3}) = d_i-(p-2)$, where $2 \leq i \leq p-4$, $\deg(j,\mathbf{M3}) = d_j - (p-2) + 1$, where $p-3 \leq j \leq p-2$, and $\deg(k,\mathbf{M3}) = d_k - (p-2) + 2$, where $p-1 \leq k \leq p$. If $\mathbf{H} = \mathbf{C}(1)\backslash\{\{1,p-3,p-1\}, \{1,p-2,p\}, \{1,p-1,p\}\}$ then $\mathbf{M3} \cup \mathbf{H}$ is a 3-graphic realisation of D.

If D4 is 3-graphic then there is a 3-graph **M4** on the set of points $\{2,3,\ldots,p\}$ such that $\deg(i,\mathbf{M4}) = d_i - (p-2)$, where $2 \leq i \leq p-3$, and $\deg(j,\mathbf{M4}) = d_j - (p-2) + 2$, where $p-2 \leq j \leq p$. $\mathbf{M4} \cup \mathbf{H}$ is a 3-graphic realisation of D if $\mathbf{H} = \mathbf{C}(1)\backslash\{\{1,p-2,p-1\}, \{1,p-2,p\}, \{1,p-1,p\}\}$.

If D5 is 3-graphic then there is a 3-graph **M5** on the set of points $\{2,3,\ldots,p\}$ such that $\deg(i,\mathbf{M5}) = d_i-(p-2)$, where $2 \leq i \leq p-4$, $\deg(j,\mathbf{M5}) = d_j - (p-2) + 1$, where $p-3 \leq j \leq p-1$, and $\deg(p,\mathbf{M5}) = d_p-(p-2) + 3$. If $\mathbf{H}=\mathbf{C}(1)\backslash\{\{1,p-3,p\},\{1,p-2,p\},\{1,p-1,p\}\}$ then $\mathbf{M5} \cup \mathbf{H}$ is a 3-graphic realisation of D.

Conversely suppose that D is 3-graphic. In any 3-graphic realisation, M, of D there will be exactly three blocks in $\mathbf{C}(1)\backslash\mathbf{M}$. These three absent blocks have one of the following five forms.

(1) $\{1,a_1,a_2\}$, $\{1,a_3,a_4\}$, and $\{1,a_5,a_6\}$.
(2) $\{1,a_1,a_2\}$, $\{1,a_1,a_3\}$, and $\{1,a_4,a_5\}$.
(3) $\{1,a_1,a_2\}$, $\{1,a_1,a_3\}$, and $\{1,a_2,a_4\}$.
(4) $\{1,a_1,a_2\}$, $\{1,a_1,a_3\}$, and $\{1,a_2,a_3\}$.
(5) $\{1,a_1,a_2\}$, $\{1,a_1,a_3\}$, and $\{1,a_1,a_4\}$.

We consider these five cases separately.

CASE 1: The absent blocks are $\{1,a_1,a_2\}$, $\{1,a_3,a_4\}$, and $\{1,a_5,a_6\}$, where $a_i = a_j$ if and only if $i = j$. By Lemma 5.5, there is a standard 3-graphic realisation, M, of D such that $A(M) = W(6) = \{p-5,p-4,p-3,p-2,p-1,p\}$. Then $M \backslash M(1)$ is a 3-graphic realisation of D1.

CASE 2: The absent blocks are $\{1,a_1,a_2\}$, $\{1,a_1,a_3\}$, and $\{1,a_4,a_5\}$, where $a_i = a_j$ if and only if $i = j$. By Lemma 5.5, there is a standard 3-graphic realisation, M, of D such that $A(M) = W(5) = \{p-4,p-3,p-2,p-1,p\}$. By Lemma 5.7, there is a standard 3-graphic realisation, M*, of D such that if $A = C(1) \backslash M^*$ then $\deg(p,A) = 2$, and for all $w \in W(5) \backslash \{p\}$, $\deg(w,A) = 1$. Then $M^* \backslash M^*(1)$ is a 3-graphic realisation of D2.

CASE 3: The absent blocks are $\{1,a_1,a_2\}$, $\{1,a_1,a_3\}$, and $\{1,a_2,a_4\}$, where $a_i = a_j$ if and only if $i = j$. By Lemma 5.5, there is a standard 3-graphic realisation, M, of D such that $A(M) = W(4) = \{p-3,p-2,p-1,p\}$. By Lemma 5.7, there is a standard 3-graphic realisation, M*, of D such that if $A = C(1) \backslash M^*$ then $\deg(p,A) = 2 = \deg(p-1,A)$, and $\deg(p-2,A) = 1 = \deg(p-3,A)$. Then $M^* \backslash M^*(1)$ is a 3-graphic realisation of D3.

CASE 4: The absent blocks are $\{1,a_1,a_2\}$, $\{1,a_1,a_3\}$, and $\{1,a_2,a_3\}$, where $a_i = a_j$ if and only if $i = j$. By Lemma 5.5, there is a standard 3-graphic realisation, M, of D such that $A(M) = W(3) = \{p-2,p-1,p\}$. Then $M \backslash M(1)$ is a 3-graphic realisation of D4.

CASE 5: The absent blocks are $\{1,a_1,a_2\}$, $\{1,a_1,a_3\}$, and $\{1,a_1,a_4\}$, where $a_i = a_j$ if and only if $i = j$. By Lemma 5.5, there is a standard 3-graphic realisation, M, of D such that $A(M) = W(4) = \{p-3,p-2,p-1,p\}$. By Lemma 5.7, there is a standard 3-graphic realisation, M*, of D such that if $A = C(1) \backslash M^*$ then $\deg(p,A) = 3$, and for all $w \in W(4) \backslash \{p\}$, $\deg(w,A) = 1$. Then $M^* \backslash M^*(1)$ is a 3-graphic realisation of D5. ■

## Section 6. Numerical Results

In sections 2, 3, 4, and 5 we have presented some conditions for degree sequences to be 3-graphic. Some idea of the power of these conditions can be gained by using them to find all the 3-graphic sequences of a given length. The table below shows the results that were obtained when the lengths of the sequences were very small.

The top ten rows record the number of sequences which might still be 3-graphic after applying some condition or combination of conditions. The next two rows record the number of sequences which can be proved to be 3-graphic after applying some condition or combination of conditions. The last row shows the number of 3-graphic degree sequences. We shall now explain the condition or conditions with which each row is associated.

| | p = 3 | p = 4 | p = 5 | p = 6 | p = 7 | p = 8 |
|---|---|---|---|---|---|---|
| A | 4 | 35 | 462 | 8 008 | 170 544 | 4 292 145 |
| AB | 2 | 13 | 156 | 2 676 | 56 862 | 1 430 739 |
| ABC | 2 | 5 | 31 | 445 | 8 390 | 190 534 |
| ABD | 2 | 5 | 35 | 476 | 8 869 | 210 176 |
| ABE | 2 | 5 | 32 | 406 | 7 246 | 164 837 |
| ABCD | 2 | 5 | 31 | 399 | 7 249 | 164 451 |
| ABCE | 2 | 5 | 31 | 379 | 6 728 | 151 336 |
| ABDE | 2 | 5 | 31 | 384 | 6 709 | 148 552 |
| ABCDE | 2 | 5 | 31 | 372 | 6 480 | 143 166 |
| ABCDEF | 2 | 5 | 31 | 371 | 6 432 | 141 988 |
| G | 2 | 5 | 31 | 369 | 6 190 | 130 677 |
| GF | 2 | 5 | 31 | 369 | 6 204 | 132 441 |
| | 2 | 5 | 31 | 369 | 6 204 | ? |

Only decreasing sequences of non-negative integers are considered. Each letter from A to G denotes a particular condition as indicated below.

A denotes the condition in Theorem 2.3(2).
B denotes the condition in Theorem 2.3(1).
C denotes the condition in Theorem 3.6.
D denotes the condition in Theorem 2.1.
E denotes the condition in Theorem 2.3(4 & 5).
F denotes the condition in Corollary 5.2.
G denotes the condition in Lemma 4.1.

The conditions with which each row is associated are indicated by the letters in its first column.

It can be shown that, for every p, the numbers in the top row are $\binom{p+m}{p}$ where $m = \binom{p-1}{2}$. Also the table shows that each of the conditions C, D, and E is needed. For each value of p, we note that the number in row ten will always be greater than or equal to the number in row twelve. If these two numbers are equal then this must be the number of 3-graphic degree sequences. When $p \geq 6$ these two numbers are not equal. When $p = 6$ the two unclassified sequences can be shown not to be 3-graphic by applying the condition in Theorem 5.3. When $p = 7$ the 228 unclassified sequences can be shown not to be 3-graphic by applying the conditions in Theorems 5.3, 5.6, 5.8, and 5.9. Thus we can determine all the 3-graphic sequences of length at most seven. However when $p = 8$ the 9547 unclassified sequences can not be classified by applying the conditions in Theorems 5.3, 5.6, 5.8, and 5.9. They could be classified by proving more theorems similar to 5.3, 5.6, 5.8, and 5.9. However, the complexity and speciality of such theorems means that they are probably not warranted.

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Computing and Information Studies,
Griffith University, Nathan,
Brisbane, Queensland,
Australia 4111.