SEVENTEEN QUADRUPLES ON SEVENTEEN POINTS

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1. Introduction.

The numbers $g^{(4)}(v)$, which give the minimum number of blocks in a pairwise balanced design with blocks of sizes 2, 3, and 4, were determined in [1] for all values except $v = 17, 18, 19$; it was also shown that $g^{(4)}(17) > 28$. In [2] and [3], it was shown that one could not obtain a minimum for the case $v = 17$ if one employed 19 or 20 quadruples. A covering of all pairs in 31 blocks is due to Stinson and Seah (cf. [2]), and it is shown in [3] that the minimality of this covering depends on excluding only three further cases. We shall, in this paper, discuss the case in which $g_2 = 1$, $g_3 = 11$, $g_4 = 17$ (one pair, 11 triples, and 17 quadruples).

As previously, in [2] and [3], we consider the defect graph consisting of all pairs not occurring in the quadruples. The points in this graph have valencies of 1, 4, 7, 10, 13, or 16, and we use a_i to designate the number of points of valence i. Then we have the following equations.

(1) $\Sigma a_i = 17,$

(2) $\Sigma i a_i = 68,$

(3) $a_4 + 2a_7 + 3a_{10} + 4a_{13} + 5a_{16} = 17.$

There can not be more than 2 triangles through a point of valence 4, not more than 3 through a point of valence 7, not more than 5 through a point of valence 10, etc. We may then count triangles to give

(4) $2a_4 + 3a_7 + 5a_{10} + 6a_{13} + 8a_{16} = 33 + y.$

Calculating (2) - 2(4) and 2(3) - (4), we obtain

(5) $a_1 + a_7 + a_{13} = 2 - 2y,$

(6) $a_7 + a_{10} + 2a_{13} + 2a_{16} = 1 - y.$

If $y = 1$, then $a_4 = 17$, that is, there are 17 points of valence 4 in the defect graph; this is impossible, since there is no way in which the 34 lines can be partitioned into eleven triangles and a pair (since a triangle through any point uses exactly 2 of the edges through that point). We thus conclude that $y = 0$, and find, from (5) and (6), that the only possibilities are:

Case 1: $a_7 = 1, a_1 = 1, a_4 = 15$;

Case 2: $a_{10} = 1, a_1 = 2, a_4 = 14$.

2. The Case of a Point of Valence 7.

We designate the point of valence 7 as A, and the point of valence 1 as B. We also set C = {1,2,3,4,5,6} and set D = {x,y,z,r,s,t,u,v,w}; the set C + D represents the fifteen points of valence 4. Then the design consists of the following blocks.

The single pair AB; three triples A12, A34, A56, together with eight other triples on the multiset C + 2D; three quadruples Addd, where the 12 elements d range over D; five quadruples Bxxx, where the 15 elements x range over the set C + D; nine quadruples xxxx, where the 36 elements x range over the multiset 3C + 2D.

Since it is not possible to have more than two blocks of the form cccd, we first consider the case when the last nine blocks (call them the set P) contain two blocks cccd, five blocks ccdd, two blocks cddd. Only one cc pair remains, and so we have one block Bccd and four blocks Bcdd. There are six triples cdd and two triples ddd.

Lemma 1. It is not possible to have a design in which there are two blocks cccd that have an element c in common.

Proof. Since the blocks cccd have an element c in common, we may take these blocks as 135d, 146d. The other seven blocks in P can be taken as 1ddd, 32dd, 36dd, 52dd, 54dd, 24dd, 6ddd (the choice of 26dd and 4dd in P is an isomorphic case under the permutation (35)(46)). Then we must also have B26d as a block. Take 1xyz as a block; if x occurs in the block 6ddd, then it can only occur with a total of five elements c. Thus the last block in P may be taken as 6uvw, and it is disjoint from the block 1xyz.

Now the elements u,v,w, must occur with 1. The only available elements 1 are in the blocks 135d, 1dd, and B1dd. Hence we may take 135u, B1vd, and1wd; then B4ud and 2ud are forced. Now consider the element d in B26d; it can only occur with 54 and with 1 in the set P. So we may take the blocks 54xd, B26x, 3xd. We next look at the appearance of y and z; they occur in two of the blocks 32dd, 36dd, 52dd, 24dd, as well as in two of the blocks B3dd, B4ud, B5dd, and in two of the triples 2ud, 4dd, 5dd, 6dd. We first note that 36yd is required; then B4uy leads to 2uy,5yd (a repeat on ud). So we must take 36yd, B5yd, 2uy, 4yd.

If we select 52zd, z must occur in two of the block 3xd, 4yd, 6dd, and this is a contradiction.

If we select 32zd, then we must take B4uz, 5zd, 6zd; take the block 36yr, and obtain B1vr, 5zr, 24rd. Now 54xd can not contain v or w (a repeat on 3 is forced). Hence we must take 54xs; this leads to 32zs, and there is no place for s in the B blocks.

Finally, if we select 24zd, we are forced to take B3zd, 5zd, 6zd. Again, 36yr leads to 5zr, B1vr, 24zr (contradiction). Thus the proof of Lemma 1 is complete.

Lemma 2. It is not possible to have a design in which the two blocks cccd contain 6 distinct elements c.

Proof. In this case, we start with the blocks 135d and 246d. The other blocks in set P may be taken as 14dd, 16dd, 32dd, 36dd, 52dd, 5ddd, 4ddd. Then B54d is a block. If an element occurs in both 5ddd and 4ddd, then it can only appear with five c-elements. Hence, we take the blocks 5xyz and 4uvw. Since x, y, and z, must occur with 4, we may take 246x, 14yd, 4zd; similarly, we get 135u, 52vd, 5wd.

If we now take 52vr, then the element r must appear in another P block (r is different from x, y, z, u, v, w). Suppose we have a block 14yr. Then we have elements s, s, t, and t appearing in the blocks 16dd, 32dd, and 36dd; this is impossible without a repeat.

If we have blocks 52vr and 16rd, then we must have B3rd and 2rd. Now consider the element d that occurs in the block B3rd; if it is a new element s, there is only one block 14yd in which it can occur. Consequently, it must be an old element, namely, w, and we have the block 14yw. Then elements s, s, t, t, appear in the three blocks 16dd, 32dd, 36dd, and we reach the same contradiction as before.

Finally, take the case when there are blocks 52vr and 36rd; then we have B1rd and 4zr as the other blocks that contain r. Now consider the blocks 14yd, 16dd, 32dd, 36rd; they contain the elements z, w, s, s, t, t. The only pairs that can occur with s and t are the combinations 16 and 32; or 14 and 32; or 14 and

36. Thus there is a symbol, say t, that occurs with 16 and 32. This forces the blocks B54t, tdd, and tdd. We also must have the block 5ws; since s occurs with four c-elements in the set P, one c-element in the B blocks, and one c-element in the triples, we see that we must have the triple tsd. This proves that we have blocks 14ys and 36rs. The remaining two quadruples in P are then either 16tz, 32tw, or 16tw, 32tz. In either case, we can easily fill in the positions of z, w, and x in the other blocks, but we fail when we subsequently try to find a place for y. This completes the proof of Lemma 2.

Lemma 3. It is not possible to have a single block cccd.

Proof. In this case, we have a single block 135d. We discuss two subcases.

Case 1. If an element from 1,3,5, occurs singly, the other blocks of the set P may be taken as 1ddd, 14dd, 32dd, 36dd, 52dd, 54dd, 26dd, 46dd (the choice of 16dd gives a set of blocks that is equivalent under the permutation (46)(35)). Pairs 16 and 24 still are needed; let x occur with 24, and we find that x must occur with 36 and with 1 in the set P. Let also y occur with 16; then y must occur in 32yd and 54yd. We let 1xuv be a block.

If B24x is a block and 16y, 5xd, are triples, then it follows from frequency considerations that u and v must each occur twice in the blocks 2dd, 3dd, and 4dd; this leads to a repeat on uv. Similarly, if B16y and B5xd are blocks, and 24x is a triple, then u and v must occur twice in the triples 3dd, 5dd, and 6dd. Again a repeat occurs on uv. Consequently, we are forced to take blocks B24x, B16y, and 5xd. There are now triples ydd and ydd. Without loss of generality, we may take 3ud, B3vd, B5ud; there are now two cases.

If 26ud is a block we get u4d, 2vd, 6vd, and 45yv. Let us consider 135r; 26ur forces 4ur, and so we must take 46rd, Brdd, 2vr, yrd. Now consider 26us; we must have 14sd, B3vs, and 5xs. Finally, consider 46rt; this forces B5ut and 32yt; the triples must be 1td and ytd (but this gives a repeat on yt).

If 46ud is a block, we similarly get u2d with v4d, v6d, and 52vd. Again we take 135r; now 46ur forces 2ur, and so we must take 26rd, and then 4vr, yrd, Brdd. Now take 46us; this forces 32ys or 52vs (the latter is followed by B3vs or 3us, either of which yields a contradiction). So we must take 32ys, Brsd, 1sd, 5xs. Then 26rt forces 14td, B5ut, and 3ut (a contradiction); or 26rt forces 54yt, 1st, and B3vt. In this latter case, w and z are the other two elements; we get 14wz, 36xw, 52vz, and this forces B5uw and 2uw (again, a contradiction).

Case 2. In this case, the element occurring singly is not one of 1,3,5. Thus, we take the set P to be 135d, 2ddd, 14dd, 16dd, 32dd, 36dd, 52dd, 54dd, 46dd.

The missing pairs are 26 and 24. Let x occur with 26; then x must occur in two of blocks in P; the only available blocks are 135d, 14dd, 54dd, and no two of these are disjoint. Hence Case 2 is impossible, and we have completed the proof of Lemma 3.

The only case remaining is when the last nine blocks are all of the form ccdd.

Lemma 4. It is not possible to have a design in which the last nine blocks all possess the form ccdd.

Proof. In this case, we take the last nine blocks in the form 14dd, 15dd, 16dd, 23dd, 24dd, 26dd, 35dd, 36dd, 45dd; the pairs remaining (namely, 13, 25, 46) form a 1-factor on C. We first look at the blocks containing 13d, 25d, 46d (these must occur in triples or with B).

Occurrence of 46z (as a triple, or in B46z) forces 15zd, 23zd; also, 13x forces 26xd, 45xd, and 25y forces 14yd, 36yd. Now at least one of 46z, 13x, 25y, must occur with B. If all three occur with B, then the B blocks are B13x, B25y, B46z, Brst, Buvw. It follows that r,s,t,u,v,w, occur twice each in the blocks cdd; hence x, y, z, occur twice each in the triples ddd, ddd. This produces a repeated block.

If only one of 13x, 25y, 46z, occurs with B, then two of these blocks occur in triples (say, 13x and 25y). This shows that x occurs with two c-elements in the triples, 1 c-element in the B blocks, and 4 c-elements in the set P. This is a contradiction. Consequently, we are led to consider the case when the B blocks are B13x, B25y, B4rs, B6tu, Bzvw, and the triples are 46z, 1dd, 2dd, 3dd, 5dd, ddd (thrice).

First, we note that the elements r and s, as well as the elements t and u, must occur singly in the triples 1dd, 2dd, 3dd, and 5dd; similarly, the elements v, v, w, w, also occur in these triples. Also, r and s must occur with 6 in the set P, and t and u must occur with 4 in P. Thus, the element from {r,s,t,u} that occurs in 5dd also occurs in 23zd, and the element that occurs in 3dd must also occur in 15zd; then, the element that occurs in 1dd must occur in 35dd, and the element that occurs in the triple 2dd must occur in 35dd. There is no loss of generality in taking 35rt, 1rd, and 2td (r also occurs in 26xr, and t also occurs in 14yt). Then we have blocks 5sd, 23zs, 16sd, as well as the blocks 3ud, 15zu, and 24ud.

There are 4 symbols v, v, w, w, to place in the set P. Putting v with 16, 24, forces 3uv and 5sv (repeated pair). Hence we must take 16sv, 45xv, 24uw, 36yw, as well as 1rw, 2tv, 3uv, 5sw.

Now the blocks Avdd, Awdd, and Azdd must contain the missing pairs vr and vy as well as wt and wx; hence the third block must be Azsu, and this is a contradiction, since z has already occurred with s and u. This completes the proof of Lemma 4.

We have now completed the discussion of the case of a point of valence 7 in the defect graph, and have proved

Theorem 1. It is not possible to have a covering in twenty-nine blocks with seventeen quadruples, eleven triples, and a pair in which the defect graph contains a point of valence 7.

3. Case of a Point of Valence 10.

In this case, we let X represent the point of valence 10, and we let A and B represent the two points of valence 1; we take the points of valence 4 to be the set C + D, where C = {1,2,,3,4,5,6,7,8,9,t} and D = {r,s,t,u}. Then the design consists of the following blocks.

The pair AB; 5 triples of the form Xcc; 6 triples of the form xxx, where x ranges over the multiset C + 2D; quadruples XArs and XBtu; 4 quadruples Axxx, where x ranges over the set C + {t,u}; 4 quadruples Bxxx, where x ranges over the set C + {r,s}; a set P consisting of 7 other quadruples xxxx, where x ranges over the multiset 2C + 2D.

The 8 elements from 2D must occur in the 6 triples xxx and in the 7 quadruples xxxx; also, at least 2 of the pairs rt, ru, st, su, must occur in the triples.

If these last four pairs all occur in the six triples, then the eight elements r, r, s, s, t, t, u, u, must occur singly in seven quadruples, and this is not possible.

If the six triples are rtc, ruc, stc, scc, ucc, ccc, then the seven quadruples must be succ, rccc, rccc, tccc, tccc, sccc, uccc. Let the first of these quadruples be su12; then 1 occurs again, say with r, in the quadruple r1cc. This proves that 1 must occur in the triple ccc; since the same argument holds for the element 2, we have a repeated pair 12. We have thus shown that two of the pairs rt, tu, st, su, must occur in triples and the other two pairs must occur in quadruples.

Suppose now that the pairs in the triples have an element in common; then the triples are rtc, ruc, tcc, ucc, scc, scc, and the quadruples are stcc, succ, tccc, uccc, rccc, rccc, cccc. Consider the quadruple st12; then the element 1 must occur in triples ruc or ucc. The same argument holds for the element 2; so we may take ru1 and u2c. Similarly, we obtain su34, rt3, and t4c. Since 1 and 3 now occur with r, u, s, and t, we see that 13 occurs in the block cccc. This gives the A blocks as XArs, Atcc, Aucc, A1cc, A3cc, and the B blocks as XBtu, Brcc, BScc, B1cc, B3cc. Also, 2 and 4 must occur in the 2 blocks rccc. Thus 2 occurs with 1, r, u, s, t, and so 2 must occur in the blocks A3cc and B3cc; this is a contradiction.

Finally, we consider the case when the two pairs in the triples form a 1-factor on D; the triples are then rtc, suc, rcc, ucc, scc, tcc, and the quadruples are rucc, stcc, rccc, uccc, sccc, tccc, cccc. Consider the quadruples ru12 and st34; we find that s1c and t2c are triples, and that r3c and u4c are triples.

Now look at the quadruple cccc; it can contain at most 2 elements from {1,2,3,4}. If we first suppose that it contains no elements, that is, it has the form 5678, then we see that the elements 5,6,7,8, occur with at most 6 elements from D in the triples; at most 2 elements from D in the A quadruples; at most 2 elements from D in the B quadruples; at most 4 elements from D in the last seven quadruples. This is a total of only 14 elements from D; however, 5,6,7,8, must occur with 16 elements from D. So we reject this possibility.

Now suppose that the quadruple cccc has the form 1567, that is, it contains only one element from 1,2,3,4. Since 1 already occurs in s1c and ru12, it follows that 1 must occur in the quadruple At1c. But then 5,6,7, can occur with at most 5 elements from D in the triples; with at most 1 element from D in the A quadruples; with at most 2 elements from D in the B quadruples; with at most 3 elements from D in the last seven quadruples. This is a total of only 11 elements from D, and we know that 5,6,7,8, must occur with 12 elements from D. So we also reject this possibility.

It follows that the quadruple cccc must contain exactly two elements from 1,2,3,4. Use of the permutation (12)(34)(ru)(st)(AB) shows that the quadruple may be taken as 1456 or as 1356. Consider first 1456. We first get the blocks At1c, A4cc, and Br4c, B1cc. In order to have 8 elements from D occurring with 5 and 6, we must take rt5 and su6; but then 6 is not able to occur with both r and t.

We are thus left with the final case in which the cccc block is 1356; in order that eight elements from D occur with 5 and 6, we must take rt5 and su6. We then get blocks At1c, Au3c, A5cc, A6cc; Br6c, Bs5c, B1cc, B3cc. This forces blocks r4cc and s2cc, as well as u5cc and t6cc. Now consider the block Br67; we immediately find that 7 must occur in u57c and s27c. But then it is not possible to place 7 in the triples. We have thus proved

Theorem 2. It is not possible to have a design with twenty-nine blocks containing seventeen quadruples , eleven triples, and a pair, in which there is a point of valence 10.

4. Conclusion.

Combining Theorems 1 and 2, we have shown that a pairwise balanced design with seventeen quadruples, eleven triples, and a pair does not exist. But this design, and the design consisting of nineteen quadruples, six triples, and four pairs, that was ruled out in [2], were the only possible designs on twenty-nine blocks. Hence we have proved

Theorem 3. The value of $g^{(4)}(17)$ is greater than 29.

It remains only to discuss the two possibilities cited in [3] that might give a value of 30 for $g^{(4)}(17)$.

REFERENCES

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