Defining sets of projective planes and biplanes and their residuals

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Abstract

Suppose S is a defining set of a symmetric 2-(v,k,λ) design D where $\lambda = 1$ or 2; that is, D is a projective plane or a biplane. In this paper, conditions under which the residual of S is a defining set of the residual of D are investigated. As a consequence, inequalities relating the sizes of smallest defining sets of D and of the residual of D are obtained. The exact sizes of smallest defining sets of $PG(2,5)$, $AG(2,5)$ and the three non-isomorphic 2-$(10,4,2)$ designs are determined.

1 Introduction

A **block design** $D = (V, \mathcal{B})$ is a set V of v elements (points), together with a set $\mathcal{B}$ of b k-subsets (blocks) of V, such that each element of V occurs in precisely r blocks, for some positive integers v, b, r, k. If $k < v$, D is said to be **incomplete**; if all the blocks of $\mathcal{B}$ are distinct, D is said to be **simple**. Henceforth, designs discussed in this paper are assumed to be simple. If

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every t-subset of V occurs in precisely λ_t blocks of $\mathcal{B}$, then D is a t-design with parameters t-(v, k, λ_t).

If $t = 2$, then the design is said to be **balanced**. Fisher's inequality states that for any balanced incomplete block design (BIBD) $b \geq v$. When equality holds, the design is said to be **symmetric**. Suppose $D = (V, \mathcal{B})$ is a 2-(v, k, λ) symmetric design with blocks $\mathcal{B} = \{B_1, B_2, \ldots, B_v\}$. Then D is necessarily **linked**, that is, $|B_i \cap B_j| = \lambda$ for $i \neq j$. The blocks $B_i^* = B_i \setminus B_v$, $i = 1, 2, \ldots v - 1$, all have (constant) size $k - \lambda$ and form the **residual design** $D^* = (V^*, \mathcal{B}^*)$ of D with respect to block B_v; see for instance Street and Wallis [16, page 388]. D^* has parameters 2-$(v - k, k - \lambda, \lambda)$. Similarly, for any collections of blocks $S \subseteq \mathcal{B}$ such that $B_v \notin S$, define S^*, the residual of S with respect to B_v, by

$$S^* = \{B_i \setminus B_v \mid B_i \in S\}.$$

There is thus a correspondence between collections of blocks $S \subseteq \mathcal{B}$ that do not contain B_v and collections of blocks $S^* \subseteq \mathcal{B}^*$. In this paper, we apply known results about this correspondence when $\lambda = 1, 2$ to the problem of finding *defining sets.*

When $\lambda = 1$, symmetric designs have parameters 2-$(q^2 + q + 1, q + 1, 1)$. A design with these parameters is generally known as a **projective plane** of order q. The subfamily of **Desarguesian** projective planes will be denoted by $PG(2, q)$. Desarguesian projective planes are **cyclic** and are known to exist whenever q is a prime or prime power. Residual designs of projective planes are **affine planes**. The subfamily of Desarguesian affine planes will be denoted by $AG(2, q)$.

When $\lambda = 2$, symmetric designs have parameters 2-$(\binom{q+2}{2} + 1, q + 2, 2)$. A design with these parameters is generally known as a **biplane** of order q. Biplanes of order q are known to exist whenever $q \in \{2, 3, 4, 7, 9, 11\}$.

Definition 1.1 (K. Gray [7]) *A set of blocks which is a subset of a unique t-(v, k, λ_t) design D is a* **defining set** *of that design. A defining set is* **smallest** *if no defining set of D has fewer blocks.*

Definition 1.2 *Let T_1 and T_2 be collections of m k-subsets of V. If T_1 and T_2 contain precisely the same (including repetitions) t-subsets, then T_1 and T_2 are said to be* **mutually t-balanced**. *If T_1 and T_2 are also disjoint then (T_1, T_2) is a (v, k, t)* **trade** *of* **volume** *$m = m(T)$.*

Definition 1.3 *The* **foundation** *of a collection of blocks X, $F(X)$, is the set of elements of V covered by X. Observe that if $T = (T_1, T_2)$ is a trade, then $F(T_1) = F(T_2)$.*

If $D = (V, \mathcal{B})$ is a t-(v, k, λ_t) design, (T^a, T^b) is a (v, k, t) trade and $T^a \subseteq \mathcal{B}$, then the single collection T^a is often referred to as a trade in D. It is simple to see that the collection of blocks $(\mathcal{B} \setminus T^a) \cup T^b$ also comprise a t-(v, k, λ_t) design. There is a natural relationship between defining sets and trades in a design.

Theorem 1.4 ([7]) *Let $D = (V, \mathcal{B})$ be a (simple) t-(v, k, λ_t) design and $S \subseteq \mathcal{B}$. Then S is a defining set of D if and only if S intersects each trade in D.* □

Example 1.5 An example of the 2-(7, 3, 1) design $PG(2, 2)$ is $F = (V, \mathcal{B})$ where $V = \{1, \ldots, 7\}$ and $\mathcal{B} = \{124, 235, 346, 457, 561, 672, 713\}$. Each of the seven collections of four blocks from $\mathcal{B}$ with an element of V omitted, is a trade of volume four. For instance, $T_1 = \{124, 156, 235, 346\} \subseteq \mathcal{B}$ trades with $T_2 = \{125, 146, 234, 356\}$. Here $F(T_1) = \{1, 2, 3, 4, 5, 6\}$. Using Theorem 1.4 or otherwise, it is simple to see that any set of three blocks of $\mathcal{B}$ not containing a common element of V is a smallest defining set of F. □

Lemma 1.6 ([7]) *If S is a defining set of $D = (V, \mathcal{B})$ and ρ is an automorphism of $\mathcal{B}$, then ρS is a defining set of D.* □

An immediate consequence of the previous lemma is that isomorphic designs have isomorphic collections of defining sets. Hence, one need only investigate defining sets of a single design in an isomorphism class.

Let D be a symmetric 2-(v, k, λ) design with $\lambda = 1$ or 2, and let S be a defining set of D. In this paper, conditions under which the residual of S is a defining set of the residual of D are investigated. Inequalities relating the sizes of smallest defining sets of D and its residual D^* are obtained. The following theorem summarises the main results of this paper.

Theorem 1.7
(A) *Assume $q > 3$. Let s_p be the cardinality of a smallest defining set of a projective plane P of order q. Let A be an affine plane (of order q) which*

is a residual of P and let s_a equal the cardinality of a smallest defining set of A. Then

$$s_a \le s_p \le s_a + q - 1.$$

(B) *Assume $q > 2$. Let s_d be the cardinality of a smallest defining set of a biplane D of order q. Let R be a residual of D and let s_r equal the cardinality of a smallest defining set of R. Then*

$$s_r - q \le s_d \le s_r + (3/2)q + 10. \qquad \square$$

The general methods developed in Sections 2 and 3 will also be used to obtain the sizes of various smallest defining sets given in Table 1. The results for the smallest defining sets of $PG(2,5)$ and $AG(2,5)$ are new and rely on some computation. The results for smallest defining sets of G_2, G_3, H_1, H_2 and H_3 use only the theory developed in Section 3. These latter results have previously been obtained by Greenhill and Street [11] using exhaustive computation. The algorithm that they used is explained in Greenhill [9, 10]. The author [4] has previously determined theoretically all non-isomorphic smallest defining sets of G_1.

Table 1: Sizes of smallest defining sets discussed in this paper

	$\lambda = 1$		$\lambda = 2$					
Parameters	2-(31, 6, 1)	2-(25, 5, 1)	2-(16, 6, 2)			2-(10, 4, 2)		
Design	$PG(2,5)$	$AG(2,5)$	G_1	G_2	G_3	H_1	H_2	H_3
Size	11	10	9	7	7	8	6	5

2 Defining sets of projective and affine planes

Throughout this section, $P = (V, \mathcal{B})$ denotes a projective plane of order q, $B_v = L \in \mathcal{B}$, and $P^* = A$ denotes the affine plane derived from P with respect to L. Without loss of generality, let $L = \{1, 2, \ldots, q, q+1\}$. Also, $\mathcal{B}$ can be written as

$$\{1X_1, 2X_2, \ldots, qX_q, (q+1)X_{q+1}, L\},$$

where iX_i denotes the collection of q blocks in $\mathcal{B}$, other than L, that contain i for each $i = 1, 2, \ldots, q+1$. In this way, the set of blocks $\mathcal{B}^*$ of A can be written as $\{X_1, \ldots, X_{q+1}\}$ where each X_i is a **parallel class**. The notation iY_i is used to represent a subset of blocks of iX_i, that is, Y_i is a subset of blocks of the parallel class X_i in A. Then $S = \bigcup_{i=1}^{q+1} iY_i$ is equivalent to $S^* = \bigcup_{i=1}^{q+1} Y_i$.

Now suppose that $T^* = (T^{a*}, T^{b*})$ is a trade and T^{a*} is in A. Let $T^{a*} = \bigcup_{i=1}^{q+1} Y_i^a$ where $Y_i^a \subseteq X_i$. Without loss of generality, we can write $T^{b*} = \bigcup_{i=1}^{q+1} Y_i^b$ for some partial parallel classes Y_i^b, and to ensure the labelling is consistent with that used for the parallel classes of A, we impose the condition that

$$(X_i \setminus Y_i^a) \cup Y_i^b \text{ is a complete parallel class.} \qquad (*)$$

Note that there is some flexibility in the labelling of the partial parallel classes Y_i^b if $Y_i^a = X_i$ for at least two distinct i. Define $T^b = \bigcup_{i=1}^{q+1} iY_i^b$ in the natural way. We show that $T = (T^a, T^b)$ is a trade. Certainly $T^a \cap T^b = \emptyset$ as T^* is a trade and so it suffices to show that T^a and T^b are mutually 2-balanced. Again, as T^* is a trade, pairs of elements $\{j, k\}$ with $j, k \notin L$ are contained in the same number of blocks of T^a and T^b. Thus we need only consider pairs of elements containing some $i \in L$. However, by $(*)$, $F(X_i \setminus Y_i^a) = F(X_i \setminus Y_i^b)$. This implies that $F(Y_i^a) = F(Y_i^b)$ and thus T is mutually 2-balanced. Thus we have proven the following result.

Lemma 2.1 *If T^{a*} is a trade in A, then T^a is a trade in P.* □

Theorem 2.2 *Suppose S is a defining set of P. Then $(S \setminus \{L\})^*$ is a defining set of A.*

Proof. By Theorem 1.4, S intersects every trade in P. Lemma 2.1 guarantees that $(S \setminus \{L\})^*$ intersects every trade in A. Thus $(S \setminus \{L\})^*$ is a defining set of A. □

For the remainder of this section, let s_p and s_a equal the cardinalities of smallest defining sets of P and A respectively.

Corollary 2.3 *For $q \geq 2$, $s_a \leq s_p$. Furthermore, if the automorphism group of P is transitive on $\mathcal{B}$, then $s_a \leq s_p - 1$.*

Proof. Let S be a smallest defining set of P. If the automorphism group of P is transitive on $\mathcal{B}$, then S can be chosen so as to contain L. By Theorem 2.2, $(S\setminus\{L\})^*$ is a defining set of A. Thus, $s_a \le |(S\setminus\{L\})^*| \le s_p$ and $s_a \le s_p - 1$ if S contains L. □

B. Gray, Hamilton and O'Keefe [5] have shown that the size of a smallest defining set of $PG(2,q)$ is at most $(q^2+3q)/2$. We can now establish the first known upper bound for the size of a smallest defining set of $AG(2,q)$.

Theorem 2.4 *The size of a smallest defining set of $AG(2,q)$ is at most $(q^2+3q-2)/2$.*

Proof. Recall that a Desarguesian projective plane is cyclic and hence its automorphism group is transitive on $\mathcal{B}$. It then follows from Corollary 2.3 and the result of [5] that the size of a smallest defining set of $AG(2,q)$ is at most $((q^2+3q)/2)-1=(q^2+3q-2)/2$. □

A trade T^a in P does not necessarily induce a trade T^{a*} in A as the following example demonstrates.

Example 2.5 A projective plane $PG(2,5)$ is $P_5=(V_5,\mathcal{B}_5)$ where

$$V_5=\{0,1\ldots,9,a,b,\ldots o,\alpha,\beta,\gamma,\delta,\epsilon,\zeta\},$$

and the blocks of $\mathcal{B}_5 = L_5 \cup (\bigcup_{i=1}^{30} B_i)$ are listed in Table 2. Let A_5 be the residual of P_5 with respect to L_5. Let $T^a=\{B_{16},B_{17},\ldots,B_{25}\}\subset \mathcal{B}_5$, let the transposition $\sigma=(\delta\epsilon)$ and let $T^b=\sigma T^a$. Then (T^a,T^b) is a $(27,6,2)$ trade of volume ten. However, $T^{a*}=T^{b*}$ and (T^{a*},T^{b*}) is certainly not a trade. □

Lemma 2.6 *If T^a is a trade in P and T^a contains neither L nor iX_i for any $i\in L$, then T^{a*} is a trade in A.*

Proof. Suppose $T=(T^a,T^b)$ is a trade. Without loss of generality, let $T^a=\bigcup_{i=1}^{q+1} iY_i^a$ and $T^b=\bigcup_{i=1}^{q+1} iY_i^b$. Note that $Y_i\neq X_i$. It is clear that $T^*=(T^{a*},T^{b*})$ is a trade if $T^{a*}\cap T^{b*}=\emptyset$. Suppose there exist blocks $C_i^a\subseteq Y_i^a$ and $C_j^b\subseteq Y_j^b$ such that $C_i^a=C_j^b$. As T is a trade, $i\neq j$. Since T^a does not contain iX_i^a, choose block $iD_i^a\in iX_i^a\setminus iY_i^a$. As $(V,(\mathcal{B}\setminus T^a)\cup T^b)$ is a projective plane, $|iD_i^a\cap jC_j^b|=1$. However, $|iD_i^a\cap jC_j^b|=|iD_i^a\cap jC_i^a|=0$ which is a contradiction. □

B_1 : $\alpha 01234$	B_{11} : $\gamma 09dhl$	B_{21} : $\epsilon 07egn$
B_2 : $\alpha 56789$	B_{12} : $\gamma 15eim$	B_{22} : $\epsilon 18aho$
B_3 : $\alpha abcde$	B_{13} : $\gamma 26ajn$	B_{23} : $\epsilon 29bik$
B_4 : $\alpha fghij$	B_{14} : $\gamma 37bfo$	B_{24} : $\epsilon 35cjl$
B_5 : $\alpha klmno$	B_{15} : $\gamma 48cgk$	B_{25} : $\epsilon 46dfm$
B_6 : $\beta 05afk$	B_{16} : $\delta 08bjm$	B_{26} : $\zeta 06cio$
B_7 : $\beta 16bgl$	B_{17} : $\delta 19cfn$	B_{27} : $\zeta 17djk$
B_8 : $\beta 27chm$	B_{18} : $\delta 25dgo$	B_{28} : $\zeta 28efl$
B_9 : $\beta 38din$	B_{19} : $\delta 36ehk$	B_{29} : $\zeta 39agm$
B_{10} : $\beta 49ejo$	B_{20} : $\delta 47ail$	B_{30} : $\zeta 45bhn$
L_5 : $\alpha\beta\gamma\delta\epsilon\zeta$		

Table 2: Blocks of the 2-$(31, 6, 1)$ design P_5

Theorem 2.7 *Suppose that S^* is a defining set of A and that $X^* \subset \mathcal{B}^*$ is chosen so that $S^* \cup X^*$ intersects each of the parallel classes of A. Then $S_P = S \cup X \cup \{L\}$ is a defining set of P.*

Proof. If T^a is a trade in P containing L or iX_i for some i, then $T^a \cap (X \cup \{L\}) \neq \emptyset$. If T^a is a trade in P which contains neither L nor iX_i for each $i \in L$, then T^{a*} is a trade in A by Lemma 2.6. But as S^* is a defining set of A, $T^{a*} \cap S^* \neq \emptyset$ which implies that $T^a \cap S \neq \emptyset$. Hence, S_P intersects every trade in P and is thus a defining set of P. □

K. Gray [8] showed that, for $q > 3$, any smallest defining set of A contains blocks from at least three distinct parallel classes.

Corollary 2.8 *For $q > 3$, $s_p \leq s_a + q - 1$.*

Proof. Construct a defining set of P, S_P, as in Theorem 2.7 with X^* chosen to contain as few blocks as possible; that is, X^* contains precisely one block from each of the parallel classes which do not intersect S^*. Then,

$$s_p \leq |S_P| \leq s_a + |X^*| + 1 \leq s_a + q - 2 + 1 = s_a + q - 1$$

as claimed. □

The main result of this section comes from combining Corollary 2.3 with an obvious generalisation of Corollary 2.8.

Theorem 2.9 *For $q > 3$,*

$$s_a \leq s_p \leq s_a + q + 2 - c \leq s_a + q - 1,$$

where c equals the maximum number of parallel classes intersected by a smallest defining set of A. □

These inequalities are consistent with the results of [8]. In particular, $s_a \geq 2q - 1$ and $s_p \geq 2q$ for $q \geq 3$, and $s_p = 2q$ for $q = 3, 4$. We show that the latter equality does not hold when $q = 5$.

2.1 Smallest defining sets of $PG(2,5)$ and $AG(2,5)$

In this subsection the sizes, $s_{p,5}$ and $s_{a,5}$, of smallest defining sets of $PG(2,5)$ and $AG(2,5)$ respectively, are determined. Recall that $PG(2,5)$, notated by P_5, was introduced in Example 2.5. Also introduced were the line L_5, and the affine plane A_5, the residual of P_5 with respect to L_5.

The computer program *bds* [2] was used to determine whether a defining set of A_5 consisting of nine blocks existed. The outcome of this exhaustive search was that any defining set of A_5 must consist of at least ten blocks, that is, $s_{a,5} \geq 10$. A defining set of P_5 of cardinality eleven is exhibited in Lemma 2.11 and it will be consequently shown that $s_{a,5} = 10$. We first state the following lemma regarding the minimum volume of a trade in a projective plane.

Lemma 2.10 ([3]) *If T is a trade in a projective plane of order q, then $m(T) \geq 2q$.* □

Lemma 2.11 *Let $S = \{B_2, B_3, B_4, B_6, B_8, B_{11}, B_{12}, B_{14}, B_{21}, B_{26}\}$. Then $S \cup \{L_5\}$ is a defining set of the projective plane P_5 in Example 2.5.*

Proof. The element α is contained in four blocks of $S \cup \{L_5\}$. The set of elements not occurring with α in $S \cup \{L_5\}$ is $\{0, 1, 2, 3, 4, k, l, m, n, o\}$. However, the pairs $0k, 0l, 0n, 0o, m2$ and $m1$ are contained in S. Therefore $\alpha, 0$ and m are not collinear and blocks $\alpha01234$ and $\alpha klmno$ are forced. As five blocks containing 0 are determined, the block $\delta08bjm$ is also forced.

Similarly, the element γ is contained in four blocks of $S \cup \{L_5\}$. The set of elements not occurring with γ in $S \cup \{L_5\}$ is $\{2, 4, 6, 8, a, c, g, j, k, n\}$.

Also, the pairs $ca, c2, c6, gn, kn$ are in known blocks (either S or blocks subsequently forced). If γ, c, n occur together in a block of P_5, then the blocks $\gamma c48jn$ and $\gamma a26kg$ are forced. But the pair $ak \in B_6$ which is in S yielding a contradiction. Thus γ, c, n are not collinear and the blocks $\gamma 26ajn$ and $\gamma 48cgk$ are forced.

Continuing in the same manner, the element m does not yet occur with elements $\zeta, 3, 9, a, g, \epsilon, 4, 6, d, f$. The pairs $ad, af, a6, 4g, 43$ are in the known blocks and so if $m, 4$ and a are collinear, then the block $m4a\zeta 9\epsilon$ is forced. This yields a contradiction as the pair $\zeta\epsilon$ is already in L_5. Consequently, $m, 4$ and a are not collinear and blocks $\epsilon 46dfm$ and $\zeta 39agm$ are forced.

Now consider the elements $\delta, 1, 9, c, n, 2, 8, e, l$ not yet occurring with f and known pairs $8c, 8\delta, 89, ce, n2, nl, ne$. It is simple to see that blocks $\zeta 28efl$ and $\delta 19cfn$ are also forced. Finally, consider the elements $1, 4, 5, 7, b, d, j, k, h, n$ not yet occurring with ζ, and the known pairs $15, 14, 1n, 5k, 4d, 7n, 7b, 7h$. This forces the blocks $\zeta 17djk$ and $\zeta 45bhn$.

Thus 22 blocks of P_5 are determined, leaving nine unknown blocks remaining. It is immediate from Lemma 2.10 that these nine blocks cannot contain a trade. Hence $S \cup \{L_5\}$ is a defining set of P_5. □

Theorem 2.12 *The set of ten blocks* $S^* = \{B_2^*, B_3^*, B_4^*, B_6^*, B_8^*, B_{11}^*, B_{12}^*, B_{14}^*, B_{21}^*, B_{26}^*\}$ *is a smallest defining set of* A_5.

Proof. That S^* is a defining set of A follows from Theorem 2.2. Hence $s_{a,5} \leq 10$. But from the computer search using *bds*, $s_{a,5} \geq 10$. Thus $s_{a,5} = 10$ and S^* is a smallest defining set of A_5. □

Theorem 2.13 *The set of eleven blocks* $S \cup \{L_5\}$ *introduced in Lemma 2.11 is a* smallest *defining set of* P_5.

Proof. Lemma 2.11 shows that $S \cup \{L_5\}$ is a defining set of P_5. As $s_{a,5} = 10$, $s_{p,5} \geq 11$ by Corollary 2.3. Hence $S \cup \{L_5\}$ is indeed a smallest defining set of P_5. □

Remark 2.14 *If* S^* *is a defining set of* A, *then* $S \cup \{L\}$ *is not necessarily a defining set of* P. *For instance, it is simple to show that* $S^* = \{B_1^*, B_2^*, B_3^*, B_4^*, B_6^*, B_7^*, B_8^*, B_{11}^*, B_{12}^*, B_{14}^*\}$ *is a (smallest) defining set of* A_5. *However,* $S \cup \{L_5\}$ *does not intersect the trade* $T^a = \{B_{16}, B_{17}, \ldots, B_{25}\}$ *of Example 2.5.*

3 Defining sets of biplanes and their residuals

Throughout this section $D = (V, \mathcal{B})$ denotes a biplane of order q with $q > 2$. The parameters of D are thus 2-$(v, q+2, 2)$ where $v = \binom{q+2}{2} + 1$. Let $L = B_v \in \mathcal{B}$ and $R = D^* = (V \setminus L, \mathcal{B}^*)$ denote the residual of $\mathcal{B}$ with respect to L. Without loss of generality, let $L = \{1, 2, \ldots, q+2\}$. For $1 \leq i < j \leq q+2$, $ijB_{i,j}$ denotes the block in $\mathcal{B}$ other than L which contains the pair of elements $\{i, j\}$. In this way,

$$\mathcal{B} = \{ijB_{i,j} \mid 1 \leq i < j \leq q+2\} \cup \{L\}.$$

Also, the set of blocks $\mathcal{B}^*$ of R can be written as $\{B_{i,j} \mid 1 \leq i < j \leq q+2\}$. Suppose $S \subseteq \mathcal{B} \setminus \{L\}$. Then, for some indexing set Λ of ordered pairs (i, j) $(i < j)$, $S = \bigcup_\Lambda \{ijB_{i,j}\}$, and equivalently, $S^* = \bigcup_\Lambda \{B_{i,j}\}$.

Example 3.1 It is well-known that a biplane $G_1 = (V_1, \mathcal{B}_1)$ of order four can be easily constructed from the array $\mathcal{A}$ below. Let $V_1 = \{1, 2, \ldots, 16\}$ and $\mathcal{B}_1 = \bigcup_{i=1}^{16} \{A_i\}$, where, for each $i \in V_1$, A_i consists of all the numbers other than i which occupy a square of $\mathcal{A}$ in the same row or column as i.

$\mathcal{A} =$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

It is a simple exercise to verify that G_1 is indeed a 2-$(16, 6, 2)$ design as claimed. Let $L_1 = A_1 = \{2, 3, 4, 5, 9, 13\}$. The design H_1, the residual of G_1 with respect to L_1, has parameters 2-$(10, 4, 2)$. □

In this section, we consider when a trade T^a in D induces a trade T^{a*} in R and vice versa.

Lemma 3.2 *Suppose that T^a is a trade in D, and that $L \notin T^a$. Further suppose that for each $x \in L \cap F(T^a)$, there exist at least three blocks containing x in $\mathcal{B} \setminus (T^a \cup \{L\})$. Then T^{a*} is a trade in R.*

Proof. Let $T^a = \bigcup_\Lambda \{ijB_{i,j}\}$ for some indexing set of ordered pairs Λ. Suppose that T^a trades with T^b. Then T^b can be written as $\bigcup_\Lambda \{ijB^b_{i,j}\}$. It is required to show that $T^* = (T^{a*}, T^{b*}) = (\bigcup_\Lambda \{B_{i,j}\}, \bigcup_\Lambda \{B^b_{i,j})\}$ is a trade.

As T is a trade, it follows that T^* is mutually 2-balanced. If T^* is not a trade, then $T^{a*} \cap T^{b*} \neq \emptyset$ which implies that $B_{l,m} = B^b_{p,q}$ for some ordered pairs $(l,m),(p,q) \in \Lambda$, $l < m$ and $p < q$. There are four cases to consider.

Case 1: $l = p, m = q$. In this case, $B_{l,m} = B^b_{l,m}$ and the block $lmB_{l,m} \in T^a \cap T^b$ which contradicts the fact T is a trade.

Case 2: $l = p, m \neq q$ or $l \neq p, m = q$. Without loss of generality, assume that $l = p, m \neq q$ and so $B_{l,m} = B_{p,m} = B^b_{p,q}$. Now $q \in F(T^a)$. Therefore, there exist at least three blocks in $Z = \mathcal{B} \setminus (T^a \cup \{L\})$ containing q. In particular, a block $jqB_{j,q}$ (or $qjB_{q,j}$) can be chosen in Z with $j \neq m$. Note that $j \neq p$ as $pqB_{p,q} \in T^a$. Now the set of blocks $(\mathcal{B} \setminus T^a) \cup T^b$ comprise a biplane containing blocks $jqB_{j,q}$ and $pqB^b_{p,q}$. But $B_{p,m} = B^b_{p,q}$, and so

$$|jqB_{j,q} \cap pqB^b_{p,q}| = |jqB_{j,q} \cap pqB_{p,m}| = 1 + |B_{j,q} \cap B_{p,m}| = 3$$

using $\{j,q\} \cap \{p,m\} = \emptyset$. This is a contradiction.

Case 3: $l < m = p < q$ or $p < q = l < m$. Without loss of generality, assume that $l < m = p < q$ and so $B_{l,m} = B_{l,p} = B^b_{p,q}$. Now $q \in F(T^a)$. Therefore, there exist at least three blocks in $Z = \mathcal{B}\setminus(T^a \cup \{L\})$ containing q. In particular, a block $jqB_{j,q}$ (or $qjB_{q,j}$) can be chosen in Z with $j \neq l$. Note that $j \neq p$ as $pqB_{p,q} \in T^a$. Now the set of blocks $(\mathcal{B} \setminus T^a) \cup T^b$ comprise a biplane containing blocks $jqB_{j,q}$ and $pqB^b_{p,q}$. But $B_{l,p} = B^b_{p,q}$, and so

$$|jqB_{j,q} \cap pqB^b_{p,q}| = |jqB_{j,q} \cap pqB_{l,p}| = 1 + |B_{j,q} \cap B_{l,p}| = 3$$

using $\{j,q\} \cap \{l,p\} = \emptyset$. This is a contradiction.

Case 4: $\{l,m\} \cap \{p,q\} = \emptyset$. Again, as there exist at least three blocks in $Z = \mathcal{B} \setminus (T^a \cup \{L\})$ containing q, there is a block $jqB_{j,q} \in Z$ (or $qjB_{q,j}$) with $j \notin \{l,m\}$. Now the set of blocks $(\mathcal{B} \setminus T^a) \cup T^b$ comprise a biplane containing blocks $jqB_{j,q}$ and $pqB^b_{p,q}$. But $B_{l,m} = B^b_{p,q}$, and so

$$|jqB_{j,q} \cap pqB^b_{p,q}| = |jqB_{j,q} \cap lqB_{l,m}| = 1 + |B_{j,q} \cap B_{l,m}| = 3$$

using $\{j,q\} \cap \{l,m\} = \emptyset$. This is a contradiction.

Thus Cases 1, 2, 3 and 4 all lead to a contradiction and so T^* is a trade. In particular, T^{a*} is a trade in R. □

Example 3.3 Let $T^a = \{A_2, A_3, A_6, A_7\} \subset \mathcal{B}_1$ of Example 3.1. Then T^a trades with T^b and T^{a*} trades with T^{b*} where, written in full,

$$T^a = \{\{1,3,4,6,10,14\},\{1,2,4,7,11,15\},\{5,7,8,2,10,14\},$$

$$\begin{aligned}
& \{5,6,8,3,11,15\}\} \\
T^b &= \{\{1,4,2,7,10,14\},\{1,4,3,6,11,15\},\{5,8,3,6,10,14\}, \\
& \{5,8,2,7,11,15\}\}, \\
T^{a*} &= \{\{1,6,10,14\},\{1,7,11,15\},\{7,8,10,14\},\{6,8,11,15\}\}, \text{ and} \\
T^{b*} &= \{\{1,7,10,14\},\{1,6,11,15\},\{6,8,10,14\},\{7,8,11,15\}\}.
\end{aligned}$$

It is interesting to verify that T^a is an example of a trade that satisfies the premises of Lemma 3.2. □

In terms of defining sets of biplanes, Lemma 3.2 yields the following result.

Theorem 3.4 *Suppose S^* is a defining set of R. If $X \subseteq \mathcal{B} \setminus \{L\}$ is such that each element of L occurs in at least three blocks of $S \cup X$, then $S \cup X \cup \{L\}$ is a defining set of D.*

Proof. Suppose $S_D = S \cup X \cup \{L\}$ is not a defining set of D. Then there exists a trade $T^a \subseteq \mathcal{B} \setminus S_D$. By Lemma 3.2, T^{a*} is a trade in R such that $S^* \cap T^{a*} = \emptyset$. But S^* is a defining set of R which is a contradiction. □

Let s_d and s_r be the sizes of smallest defining sets of D and R respectively. Let $\lceil x \rceil$ denote the least integer greater than or equal to x.

Corollary 3.5 *For $q > 2$, $s_d \leq s_r + (3/2)q + 10$.*

Proof. Partition the set V into as many subsets of size four as possible. Note that $v \equiv 0 \pmod 4$ for $q \equiv 1,4 \pmod 8$. For $q \equiv 0,2,3,5,6,7 \pmod 8$, we allow one group of size $2,3,3,2,1,1$ respectively.

Without loss of generality let one subset of V of size four be $G = \{1,2,3,4\}$. Each element of G is contained in three blocks of

$$X_G = \{12B_{1,2}, 13B_{1,3}, 14B_{1,4}, 23B_{2,3}, 24B_{2,4}, 34B_{3,4}\}.$$

For each subset R of size four (or less), construct a set X_R in a simliar way. Let $X = \bigcup_R X_R$ where R varies over all the subsets in the partition of V. Then $X \subseteq \mathcal{B} \setminus \{L\}$ and each element of L is contained in at least three blocks of X. By Theorem 3.4, if S^* is a defining set of R, then $S \cup X \cup \{L\}$ is a defining set of D.

Now $|X| \leq 6\lceil (q+2)/4 \rceil$. Hence,

$$s_d \leq s_r + |X| + 1 \leq s_r + 6((q+2)/4 + 1) + 1 = s_r + (3/2)q + 10. \quad □$$

Of course the upper bound in Corollary 3.5 can be tightened considerably when the structure of a particular smallest defining set of R is known. The result of this generalisation is stated in Theorem 3.11.

As in Street and Street [15, page 261], the blocks of the residual of a biplane can be placed in a $(q+1) \times (q+1)$ array Δ with empty diagonal cells so that any two blocks have precisely one variety in common if they appear in the same row or column of Δ. In fact, a row of Δ is obtained from the set of residuals of all the blocks of D containing a common element of L. The next example illlustrates this idea.

Example 3.6 Up to isomorphism, there is a unique biplane of order three with parameters 2-$(11, 5, 2)$. This design is cyclic. Let D be an example of this design, based on the elements $\{0, 1, 2, \ldots, 9, a\}$, with starter block $L = 1237a$. The residual of D with respect to L, is the 2-$(6, 3, 2)$ design with blocks

$$\mathcal{B}^* = \{459, 456, 056, 468, 589, 689, 049, 058, 069, 048\}.$$

We arrange these blocks in the array Δ below so that any two blocks have precisely one variety in common if they appear in the same row or column of Δ.

$\Delta =$

*	459	468	058	069
459	*	056	689	048
468	056	*	049	589
058	689	049	*	456
069	048	589	456	*

Let $T^{a*} = \{459, 468, 058, 069\}$. Then T^{a*} trades with $T^{b*} = \{059, 068, 458, 469\}$. T^{a*} extends to $T^a = \{13459, 17468, 1a058, 12069\}$.

We now recreate all the blocks of D using Δ. Recall that $L = 1237a$ and consider, for instance, the possible extensions of the block $056 \in \mathcal{B}^*$. Such an extension necessarily intersects both 13459 and 17468 in two elements. Five blocks in D containing the element 1 are known and hence 056 is forced to extend to 37056. All the remaining blocks of D can be recreated from Δ in this way. This illustrates the Hall-Connor theorem; see [15]. □

The reader can easily verify that in the previous example, T^a is not a trade in D. In fact, how would one extend T^{b*} to provide a possible trade mate T^b for T^a? We see in the following lemma that, as long as one complete row of Δ is disjoint from a trade T^{a*} in R, then it is indeed possible to extend T^{b*} so as to induce a trade (T^a, T^b) with T^a in D.

Lemma 3.7 *Suppose that $q > 2$, that T^{a*} is a trade in $R = (V^*, \mathcal{B}^*)$ and that there exists a row of the array Δ associated with R disjoint from T^{a*}. Then T^a is a trade in D.*

Proof. Let $\mathcal{R}$ be a row of Δ disjoint from T^{a*}. The extension of blocks of $\mathcal{R}$ to blocks in D is known. The previous example illustrates how this forces the remaining blocks in R to extend to blocks in D uniquely. Also, $\mathcal{R}$ corresponds to a complete row in the array Δ^b corresponding to the design $(R \setminus T^{a*}) \cup T^{b*}$. Similarly, as $\mathcal{R}$ is disjoint from T^{b*}, the extension of $(R\setminus T^{a*})\cup T^{b*}$ to $(D\setminus T^a)\cup T^b$ is forced (uniquely). Clearly $T^a \cap T^b = \emptyset$ and so T^a is a trade in D. □

Corollary 3.8 *Suppose $q > 2$ and that T^{a*} is a trade in R with $m(T^{a*}) < q + 1$. Then T^a is a trade in D.*

Proof. Construct the array Δ from the blocks of $\mathcal{B}$. As Δ contains $q + 1$ rows and columns, there must exist a complete row of Δ disjoint from T^{a*}. The result follws from Lemma 3.7. □

Theorem 3.9 *Suppose S is a defining set of $D = (V, \mathcal{B})$. Let x be an arbitary element of L. Choose $X \subseteq \mathcal{B} \setminus \{L\}$ so that $(S \cup X) \setminus \{L\}$ contains the $q + 1$ blocks of D (other than L) which contain the element x. Then $(S \setminus \{L\})^* \cup X^*$ is a defining set of R.*

Proof. Suppose $(S\setminus\{L\})^* \cup X^*$ is not a defining set of R. Then there exists a trade T^{a*} in R disjoint from $(S \setminus \{L\})^* \cup X^*$. However, as $(S \cup X) \setminus \{L\}$ contains the $q+1$ blocks of D with element x (other than L), there is a row in Δ disjoint from T^{a*}. Hence T^a is a trade in D disjoint from S which is a contradiction since S is a defining set of D. □

Corollary 3.10 *For $q > 2$, $s_r \leq s_d + q$.*

Proof. Let S be a smallest defining set of D. Some element of L, x say, will be contained in at least one block of $S \setminus \{L\}$. Choose X so that $(S \cup X) \setminus \{L\}$ contains the $q + 1$ blocks of D (other than L) with element

x. Then $(S \setminus \{L\})^* \cup X^*$ is a defining set of R with cardinality no greater than $s_d + q$. □

The main result of this section comes from generalising Corollaries 3.5 and 3.10.

Theorem 3.11 *Let c_1 be the maximum number of blocks in a smallest defining set of D which contain a common element of L. Let S^* be a smallest defining set of R. Let c_2 be the minimum number of blocks in a set X such that each element of L occurs in at least three blocks of $S \cup X$. Then for $q > 2$,*

$$s_r - q \leq s_r - (q + 1 - c_1) \leq s_d \leq s_r + c_2 + 1 \leq s_r + (3/2)q + 10. \quad \square$$

3.1 Biplanes of order four and their residuals

There are three non-isomorphic 2-(16, 6, 2) designs. We denote these designs by G_1, G_2 and G_3; their blocks are given in Example 3.1 and Table 4. The automorphism group of each of these designs is transitive on the sets of blocks and points. The residuals of these designs are denoted by H_1, H_2 and H_3. For $i = 1, 2, 3$, let g_i and h_i denote the size of a smallest defining set of G_i and H_i respectively. A summary of known theoretical results [11, 4] regarding the sizes of smallest defining sets of these six designs is given in Table 3.

Table 3: Known theoretical results for the sizes of smallest defining sets

Design	G_1	G_2	G_3	H_1	H_2	H_3
Size	$g_1 = 9$	$g_2 \leq 7$	$g_3 \leq 7$	$h_1 \leq 8$	$h_2 \leq 6$	$h_3 \leq 5$

Greenhill [9] used the computer program *bds* to verify that these bounds gave the exact sizes of smallest defining sets of these designs. It would be of considerable interest to verify these results theoretically as was done in [4] for the design G_1. Using the results developed in Section 3, theoretical proofs for the sizes of smallest defining sets of G_2, H_1, H_2 and H_3 will be presented. Unfortunately, the exact size of a smallest defining set of G_3 cannot be determined using only this method. The reason for this is discussed at the end of this section.

Table 4: The blocks of the 2-$(16, 6, 2)$ designs G_2 and G_3

G_2						G_3					
1	2	3	4	5	6	1	2	3	4	5	6
1	2	7	8	9	10	1	2	7	8	9	10
1	3	7	11	12	13	1	3	7	11	12	13
1	4	8	11	14	15	1	4	8	11	14	15
1	5	9	12	14	16	1	5	9	12	14	16
1	6	10	13	15	16	1	6	10	13	15	16
3	9	15	6	8	12	3	8	16	5	10	11
3	10	14	5	8	13	3	9	15	6	8	12
4	9	13	6	7	14	3	10	14	4	9	13
4	10	12	5	7	15	4	7	16	6	9	11
3	16	2	7	14	15	4	10	12	5	7	15
3	11	16	4	9	10	5	8	13	6	7	14
2	4	8	12	13	16	2	3	7	14	15	16
5	6	7	8	11	16	2	4	8	12	13	16
2	5	9	11	13	15	2	5	9	11	13	15
2	6	10	11	12	14	2	6	10	11	12	14

We commence with a classification of trades of volume four in symmetric designs by Khosrovshahi, Majumdar and Widel [12]. The form in which the following lemma is stated is easily inferred from [4].

Theorem 3.12 ([12],[4]) *Suppose T_1 is a subset of four blocks of a symmetric 2-(v,k,λ) design. Then (T_1,T_2) is a trade of volume four if and only if T_1 can be written as*

$$\begin{aligned} T_1 &= \{S_0S_1S_3S_5, S_0S_1S_4S_6, S_0S_2S_4S_5, S_0S_2S_3S_6\}, \text{ in which case} \\ T_2 &= \{S_0S_1S_4S_5, S_0S_1S_3S_6, S_0S_2S_3S_5, S_0S_2S_4S_6\}, \end{aligned}$$

where $S_i \cap S_j = \emptyset$ $(i \neq j)$, $|S_1| = |S_2| = \ldots = |S_6| = (k-\lambda)/2$ and $|S_0| = (3\lambda - k)/2$. □

One of the immediate consequences of this theorem is that a symmetric 2-(v,k,λ) design contains a trade of volume four only if $k \equiv \lambda \pmod 2$ and $3\lambda \geq k$. We shall investigate the case $3\lambda = k$ and the next corollary is an immediate consequence of Theorem 3.12.

Corollary 3.13 *Suppose T_1 is a subset of four blocks of a symmetric 2-$(v,3\lambda,\lambda)$ design D. Then T_1 is a trade of volume four if and only if T_1 can be written as*

$$T_1 = \{S_1S_3S_5, S_1S_4S_6, S_2S_4S_5, S_2S_3S_6\},$$

where $S_i \cap S_j = \emptyset$ $(i \neq j)$ and $|S_1| = |S_2| = \ldots = |S_6| = \lambda$. □

From a symmetric design $D = (V, \mathcal{B})$, it is possible to construct the **dual** design $D^\perp$. Suppose the blocks of $\mathcal{B}$ are labelled with the elements of V. For each element $v \in V$, define $B_v^\perp$ to be the set of labels of blocks in B which contain v. Define $\mathcal{B}^\perp = \bigcup_v B_v^\perp$ and $D^\perp = (V, \mathcal{B}^\perp)$. Clearly $D^\perp$ and D have the same parameters. The design D is said to be **self-dual** if D is isomorphic to $D^\perp$. For a set X of blocks (points) in a symmetric design D, let $X^\perp$ denote the corresponding set of points (blocks) in the dual of D, $D^\perp$.

The concept of ovals in symmetric designs was introduced by Assmus and van Lint [1].

Definition 3.14 *Let D be a symmetric design with parameters* 2-(v,k,λ). *The* **order** *of D is $k - \lambda$. A collection S of points of the design is an* **arc** *if no three points of S lie on a block. If D is of even order with λ dividing k, then an arc with $(k+\lambda)/\lambda$ points is called an* **oval**.

Assmus and van Lint define ovals for symmetric designs of other orders but we will only be interested in symmetric designs with parameters 2-$(v, 3\lambda, \lambda)$. Ovals in these designs consist of four points. In particular, ovals in a design with these parameters will be shown to correspond to trades of volume four in the dual design.

Lemma 3.15 *Let D be a symmetric 2-$(v, 3\lambda, \lambda)$ design. Then T is a trade of volume four in D if and only if $T^{\perp}$ is an oval in $D^{\perp}$.*

Proof. Let $T = \{B_1, B_2, B_3, B_4\}$. That $B_i \cap B_j \cap B_k = \emptyset$ for distinct i, j, k is immediate from Corollary 3.13. It follows that $T^{\perp}$ is an oval in $D^{\perp}$.

Conversely, suppose that $\mathcal{O} = \{1, 2, 3, 4\}$ is an oval in $D^{\perp}$. Let $\mathcal{O}^{\perp} = T = \{B_1, B_2, B_3, B_4\}$. As D is linked, $|B_i \cap B_j| = \lambda$ for distinct i, j. As $\mathcal{O}$ is an oval, $B_i \cap B_j \cap B_k = \emptyset$ for distinct i, j, k. Let $S_1 = B_1 \cap B_2, S_5 = B_1 \cap B_3, S_3 = B_1 \cap B_4, S_4 = B_2 \cap B_3, S_6 = B_2 \cap B_4$ and $S_2 = B_3 \cap B_4$. It follows that $|S_i| = \lambda$ and $S_i \cap S_j = \emptyset$ $(i \neq j)$. Also T can be written as $\{S_1S_3S_5, S_1S_4S_6, S_2S_4S_5, S_2S_3S_6\}$, and is thus a trade of volume four in D by Corollary 3.13. □

If T_1 is a trade of volume four in a symmetric 2-(v, k, λ) design D with $3\lambda > k$, then by Theorem 3.12 there exists a non-empty set S_0 contained in each of the blocks of T_1. Thus when $3\lambda > k$, $T_1^{\perp}$ is certainly not a subset of an oval in $D^{\perp}$.

We now focus our attention on the three 2-$(16, 6, 2)$ designs G_1, G_2 and G_3. Each of these designs is self-dual. The structure of ovals in these designs has been extensively studied. The classification used here is that of Roghelia and Sane [14]. The number of ovals in G_1, G_2 and G_3 is $60, 28$ and 12 respectively. The ovals in G_2 and G_3 are listed in Table 5. G_3 contains the ovals in the columns headed by Q_1, Q_2 and Q_3. G_2 contains all the ovals listed in Table 5.

Henceforth, X_1, X_2 and X_3 are sets of points that intersect every oval in G_1, G_2 and G_3 respectively. We show that $|X_1| \geq 9$, $|X_2| \geq 7$ and $|X_3| \geq 6$. As each of G_1, G_2 and G_3 is self-dual, it follows immediately that $g_1 \geq 9, g_2 \geq 7$ and $g_3 \geq 6$. Let H_i be the residual of G_i with respect to the block L_i. Each trade of volume four in G_i, which does not contain L_i, induces a trade of volume four in H_i. This follows from Lemma 3.2 and Corollary 3.13 (and conversely by Corollary 3.8). Thus we conclude that $h_1 \geq 8, h_2 \geq 6$ and $h_3 \geq 5$. These results when combined with the results

Table 5: The ovals of G_2 and G_3

Q_1	Q_2	Q_3
1 2 11 16	1 2 12 15	1 2 13 14
12 13 14 15	11 13 14 16	11 12 15 16
5 6 9 10	4 6 8 10	4 5 8 9
3 4 7 8	3 5 7 9	3 6 7 10

Q_4	Q_5	Q_6	Q_7
1 6 9 11	1 5 10 11	1 3 8 16	1 4 7 16
2 5 10 16	2 6 9 16	2 4 7 11	2 3 8 11
7 8 12 14	7 8 13 15	9 10 14 15	9 10 12 13
3 4 13 15	3 4 12 14	5 6 12 13	5 6 14 15

summarised in Table 3 show that $g_1 = 9, g_2 = 7, h_1 = 8, h_2 = 6, h_3 = 5$ and that $6 \leq g_3 \leq 7$.

Lemma 3.16 ([4]) $|X_1| \geq 9$.

Proof. Defining sets and trades of volume four in G_1 are discussed in [4]. It is shown that there are precisely two non-isomorphic smallest defining sets of G_1 with nine blocks. In proving this result, it is shown that at least nine blocks are required to intersect all the trades of volume four in G_1. Thus $|X_1| \geq 9$ by Lemma 3.15. □

Lemma 3.17 $|X_3| \geq 6$ *and* $|X_2| \geq 7$.

Proof. Let $I_1 = \{\{1,2\},\{11,16\},\{12,15\},\{13,14\}\}$, $I_2 = \{\{5,9\}, \{6,10\}, \{3,7\}, \{4,8\}\}$. Observe how the ovals of G_3 can be decomposed into pairs of members from each of I_1 and I_2. For instance, the first oval of Q_1, namely $\{1,2,11,16\}$, can be decomposed into $\{1,2\} \cup \{11,16\}$. Furthermore, any pair of members of I_1 comprise an oval in Q_1, Q_2 or Q_3 and similarly for any pair of members of I_2. It follows that X_3 must contain elements from at least three of the members in each of I_1 and I_2. That is, $|X_3| \geq 6$.

In a similar manner, the ovals of G_2 can be decomposed into pairs of members from the collections I_1 and I_2, together with $J_1 = \{\{1,11\}, \{6,9\}, \{5,10\}, \{2,16\}\}, J_2 = \{\{7,8\}, \{12,14\}, \{13,15\}, \{3,4\}\}, K_1 = \{\{1,16\}, \{3,8\}, \{4,7\}, \{2,11\}\}, K_2 = \{\{9,10\}, \{14,15\}, \{12,13\}, \{5,6\}\}$.

For instance, the first oval of Q_4, $\{1,11,6,9\}$ can be decomposed into $\{1,11\} \cup \{6,9\}$ from J_1. Note that each of the ovals of Q_1 can be decomposed in three different ways; for example,

$$\{1,2,11,16\} = \{1,11\} \cup \{2,16\} = \{1,16\} \cup \{2,11\} = \{1,2\} \cup \{11,16\}.$$

We remark that a pair of distinct members chosen from one of the collections $I_1, I_2, J_1, J_2, K_1, K_2$ comprises an oval in G_2. As each of the ovals of Q_1 can be decomposed in three ways, the total number of ovals in G_2 (still) equals $6\binom{4}{2} - 2 \times 4 = 28$.

Recall the four ovals of Q_1:

$\{1,2,11,16\}$	$\{12,13,14,15\}$	$\{5,6,9,10\}$	$\{3,4,7,8\}$

X_2 clearly contains at least one element from each of these ovals, for if not, there is an oval of Q_1 disjoint from X_2. Suppose X_2 contains precisely one element from one of these ovals. As the automorphism group of G_2 is transitive on the points, assume, without loss of generality, that this is the element 1 say. Consider the ovals in G_2 which do not contain the element 1. Some of these are listed in the following array:

$\{11,16,12,15\}$	$\{2,16,5,10\}$	$\{2,11,3,8\}$
$\{11,16,13,14\}$	$\{2,16,6,9\}$	$\{2,11,4,7\}$

Clearly, at least two elements from each of the second, third and fourth ovals of Q_1 must be contained in X_2. Thus $|X_2| \geq 7$ as required. □

It is of interest to count the total number of smallest defining sets of G_2. We illustrate how this is done utilising the symmetries in G_2, where the details of the symmetry arguments are omitted. As before, suppose that X_2 is a set of 7 points that intersects all the ovals of G_2. Then X_2 contains precisely one, two, two and two points from the four ovals of Q_1. Without loss of generality, as the automorphism group of G_2 is transitive, suppose 1 is the only member of X_2 from the first oval.

Now observe that there are four possible choices of two elements from the second oval of Q_1 to be contained in X_2, namely $\{12,13\}, \{12,14\}, \{13,15\}$ and $\{14,15\}$. The choices $\{13,14\}$ and $\{12,15\}$ are forbidden as $\{11,13,$ $14, 16\}$ and $\{11,12,15,16\}$ are ovals of G_2 and 1 is the only member of X_2 from the first oval of Q_1. By symmetry arguments, we choose without loss of generality, $12, 13 \in X_2$.

Now consider the ovals in the previous array, together with the ovals $\{14,15,9,10\}$ and $\{14,15,5,6\}$. These imply that there are only two choices for the two elements of the third oval of Q_1; namely, $\{5,9\}$ and $\{6,10\}$. Again, utilising the symmetries of G_2 we can suppose that $5,9 \in X_2$.

Finally, consider the ovals in the previous array, together with $\{6,10,3,7\}$ and $\{6,10,4,8\}$. It is simple to see that again there are only two choices for the two elements of the fourth oval of Q_1; namely, $\{3,4\}$ and $\{7,8\}$.

Hence, in total, there are $16 \times 4 \times 2 \times 2 = 256$ possible ways X_2 could be constructed (all of which are isomorphic). More formally, we have the following result which was also stated in [11]. The method employed in [11] used extensive computation and some group theory.

Theorem 3.18 *A smallest defining set of G_2 contains seven blocks, is unique to isomorphism and has 256 copies in G_2.* □

Finally, we show in the following example why some of the premises of Lemma 3.2 are necessary.

Example 3.19 Let $T = (T^a, T^b)$ be the trade in G_3 induced by the permutation $\sigma = (15)$. So $T^b = \sigma T^a$. We show that T^{a*} is not a trade in H_3, the residual of G_3 with respect to the block $\{1,2,3,4,5,6\}$. Explicitly,

$$\begin{aligned} T^a &= \{\{1,2,7,8,9,10\},\{1,3,7,11,12,13\},\{1,4,8,11,14,15\},\\ &\quad \{1,6,10,13,15,16\},\{4,5,7,10,12,15\},\{3,5,8,10,11,16\},\\ &\quad \{5,6,7,8,13,14\},\{2,5,9,11,13,15\}\},\\ T^{a*} &= \{\{7,8,9,10\},\{7,11,12,13\},\{8,11,14,15\},\{10,13,15,16\},\\ &\quad \{7,10,12,15\},\{8,10,11,16\},\{7,8,13,14\},\{9,11,13,15\}\}\\ &= T^{b*}. \end{aligned}$$

Certainly, (T^{a*}, T^{b*}) is not a trade. We show that there is no trade mate T^{c*} such that (T^{a*}, T^{c*}) is a trade. Suppose that such a T^{c*} exists.

First consider the fact that elements 7 and 11 occur in four blocks of T^{a*} each, and the pair $\{7,11\}$ occurs in one block of T^{a*}. This forces partial blocks $\{7,11\}, \{7\}, \{7\}, \{7\}, \{11\}, \{11\}, \{11\}$ to be in T^{c*}. Next consider the blocks in T^{a*} containing elements $12, 16$ and 10. It is clear that at the very least, the partial blocks $\{7,11,12\}, \{7,12\}, \{7\}, \{7\}, \{11\}, \{11\}, \{11,16,10\}, \{16,10\}$ are forced to be in T^{c*}.

Now the pair $\{8, 16\}$ must occur in a block of T^{c*}. However, the pair $\{8, 12\}$ does not occur in a block of T^{a*}. Hence, by balancing the occurrences of 8 with elements 7 and 11 in T^{c*}, we derive the contradiction $\{8, 11, 10, 16\} \in T^{c*}$. Thus T^{a*} has no trade mate as claimed. □

It is this type of *single-transposition* trade, which occurs in G_3 but does not induce a trade in H_3, that causes $g_3 > |X_3|$; the details of a complete analysis of this case can be found in [6].

4 Conclusion

If D_d is a design with a residual D_d^*, then let μ_d and μ_d^* represent the fraction of blocks in the smallest defining sets of D_d and D_d^* respectively. When D_d is the symmetric block design formed from the points and hyperplanes of $PG(d, 2)$, then it has been shown by the author in [3] that $\lim_{d\to\infty} \mu_d = \lim_{d\to\infty} \mu_d^* = 1$. When $D_d = PG(2, d)$ and if $\lim_{d\to\infty} \mu_d$ exists, then it follows from Theorem 2.9 that $\lim_{d\to\infty} \mu_d = \lim_{d\to\infty} \mu_d^*$. Theorem 3.11 suggests a similar result for families of biplanes (if an infinite family of these designs can be found!). The following two questions are posed:

1. Is there a family of symmetric designs D_d with $\lim_{d\to\infty}(\mu_d - \mu_d^*) \neq 0$?
2. Is there a symmetric design D such that the cardinality of a smallest defining set of D is strictly less than the cardinality of a smallest defining set of D^*?

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