On Avoiding Arithmetic Progressions Whose Common Differences Belong to a Given Small Set

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ABSTRACT. It is well-known that if D is any finite set of integers then there is an n large enough so that there exists a 2-coloring of the positive integers that avoids any monochromatic n-term arithmetic progressions whose common differences belong to D. If $\vec{d} = (d_1, \dots, d_k)$ and $\vec{n} = (n_1, \dots, n_k)$ are k-tuples of positive integers, denote by $f_{\vec{d}}(\vec{n})$ the least positive integer N, if it exists, such that for every 2-coloring of $[1, N]$ there is, for some i, a monochromatic n_i-term arithmetic progression with common difference d_i. This paper looks at the problem of determining when $f_{\vec{d}}(\vec{n})$ exists, and it's value when it does exist, for $k \leq 3$. A complete answer is given for $k = 2$. A partial answer is given for $k = 3$, including the fact that for all ordered triples $\vec{d}$, $f_{\vec{d}}(4, 4, 4)$ does not exist.

1 Introduction and Terminology

Van der Waerden's theorem on arithmetic progressions [11] tells us that for each positive integer n, there is a least positive integer $N = w(n)$, such that whenever $[1, N] = \{1, \dots, N\}$ is 2-colored, there must be a monochromatic n-term arithmetic progression. Analogues of van der Waerden's theorem have been considered, where the collection of all arithmetic progressions, A, is replaced by some other collection of sequences B. If, for a given collection B, $w'(n)$ exists for all n, then B is said to have the *Ramsey property*. When, for a given n, $w'(n)$ does not exist, we write $w'(n) = \infty$.

Of course if B is a superset of A, then by van der Waerden's theorem B has the Ramsey property. The associated Ramsey functions $w'(n)$ have been studied for a variety of such B; examples can be found in [2,5-7]. In [4,8-10] the authors considered certain analogues of van der Waerden's theorem involving arithmetic progressions (mod m), i.e., increasing sequences $\{x_i : 1 \leq i \leq n\}$ where for some d, $1 \leq d \leq m-1$, we have $x_i - x_{i-1} \equiv d$

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(mod m) for all i, $2 \leq i \leq n$. One interesting result is that, for every $m \geq 2$, the collection of all arithmetic progressions (mod m) does not have the Ramsey property; in fact, in [4] it was proven that for this collection of sequences,

$$w'(n) = \infty \text{ whenever } n > \lceil m/2 \rceil. \tag{1}$$

For an arithmetic progression $S = \{x + id : 0 \leq i \leq n-1\}$, we call d the *common difference* of S. For convenience, we will sometimes refer to S as a *d-a.p. of length n*. In a recent article, Brown, Graham, and Landman [3] determined whether certain subcollections of A have the Ramsey property. Specifically, let D be a set of positive integers, and let $N = f(D,n)$ be the least positive integer such that for every 2-coloring of $[1,N]$ there is a monochromatic n-term arithmetic progression whose common difference belongs to D. Certain conditions on D were shown to be sufficient for the Ramsey property. For example, if D is a set of positive integers containing k-cubes for arbitrarily large k, then $f(D,n) < \infty$ for all n (in fact, the Ramsey property is satisfied not only for two colors, but for r colors, for all r). A deep paper of Bergelson and Leibman [1] contains a theorem which implies, in particular, that if D is the range of any polynomial p with integer coefficients such that the leading coefficient is positive and $p(0) = 0$, then $f(D,n) < \infty$ for all n.

On the other hand if D is too sparse, then the corresponding collection of arithmetic progressions will not have the Ramsey property [3]. In particular, this is the case for any finite set D. The main purpose of this article is to introduce a related problem that, apparently, has not been studied before. Namely, we look at the situation in which D is a given finite set, and ask: for which n does $f(D,n)$ exist?

Although we can be sure that, for a given finite set D, $f(D,n)$ does not exist if n is large enough, it is not clear how large n must be. The answer seems to depend not only on the size of the set D, but also on the specific elements of D. For example, the third van der Waerden number, $w(3)$, is known to equal nine. Obviously, any 3-term arithmetic progression that is contained in $[1,9]$ will have its common difference belonging to $D = \{1,2,3,4\}$. Hence, for this choice of D, $f(D,3) = 9$. Meanwhile, if $E = \{1,3,5,7\}$, then the 2-coloring of the positive integers 101010... clearly avoids any monochromatic pair of elements whose difference belongs to E; hence, $f(E,2) = \infty$. In fact, this coloring shows shows that $f(E,2) = \infty$ for any set E consisting only of odd numbers.

In this note, we look at the function $f(D,n)$ for the cases in which $|D| \leq 3$. We can examine the problem in more detail by using the following definition.

Definition 1. *Let $n_1, \ldots, n_k$ be positive integers, and let $d_1, \ldots, d_k$ be distinct positive integers. Let $\vec{n} = (n_1, \ldots, n_k)$ and $\vec{d} = (d_1, \ldots, d_k)$. Then*

$f_{\vec{d}}(\vec{n})$ is the least positive integer N, if it exists, such that for every 2-coloring of $[1,N]$ there will be, for some i, a monochromatic d_i-a.p. of length n_i. If no such N exists, we write $f_{\vec{d}}(\vec{n}) = \infty$.

Thus if $D = \{d_1, \dots, d_k\}$, then $f_{\vec{d}}(\vec{n}) \leq f(D, \max_{1\leq i\leq k} n_i)$.

If χ is a coloring of $[1, M]$ such that, for each i, there is no monochromatic d_i-a.p. of length n_i, we say that χ is $f_{\vec{d}}(\vec{n})$-*valid* on $[1, M]$; or, when it is clear what $\vec{d}$ and $\vec{n}$ are, we simply say that χ is *valid* on $[1, M]$.

2 Results

The case in which $|D| = 1$ is easy. In [9] it was noted that, in this case, $f(D, 2) = \infty$ (any 2-coloring χ of Z^+ having the property that for all x, $\chi(x) \neq \chi(x + d)$, avoids 2-term monochromatic d-a.p.'s).

We next examine what happens when $|D| = 2$. We begin with a lemma that enables us to assume gcd $(d_1, \dots, d_k) = 1$.

Lemma 1 *Let $\vec{d}$ and $\vec{n}$ be as in Definition 1, and let r be a positive integer. Then $f_{r\vec{d}}(\vec{n}) = r[f_{\vec{d}}(\vec{n}) - 1] + 1$ (if $f_{\vec{d}}(\vec{n}) = \infty$, then $f_{r\vec{d}}(\vec{n}) = \infty$).*

Proof: Assume $f_{\vec{d}}(\vec{n}) < \infty$ and let $M = f_{\vec{d}}(\vec{n})$. Let χ be any 2-coloring of $[1, r(M - 1) + 1]$. Define χ' on $[1, M]$ as follows:

$$\chi'(x) = \chi(r(x - 1) + 1).$$

By the definition of M, for some i, $1 \leq i \leq k$, there is a d_i-a.p., $S = \{s_1, \dots, s_{n_i}\} \subseteq [1, M]$, that is monochromatic under χ'. Then $\{r(s_j - 1) + 1 : 1 \leq j \leq n_i\}$ is monochromatic under χ and is an n_i-term rd_i-a.p. Thus, $f_{r\vec{d}}(\vec{n}) \leq r(M - 1) + 1$.

To obtain the reverse inequality, we know that there exists an $f_{\vec{d}}(\vec{n})$-valid 2-coloring ϕ of $[1, M - 1]$. Define ϕ' on $[1, r(M - 1)]$ as follows:

$$\phi'[r(j - 1) + 1, rj] = \phi(j) \text{ for each } j = 1, \dots, M - 1.$$

Then ϕ' is $f_{r\vec{d}}(\vec{n})$-valid on $[1, r(M - 1)]$. Hence $f_{r\vec{d}}(\vec{n}) \geq r(M - 1) + 1$. It is clear that this argument also takes care of the case in which $f_{\vec{d}}(\vec{n}) = \infty$. □

For $|D| = 2$ we have a complete answer. For convenience, if $\vec{d} = (a, b)$ we will write $f_{\vec{d}}$ as $f_{a,b}$.

Theorem 1 *Let a and b be distinct positive integers with $g = gcd(a, b)$, and assume $m \leq n$. Then $f_{a,b}(m, n) = \infty$ unless all of the following are true:*
(i) *a/g and b/g are not both odd*
(ii) $m = 2$
(iii) $a = g$ *or* $n = 2$.
Furthermore, if $f_{a,b}(m, n) < \infty$, then $f_{a,b}(m, n) = (n - 1)(a + b - g) + 1$.

Proof: We shall assume that $g = 1$ because, by Lemma 1, if the theorem is true when $g = 1$, then if $d_1 = a/g$, $d_2 = b/g$, we have $f_{a,b}(m,n) = g[f_{d_1,d_2}(m,n) - 1] + 1 = (n-1)[a+b-g] + 1$.

We begin with the cases in which $f_{a,b}(m,n)$ is finite. First assume a and b are not both odd, and $m = n = 2$. By way of contradiction, let χ be an $f_{a,b}(2,2)$-valid coloring of $[1, a+b]$. Let $\oplus$ represent addition modulo $a+b$. Then $\chi(i) \neq \chi(i \oplus a)$ for all $i \in [1, a+b]$ because $| i \oplus a - i | = a$ or b. Thus

$$\chi(1+b) = \chi(1 \oplus (a+b-1)a) = \chi(1)$$

because $a + b - 1$ is even, a contradiction. Hence no such valid coloring exists and $f_{a,b}(2,2) \leq a+b$.

To show the reverse inequality, define χ on $[1, a+b-1]$ by

$$\chi(i) = \begin{cases} 1 & \text{if } i \equiv ea(\text{mod}(a+b)) \text{ with } e \text{ even} \\ 0 & \text{if } i \equiv ua(\text{mod}(a+b)) \text{ with } u \text{ odd} \end{cases}$$

Since $g = 1$, χ is a well-defined coloring of $[1, a+b-1]$. Now assume $y, z \in [1, a+b-1]$ with $z = y+a$. Then $z \not\equiv a(\text{ mod } (a+b))$. Therefore $\chi(y) \neq \chi(z)$. Also, if $y, z \in [1, a+b-1]$ with $z = y+b$, then $y \equiv z \oplus a(\text{mod}(a+b))$, so that $y \equiv ta(\text{mod}(a+b))$ where $t \neq 0$, $t \neq 1$. It follows that $\chi(y) \neq \chi(z)$. Hence χ is an $f_{a,b}(2,2)$-valid coloring of $[1, a+b-1]$.

Now assume $a = 1$, b is even, and $m = 2$. For each $i \in [1, b(n-1)]$, let

$$\gamma(i) = \begin{cases} 1 & \text{if } i \text{ odd} \\ 0 & \text{if } i \text{ even} \end{cases}$$

Under γ there is no 2-term monochromatic 1-a.p. and no n-term monochromatic b-a.p. So $f_{1,b}(2,n) \geq b(n-1)+1$. On the other hand if ϕ is any 2-coloring of $[1, b(n-1)+1]$ with no monochromatic 2-term 1-a.p., then $\phi(i) = \gamma(i)$ for each $i \in [1, b(n-1)]$. Now if $\phi(b(n-1)+1) = \phi(b(n-1))$, then we have a 2-term monochromatic 1-a.p., while otherwise $\{1 + i(n-1) : i = 0, \ldots, n-1\}$ is a monochromatic n-term b-a.p. Thus, $f_{1,b}(2,n) = b(n-1)+1$.

We now do the cases in which $f_{a,b}$ is infinite. The case in which a and b are both odd was mentioned in the introduction. The only case remaining is that in which $m \geq 2$, $n \geq 3$, $a \neq 1$, and a and b are not both odd (notice that this covers the case of $a = 1$, $m \geq 3$, b even). To prove that $f_{a,b}(m,n) = \infty$, we will exhibit an $f_{a,b}(m,n)$-valid 2-coloring χ of Z^+ that is periodic with period $2a$. Note that if $b \equiv i(\text{mod} 2a)$ where $a \leq i < 2a$, then if $X = \{x, x+b, x+2b\}$ were monochromatic under χ, then for any $t > 0$ satisfying $2ta > b$, the set

$$Y = \{x + 4ta, x + 2ta + b, x + 2b\}$$

would be a monochromatic arithmetic progression with common difference $2ta - b \equiv -i(\text{mod}2a)$. Hence we may assume $b \equiv i(\text{mod}2a)$ with $1 \leq i \leq a$. Also, since χ has period $2a$, we may assume $1 \leq b \leq a$.

We consider 3 subcases:

(i) a even, b odd, $1 \leq b < a/2$.

(ii) a odd, b even, $1 < b < a/2$.

(iii) $a/2 \leq b < a$.

Subcase (i). For each $i \leq a$, let $\chi(i) = 1$ if i is odd, and $\chi(i) = 0$ if i is even; and for each $i > a$, let $\chi(i) \neq \chi(i-a)$. Then there is no 2-term monochromatic a-a.p., and χ has period $2a$. Let B_j denote the interval $[(j-1)a+1, ja]$. If $\{x, x+b\}$ is monochromatic with $x \in B_j$, then by the way χ is defined, since b is odd, $x+b \in B_{j+1}$. Since $x+b \leq ja + (a/2)$, we have $x + 2b \in B_{j+1}$, so that $\chi(x+2b) \neq \chi(x+b)$. Thus there is no 3-term monochromatic b-a.p. in Z^+, and χ is $f_{a,b}(2,3)$-valid on Z^+.

Subcase (ii). Define χ as follows:

$\chi(i) = 1$ for $1 \leq i \leq b$

$\chi(i) \neq \chi(i-b)$ for $b < i \leq a$

$\chi(i) \neq \chi(i-a)$ for $i > a$.

It is clear that there is no 2-term monochromatic a-a.p. If $\{x, x+b\}$ is monochromatic, then $x \in B_j$, $x+b \in B_{j+1}$ for some j. Thus $x+2b \in B_{j+1}$, so that $\chi(x+2b) \neq \chi(x+b)$. Hence there is no monochromatic 3-term b-a.p.

Subcase (iii). For every i, let $\chi(B_i) = 1$ for i odd and $\chi(B_i) = 0$ for i even. Clearly there is no monochromatic 2-term a-a.p. Also, if $\{x, x+b\}$ is monochromatic, with $x \in B_i$, then $x+b$ must belong to B_i. Thus $x + 2b \in B_{i+1}$ (or else $2b < a$), so that $\chi(x+2b) \neq \chi(x)$. So χ is valid. □

Theorem 1 says, in particular, that $f(D,3) = \infty$ whenever $|D| = 2$. Although we do not have quite as complete an answer for the case of $|D| = 3$ as we do for $|D| = 2$, the following theorem does give us the comparable result that $f(D,4) = \infty$ for every 3-element set D.

Theorem 2 *Let a, b, c be positive integers with $a = \min\{a,b,c\}$. Then $f_{a,b,c}(4,3,3) = \infty$.*

Proof: The proof splits naturally into two cases:

Case 1. $2b \leq c$.

In this case let χ be the $2c$-periodic coloring of Z^+ defined recursively as follows:

$\chi(i) = 1$ for $1 \leq i \leq a$

$\chi(i) \neq \chi(i-a)$ for $a < i \leq b$

$\chi(i) \neq \chi(i-b)$ for $b < i \leq c$

$\chi(i) \neq \chi(i-c)$ for $i > c$.

Let $j = \lceil \frac{c}{b} \rceil$. For $1 \le i < j$, let $B_i = [(i-1)b+1, ib]$ and let $B_j = [(j-1)b+1, c]$. Clearly, in each B_i, $1 \le i \le j$, there is no monochromatic 2-term a-a.p. Therefore there is no monochromatic 3-term a-a.p. in $[1, c]$. Notice that there is no monochromatic 2-term a-a.p. in $[1, 2a]$, and so there are none in $[kc+1, kc+2a]$ for each non-negative integer k. Thus in Z^+ there is no monochromatic 4-term a-a.p. Similarly, there is no monochromatic 3-term b-a.p. Finally, it is clear that there is no monochromatic 2-term c-a.p. This implies that $f_{a,b,c}(4,3,3) = \infty$ (in fact we have shown that $f_{a,b,c}(4,3,2) = \infty$).

Case 2. $2b > c$.
Let χ' be the $4c$-periodic coloring of Z^+ defined by:

$\chi'(i) = 1$ for $1 \le i \le a$
$\chi'(i) \ne \chi'(i-a)$ for $a+1 \le i \le b$
$\chi'(i) \ne \chi'(i-b)$ for $b+1 \le i \le 2c$
$\chi'(i) \ne \chi'(i-2c)$ for $i > 2c$.

Letting $d = 2c$, then since $2b \le d$, by the method used in Case 1 we see that under χ' there is no monochromatic 4-term a-a.p., no monochromatic 3-term b-a.p., and no monochromatic 2-term d-a.p. Hence there is no monochromatic 3-term c-a.p. This completes the proof. □

We believe that Theorem 2 can be improved. Namely, we suspect that $f(D,3) = \infty$ whenever $|D| = 3$. This may not be difficult, but we have not been able to find valid colorings for each of the cases. We do have a bit more information about the function $f_{\vec{d}}(2,2,3)$.

Proposition 1 *Let $u = gcd\,(a,b,c)$ and $g = gcd\,(a,b)$. Then $f_{a,b,c}(2,2,3) = \infty$ if either (i) a/u and b/u are odd, c/u is even, and $g \ne 1$ or (ii) all of a/g, b/g, and c/u are odd.*

Proof: We assume that $u = 1$, since the proposition will then follow from Lemma 1.

First consider the case in which c is even, a and b are odd, and $g \ne 1$. By assumption $g \not| c$. Hence by Theorem 1 there is a coloring χ of Z^+ that is $f_{g,c}(2,3)$-valid. Then for all $x \ge 1$, $\chi(x) \ne \chi(x+ig)$ whenever i is odd, so that χ is $f_{a,b,c}(2,2,3)$-valid.

Now assume a/g, b/g, and c are odd. If a, b, and c are all odd, then, as we have mentioned before, $f_{a,b,c}(2,2,2) = \infty$. So we may assume that g is even. As in the first case, we can apply Theorem 1 to the pair (g,c) and, in the same way, this yields an $f_{g,c}(2,3)$-valid coloring. □

Remark. Computer data suggests that all choices of a, b, c not covered by Proposition 1 do yield a finite value for $f_{a,b,c}(2,2,3)$. Certain cases are immediate from Theorem 1. For example, if exactly one of a/g and b/g is even, then $f_{a,b,c}(2,2,3) \le f_{a,b}(2,2) \le a+b-g+1$. If, in addition,

$c = \max\{a, b, c\}$, then $f_{a,b,c}(2,2,3) = a + b - g + 1$, since $[1, a + b - g + 1]$ could not contain a 3-term c-a.p. There are some other cases for which we are able to give an exact formula for $f_{a,b,c}(2,2,3)$. One such case is the following.

Proposition 2 *Let a and b be odd and c even. Let $gcd\ (a, b) = 1$, and assume $c > a, b$. Then $f_{a,b,c}(2,2,3) = 2c + 1$.*

Proof: Let $f = f_{a,b,c}(2,2,3)$. It is obvious that the coloring 101010...10 of $[1, 2c]$ avoids monochromatic 2-term a-a.p.'s and b-a.p.'s and monochromatic 3-term c-a.p.'s. Hence $f \geq 2c + 1$.

To show that $f \leq 2c+1$, assume that χ is a valid 2-coloring of $[1, 2c+1]$. Since $g = 1$, there exist positive integers r_0, s_0 such that $r_0 a - s_0 b = c$. We now define a sequence recursively as follows. Let $x_1 = 1 + r_1 a - s_0 b$ where r_1 is the least integer such that $r_1 a - s_0 b \geq 0$. Once x_i, $i \geq 1$, has been defined, let $y_i = 1 + r_i a - s_0 b$, where s_i is the least integer such that $r_i a - s_i b \leq 2c$. Then once y_i has been defined, let $x_{i+1} = 1 + r_{i+1} a - s_i b$ where r_{i+1} is the least integer such that $r_{i+1} a - s_i b \geq 0$. Notice that $s_{i-1} > s_i$ for all $i \geq 1$, for otherwise, since $r_i a - (s_i - 1)b > 2c$, we would have $r_i a - s_{i-1} b > c$, contradicting the meaning of r_i.

Now let k be the least positive integer such that $s_k \leq 0$. Then $1 \leq r_k a \leq 2c+1$. Now consider the sequence $S = \{1, 1+r_k a, x_k, y_{k-1}, x_{k-1}, y_{k-2}, ..., x_2, y_1, x_1, c+1\}$. It is clear that $S \subseteq [1, 2c+1]$. By our assumption about χ, $\chi(x) \neq \chi(x+a)$ whenever $x, x+a \in [1, 2c+1]$. Hence, if $x, x+ja \in [1, 2c+1]$, then $\chi(x) = \chi(x+ja)$ if and only if j is even. The same is true if a is replaced by b. Thus, since c is odd, and since each pair of adjacent elements of S differ by either a multiple of a or a multiple of b, we see that $\chi(1+c) = \chi(1)$.

By an argument symmetric to the one just used, one can show that $\chi(c + 1) = \chi(2c + 1)$. We omit the details [briefly: use $2c + 1$ in place of 1, define $x_1 = 2c + 1 - (r_1 a - s_0 b)$, where r_1 is the least such that $r_1 a - s_0 b \geq 0$, etc.]. Then $\{1, c + 1, 2c + 1\}$ is a monochromatic 3-term c-a.p., a contradiction, and the proof is complete. □

3 Remarks and Questions

When $|D| = 3$, there are certain other cases in which a precise formula for $f_D(n_1, n_2, n_3)$ can be easily determined. For example, if $D = (1, b, c)$ where exactly one of b and c is odd, say c, then by Theorem 1, $f_{\vec{d}}(2, n_2, n_3) \leq (n_2 - 1)b + 1$, while the coloring 1010...10 of length $(n_2 - 1)b$ shows that the reverse inequality also holds.

For each positive integer k, let $m = m(k)$ denote the least positive such that for all D with $|D| = k$, $f(D, m) = \infty$. We would very much like to know about the function $m(k)$. The results of this paper show that

$m(1) = 2$, $m(2) = 3$, and $m(3) = 3$ or 4. Based on computer output, we conjecture that $m(3) = 3$. By (1), we see that $f(D, 3) = \infty$ for those D which contain no multiples of 3 and for those which contain no multiples of 4. Perhaps $m(k) \leq k + 1$ for all k.

In the remark preceding Proposition 2, it was noted that $f_{a,b,c}(2,2,3) \leq a + b - g + 1$ if $g = \gcd(a,b)$ and exactly one of a/g and b/g is even. We would like to have a precise formula for the function $f_{\vec{d}}(2,2,3)$. Even letting gcd $c = 1$, computer data does not suggest an obvious formula. For example, fixing $a = 8$, and varying b, we found the following values of $f = f_{a,b,c}(2,2,3)$ and $a + b - f$.

b:	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
f:	9	9	13	10	11	19	19	23	25	25	29	25	27	35	35	39
$a+b-f$:	0	2	0	5	6	0	2	0	0	2	0	6	6	0	2	0

We would also like to characterize those triples $\vec{d} = (a,b,c)$ for which $f_{\vec{d}}(2,3,3) < \infty$.

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References

[1] V. Bergelson and A. Leibman, Polynomial extensions of van der Waerden's and Szemerédi's theorems, *J. Amer. Math. Soc.* **9** (1996), 725–753.

[2] T.C. Brown, P. Erdős, and A.R. Freedman, Quasi-progressions and descending waves, *J. Comb. Theory (A)* **53** (1990), 81–95.

[3] T.C. Brown, R.L. Graham, and B.M. Landman, On the set of common differences in van der Waerden's theorem on arithmetic progressions, preprint.

[4] T.C. Brown and B.M. Landman, The Ramsey property for collections of sequences not containing all arithmetic progressions, *Graphs Comb.* **12** (1996), 149–161.

[5] R.N. Greenwell and B.M. Landman, On the existence of a reasonable upper bound for the van der Waerden numbers, *J. Comb. Theory (A)* **50** (1989), 82–86.

[6] B.M. Landman, Ramsey functions related to the van der Waerden numbers, *Discrete Math.* **102** (1992), 265–278.

[7] B.M. Landman, Ramsey functions for quasi-progressions, *Graphs Comb.*, to appear.

[8] B.M. Landman, Avoiding arithmetic progressions (mod m) and arithmetic progressions, *Utilitas Math.* **52** (1997), 173-182.

[9] B.M. Landman and A.F. Long, Ramsey functions for sequences with adjacent differences in a specified congruence class, *Congressus Numerantium* **103** (1994), 3–20.

[10] B.M. Landman and B. Wysocka, Collections of sequences having the Ramsey property only for few colors, *Bull. Australian Math. Society* **55** (1997), 19–28.

[11] B.L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw Arch. Wisk* **15** (1927), 212–216.