Independent Edges of Certain Graphs

Mao-Ting Chien
Department of Mathematics
Soochow University
Taipei, Taiwan 11102
email: mtchien@math.math.scu.edu.tw

ABSTRACT. The numbers of the sets of independent edge sets in 2-lattice graphs, wheel graphs and circuit graphs are computed.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. An *independent set* of G is a subset of V such that no two of its vertices are adjacent, and the *Fibonacci number* $f(G)$ of G is defined to be the number of the set of independent sets in G. In [2] and [3], it is observed that the Fibonacci number of the graph P_n ,as in Figure 1, of a path on $\{1, 2, \cdots, n\}$ is f_{n+1}, the $(n+1)$th Fibonacci sequence, which is defined by $f_0 = f_1 = 1, f_n = f_{n-1} + f_{n-2}$.

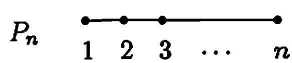

FIGURE 1.

They also dealt with the lattice graph L_{2n} as in Figure 2, and obtained that

$$f(L_{2n}) = \frac{1}{2}(1+\sqrt{2})^{n+1} + \frac{1}{2}(1-\sqrt{2})^{n+1}$$

which satisfies recursion

$$f(L_{2n}) = 2f(L_{2(n-1)}) + f(L_{2(n-2)}). \tag{1}$$

Although it may difficult to find a formula for the Fibonacci number of a general m-lattice graph L_{mn}, Yeh [4] gave a computer algorithm that counts the Fibonacci number $f(L_{mn})$.

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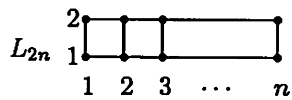

FIGURE 2.

An *independent edge set* of $G = (V, E)$ is defined to be a subset of E such that no two of its edges are adjacent. The number of the set of independent edge sets in G is denoted by $f_E(G)$, and for each nonnegative integer k, the number of the set of k-independent edge sets in G is denoted by $f_E(G, k)$. It is obvious that $f_E(G) = \sum_{k\geq 0} f_E(G, k)$. From the viewpoint of the matching polynomial of a graph, it was shown by Farrell [1] that the coefficients of the matching polynomial are the numbers of the sets of independent edge sets of various cardinalities in the graph. In this paper, we count the numbers of the sets of independent edge sets in 2-lattice graphs, wheel graphs and circuit graphs.

Theorem 1. $f_E(P_n) = f_n$.

Proof: It is clear that $f_E(P_0) = f_E(P_1) = 1$, and the independent edge sets of P_n may be divided into those which conatin the last edge $\{n-1, n\}$ and those which do not contain the last edge in P_n, thus we have $f_E(P_n) = f_E(P_{n-2}) + f_E(P_{n-1})$.

To compute $f_E(L_{2n})$, we need two auxiliary graphs B_n and C_n as in Figure 3.

FIGURE 3.

Let a_n, b_n and c_n denote the numbers of the sets of independent edge sets in L_{2n}, B_n and C_n respectively.

Theorem 2. *The numbers of the sets of independent edge sets a_n, b_n and c_n satisfy the recurrence relation*

$$x_{n+1} = 3x_n + x_{n-1} - x_{n-2}. \tag{2}$$

Numerically,

$$a_n = .664591(3.21432)^n + .255972(-.675131)^n + .079437(.460811)^n. \tag{3}$$

Proof: The number of the set of independent edge sets in L_{2n} can be computed by counting those which contain the nth stalk edge in Figure 2 and those which do not contain the nth stalk edge. We obtain that

$$a_n = a_{n-1} + (b_n + c_n), \tag{4}$$

$$b_n = a_{n-1} + b_{n-1}, \tag{5}$$

$$c_n = a_{n-2} + b_{n-1}. \tag{6}$$

Substituting (6) in (4), we have

$$a_n = c_{n+1} + c_n. \tag{7}$$

From (6)

$$b_{n-1} = c_n - a_{n-2} = c_n - c_{n-1} - c_{n-2}. \tag{8}$$

Substituting (7) and (8) into (5), we obtain

$$c_{n+1} - c_n - c_{n-1} = (c_n + c_{n-1}) + (c_n - c_{n-1} - c_{n-2}),$$

and thus the recursion (2) holds for c_n. From (7), a_n is a linear sum of c_{n+1} and c_n, it follows that a_n satisfies (2), and then by (6), b_n satisfies (2).

The characteristic equation of recurrence relation (2) is

$$x^3 - 3x^2 - x + 1 = 0$$

which has numerical solutions 3.21432, -.675131 and .460811. Compute the inital values of a_n, we have $a_0 = 1, a_1 = 2$ and $a_2 = 7$. By solving the linear homogeneous recurrence (2), the general solution (3) follows.

Notice that, by Theorem 1, $f_E(P_n) \leq f(P_n)$ for all n, however, by comparing (1) and (2), we have $f_E(L_{2n}) \geq f(L_{2n})$ whenever $n \geq 2$. Extension of Theorem 2 to compute $f(L_{mn})$ of a general m-lattice product graph L_{mn} seems worthy of further study. Consider the wheel graph W_n and two auxiliary graphs D_n and the circuit graph E_n as in Figure 4. The center vertex is denoted by w, the numbers of the sets of independent edge sets in W_n, D_n and E_n are denoted by w_n, d_n and e_n respectively.

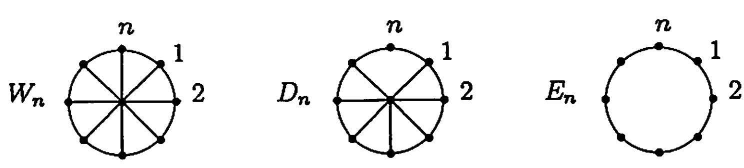

FIGURE 4.

We first count $F_E(P_n, k)$, and obtain a well-known combinatorial identity.

Theorem 3. *For each nonnegative integer k,*

(i) $f_E(P_n, k) = \binom{n-k}{k}$;

(ii) $f_n = \sum_{k\geq 0} \binom{n-k}{k}$.

Proof: Let $p(n, k)$ denote the number $f_E(P_n, k)$. Then it is obvious that

$$p(n,k) = p(n-2, k-1) + p(n-1, k). \tag{9}$$

Let $g(x, y)$ be the generating function of the sequence $p(n, k)$. We compute that

$$\begin{aligned}
g(x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} p(n,k)x^n y^k \\
&= \sum_{n=2}^{\infty}\sum_{k=1}^{n} p(n,k)x^n y^k + p(1,1)xy + \sum_{n=0}^{\infty} p(n,0)x^n \\
&= \sum_{n=2}^{\infty}\sum_{k=1}^{n} p(n,k)x^n y^k + \frac{1}{1-x}.
\end{aligned} \tag{10}$$

Substituting (9) into (10), we have

$$\begin{aligned}
g(x,y) &= \sum_{n=2}^{\infty}\sum_{k=1}^{n} p(n-2,k-1)x^n y^k + \sum_{n=2}^{\infty}\sum_{k=1}^{n} p(n-1,k)x^n y^k + \frac{1}{1-x} \\
&= x^2 y \sum_{n=0}^{\infty}\sum_{k=0}^{n} p(n,k)x^n y^k + \\
&\quad x\Big(\sum_{n=0}^{\infty}\sum_{k=0}^{n} p(n,k)x^n y^k - \sum_{n=0}^{\infty} p(n,0)x^n\Big) + \frac{1}{1-x} \\
&= x^2 y g(x,y) + x g(x,y) + 1.
\end{aligned} \tag{11}$$

From (11),

$$
\begin{aligned}
g(x,y) &= \frac{1}{1-x-x^2y} \\
&= \frac{1}{1-x}\frac{1}{1-\frac{x^2y}{1-x}} \\
&= \frac{1}{1-x}\sum_{i=0}^{\infty} x^{2i}y^i(1-x)^{-i} \\
&= \sum_{i=0}^{\infty} x^{2i}y^i(1-x)^{-(i+1)} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j} x^{2i+j}y^i. \qquad (12)
\end{aligned}
$$

Collecting the coefficeint of x^ny^k of $g(x,y)$ in (12), we conclude that

$$p(n,k) = \binom{n-k}{n-2k} = \binom{n-k}{k},$$

and this proves (i). Taking the sum over k on (i), the result (ii) follows.

Hopkins and Staton [2] determined the Fibonacci number $f(W_n) = (-1)^{n+1} + (1+\sqrt{2})^n + (1-\sqrt{2})^n$. In the following we compute $f_E(W_n)$ and $f_E(E_n)$.

Theorem 4.

(i) $f_E(W_n,k) = f_E(P_n,k) + nf_E(P_{n-1},k-1) + f_E(P_{n-2},k-1)$;

(ii) $f_E(E_n,k) = f_E(P_{n-2},k-1) + f_E(P_n,k)$;

(iii) $f_E(W_n) = f_n + nf_{n-1} + f_{n-2}$;

(iv) $f_E(E_n) = f_{n-2} + f_n$.

Proof: Let $w(n,k), p(n,k), d(n,k)$ and $e(n,k)$ denote the numbers $f_E(W_n,k), f_E(P_n,k), f_E(D_n,k)$ and $f_E(E_n,k)$ respectively. The number $w(n,k)$ can be computed by counting those k-independent edge sets which contain the edge $\{w,n\}$ and those which do not contain the edge $\{w,n\}$ in W_n. Then we have

$$w(n,k) = p(n-1,k-1) + d(n,k). \qquad (13)$$

We compute $d(n,k)$ by counting those which contain the edge $\{w,n-1\}$ and those which do not contain $\{w,n-1\}$ in D_n,

$$d(n,k) = p(n-1,k-1) + d'(n,k), \qquad (14)$$

where $d'(n,k)$ is the number of the set of k-independent edge sets in the graph by removing the edge $\{w, n-1\}$ from D_n. Continue the counting in this way on $d'(n,k)$, we obtain from (13) and (14) that

$$w(n,k) = np(n-1,k-1) + e(n,k).$$

By similar counting on $e(n,k)$, we have $e(n,k) = p(n-2,k-1) + p(n,k)$, and this proves (i) and (ii).

By summing over k on equations (i) and (ii), and together with Theorem 1, the result (iii) and (iv) follow.

References

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