Vulnerability in Graphs - a Comparative Survey

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ABSTRACT. The concept of tenacity of a graph G was introduced in References [5,6] as a useful measure of the "vulnerabilty" of G. In assessing the "vulnerability" of a graph one determines the extent to which the graph retains certain properties after the removal of vertices or edges. In this paper we will compare different measures of vulnerability with tenacity for several classes of graphs.

INTRODUCTION

If we think of the graph as modeling a network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. In this survey we will restrict ourselves to the study of vertex versions of vulnerability.

Throughout this paper we will let n be the number of vertices of G, and we use $\alpha(G)$ to denote the independence number of G. Let A be a subset of V(G). The neighborhood of A, N(A), consists of all vertices of G adjacent to at least one vertex of A. We define G-A to be the graph induced by the vertices of V-A. Also, for any graph G, $\tau(G)$ is the number of vertices in a largest component of G and $\omega(G)$ is the number of components of G. A cutset of a connected graph G is a collection of vertices whose removal results in a disconnected graph.

The connectivity of G, $\kappa = \kappa(G)$ is the minimum order of a cutset of G.

The binding number of a graph G was introduced by Woodall in [17] and is defined as $\text{bind}(G) = \min\{\frac{|N(A)|}{|A|}\}$, where the minimum is taken over all $A \subseteq V(G)$ with $A \neq \phi$ and $N(A) \neq V(G)$. The binding number has also been studied in [8,10,11,12,13,16] among others.

The concept of integrity of a graph G was introduced in [2] as a useful measure of the vulnerability of a graph G. The integrity of a graph G is

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defined as $I(G) = \min \{|A| + \tau(G-A)\}$, where the minimum is taken over all $A \subseteq V(G)$. The integrity is a measure which deals with the first fundamental question. How many vertices can still communicate? Integrity has been studied in numerous papers, including [1,4].

The toughness of a graph G was introduced by Chvàtal in [3], where he obtained several results regarding this invarient including the relationship between this parameter and the existence of Hamilton cycles. The toughness of a graph, denoted by t(G), is defined as $t(G) = \min\{\frac{|A|}{\omega(G-A)}\}$, where the minimum is taken over all cutsets A in V(G). The toughness deals with the second fundemental question, namely, how difficult is it to reconnect the graph ? The toughness has been studied extensively; see for example [7,15].

The tenacity is another vulnerability measure, incorporating ideas of both toughness and integrity and dealing with both of the above questions. The tenacity of a graph G, T(G), is defined by $T(G) = \min\{\frac{|A|+\tau(G-A)}{\omega(G-A)}\}$, where the minimum is taken over all vertex cutsets A of G, G-A is the graph induced by the vertices of V-A, $\tau(G-A)$ is the number of vertices in the largest component of the graph induced by G-A and $\omega(G-A)$ is the number of components of G-A. A connected graph G is called T-tenacious if $|A| + \tau(G-A) \geq T\omega(G-A)$ holds for any subset A of vertices of G with $\omega(G-A) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p ($p \geq 1$). A set $A \subseteq V(G)$ is said to be a T-set of G if $T(G) = \frac{|A|+\tau(G-A)}{\omega(G-A)}$.

We will compare integrity, connectivity, binding number, toughness and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of vulnerabilty in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability.

VULNERABILITY CALCULATION

Let $C_n = (v_1 v_2 \cdots v_n)$ be the n-cycle and define the k-th power of the n-cycle, C_n^k, by

$$C_n^k = C_n + \{v_i v_j \mid |i-j| \leq k\}.$$

We will calculate the five measures of vulnerability for the complete bipartite graph $K_{k,n-k}$, $k \leq n-k$, powers C_n^k of the n-cycle, and the graph $G(n,k)$, $1 \leq k \leq \lfloor\frac{n-1}{2}\rfloor$, which has n vertices and vertex v which is adjacent to all vertices of the two complete subgraphs, copies of K_k and K_{n-k-1}, i.e $G_{n,k} \equiv K_1 + (K_k \cup K_{n-k-1})$.

These graphs were purposefully chosen, because they exhibit the widest possible range of edge density and because they illustrate where the different measures of vulnerability differ in their effectiveness in measuring important structural characteristics of graphs.

Theorem 1 : (Chvàtal [3]). For all graphs G, $\frac{\kappa(G)}{\alpha(G)} \leq t(G) \leq \frac{1}{2}\kappa(G)$.

Theorem 2 : (Woodall [17]). For all graphs G, $bind(G) \leq \frac{n+\kappa(G)}{n-\kappa(G)}$.
Theorem 3 : (Woodall [17]). For all graphs G, $bind(G) \leq t(G)+1$.

The following four proposition were proved in [6].

Proposition 1 : If G is a spanning subgraph of H, then $T(G) \leq T(H)$.
Proposition 2 : For any graph G, $T(G) \geq \frac{\kappa(G)+1}{\alpha(G)}$.
Proposition 3 : If G is not complete, then $T(G) \leq \frac{n-\alpha(G)+1}{\alpha(G)}$.
Proposition 4 : If $k \leq n-k$, then $T(K_{k,n-k}) = \frac{k+1}{n-k}$.

Lemma 1 : If A is a minimal T-set for C_n^k, then A consists of the union of sets of k consecutive vertices such that there exists at least one vertex not in A between any two sets of consecutive vertices in A.

Proof : We assume C_n^k is labeled by $0,1,2,\cdots,n-1$. Let A be a minimal T-set for C_n^k and j be the least integer such that $S=\{j,j+1,\cdots,j+t-1\}$ is a maximal set of consecutive vertices such that $S \subseteq A$. Re-label the vertices of C_n^k as $v_1=j, v_2=j+1,\cdots,v_t=j+t-1,\cdots,v_n$. Since $A \neq V(C_n^k)$, $S \neq V(C_n^k)$ so v_n does not belong to A. Since A must leave at least two components, $t \neq n-1$, so $v_{t+1} \neq v_n$. Therefore $\{v_{t+1}, v_n\} \cap A = \phi$. Now suppose $t<k$. Choose v_i such that $1 \leq i \leq t$, and delete v_i from A yielding a new set $A' = A - \{v_i\}$ with $|A'| = |A|-1$. The edges v_iv_n and v_iv_t+1 are in $C_n^k - A'$. Consider a vertex v_p adjacent to v_i in $C_n^k - A'$, then either $t+1 \leq p < t+k$, so v_p is also adjacent to v_{t+1} in $C_n^k - A'$, or $n-k+1 \leq p < n$ and v_p is also adjacent to v_n in $C_n^k - A'$. Since $t<k$, then v_n and v_{t+1} are adjacent in $C_n^k - A$. Therefore we can conclude that deleting vertex v_i from A does not change the number of components, and so $\omega(C_n^k - A) = \omega(C_n^k - A')$ and the maximum order of a component of $C_n^k - A$ is $\tau(C_n^k - A') \leq \tau(C_n^k - A)+1$.

Therefore $\frac{|A'|+\tau(C_n^k-A')}{\omega(C_n^k-A')} \leq \frac{|A|-1+\tau(C_n^k-A)+1}{\omega(C_n^k-A)} = T(C_n^k)$, contrary to our choice of A. Thus we must have $t \geq k$.

Now suppose $t>k$. Delete v_t from the set A yielding a new set $A_1 = A - \{v_t\}$. Since $t>k$, the edge v_tv_n is not in $C_n^k - A_1$. Consider a vertex v_p adjacent to v_t in $C_n^k - A_1$. Then $p \geq t+1$ and $p \leq t+k$. So v_p is also adjacent to v_t+1 in $C_n^k - A_1$. Therefore deleting v_t from A yields $\omega(C_n^k - A) = \omega(C_n^k - A_1)$, $\tau(C_n^k - A_1) \leq \tau(C_n^k - A)+1$. Therefore $\frac{|A_1|+\tau(C_n^k-A_1)}{\omega(C_n^k-A_1)} \leq \frac{|A|-1+\tau(C_n^k-A)+1}{\omega(C_n^k-A)}$, again contrary to our choice of A. Thus $t=k$ and so A consists of the union of sets of exactly k consecutive vertices.

Lemma 1 gives us an indication of the size of the cut-set for the tenacity of C_n^k ; the next lemma gives us the size of the largest component.

Lemma 2 : There is a T-set, A, for C_n^k such that all components of $C_n^k - A$ have order $\tau(C_n^k - A)$ or $\tau(C_n^k - A)-1$.

Proof: Among all minimum order T-sets, consider those sets B with maximum order, s, of the minimum order component of $C_n^k - B$. Among these sets let A be one with the fewest components of order s in $C_n^k - A$. Suppose $s \le \tau(C_n^k - A) - 2$. Note that all of the components must be sets of consecutive vertices. Suppose C_p is a smallest component, so $| V(C_p) |= s$, and without loss of generality let $C_p = \{v_1, v_2, \cdots, v_s\}$. Suppose C_l is a largest component, so $| V(C_l) |= \tau(C_n^k - A) = m$, and $C_l = \{v_j, \cdots, v_{j+m-1}\}$. Let $C_1, C_2, \cdots, C_a$ be components with vertices between v_s and v_j, such that $| C_i |= n_i$ for $1 \le i \le a$ and $C_i = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{n_i}}\}$. Now construct A' as follows, $A' = A - \{v_{s+1}, v_{1_{n_1}+1}, v_{2_{n_2}+1}, \cdots, v_{a_{n_a}+1}\} \cup \{v_{1_1}, v_{2_1}, \cdots, v_{a_1}, v_j\}$. Therefore $| A' |=| A |$, $\tau(C_n^k - A') \le \tau(C_n^k - A)$ and $\omega(C_n^k - A') = \omega(C_n^k - A)$. So, $\frac{|A'| + \tau(C_n^k - A')}{\omega(C_n^k - A')} \le \frac{|A| + \tau(C_n^k - A)}{\omega(C_n^k - A)}$. Therefore $\tau(C_n^k - A') = \tau(C_n^k - A)$. But $C_n^k - A'$ has one less component of order s than $C_n^k - A$, and this is a contradiction. Thus all components of $C_n^k - A$ have order $\tau(C_n^k - A)$ or $\tau(C_n^k - A) - 1$. So $\tau(C_n^k) = \lceil \frac{n - k\omega}{\omega} \rceil$.

These two lemmas allow us to determine precisely the tenacity of the power of cycles.

Theorem 4 : Let C_n^k be a power of cycles and $n = r(k+1) + s$, for $0 \le s < k+1$. Then $T(C_n^k) = k + \frac{1 + \lceil \frac{s}{r} \rceil}{r}$.

Proof : Let A be a minimal T-set of C_n^k. By Lemma 1 and Lemma 2, $| A |= k\omega$, and $\tau(C_n^k - A) = \lceil \frac{n - k\omega}{\omega} \rceil$. Thus, from the definition of tenacity we have

$$T = min\{\frac{k\omega + \lceil \frac{n-k\omega}{\omega} \rceil}{\omega} \mid 2 \le \omega \le r\}.$$

Now consider the function $f(\omega) = \frac{k\omega + \lceil \frac{n-k\omega}{\omega} \rceil}{\omega} = k + \frac{\lceil \frac{n}{\omega} - k \rceil}{\omega}$. Let ω_1 and ω_2 be any two integers in [2 , r] with $\omega_1 \le \omega_2$, then $\lceil \frac{n}{\omega_2} \rceil \le \lceil \frac{n}{\omega_1} \rceil$. Thus $f(\omega_2) = k + \frac{\lceil \frac{n}{\omega_2} - k \rceil}{\omega_2} \le k + \frac{\lceil \frac{n}{\omega_1} - k \rceil}{\omega_1} = f(\omega_1)$. Hence the function $f(\omega)$ is a nonincreasing function and the minimum value occurs at the boundary. Thus $\omega = r$ and $\lceil \frac{n-k\omega}{\omega} \rceil = \lceil \frac{r(k+1)+s-kr}{r} \rceil = 1 + \lceil \frac{s}{r} \rceil$. Therefore, $T(C_n^k) = k + \frac{1 + \lceil \frac{s}{r} \rceil}{r}$.

DISCUSSION

Now consider the complete bipartite graph $K_{k,n-k}$. In [17], the binding number for a complete bipartite graph was calculated by Woodall, where he gives the result $bind(K_{a,b}) = min\{\frac{a}{b}, \frac{b}{a}\}$ for $a \ge 1$ and $b \ge 1$. Thus if $k \le n-k$, then $bind(K_{k,n-k}) = \frac{k}{n-k}$. The connectivity of $K_{k,n-k}$ obviously is equal to k. From [3], we have $t(K_{k,n-k}) = \frac{k}{n-k}$. It is shown in [2] that $K_{k,n-k}$ has integrity equal to k+1. By proposition 4, $T(K_{k,n-k}) = \frac{k+1}{n-k}$.

Thus we have the following results for $G = K_{k,n-k}$:

$$\kappa(G) = k$$

$$t(G) = \frac{k}{n-k}$$

$$bind(G) = \frac{k}{n-k}$$

$$I(G) = k+1$$

$$T(G) = \frac{k+1}{n-k}$$

The binding number implies that the neighborhood of subset $A \subseteq V(K_{k,n-k})$ has order k and $|A| = n-k$. The value of $\kappa(K_{k,n-k})$ shows us that at least k vertices must be destroyed in order to break a complete bipartite graph. But these two measures do not indicate how many components exist after removing the cutset from the graph. Since the toughness of $K_{n,n-k}$ is equal to $\frac{k}{n-k}$, the cardinality of the cutset and the number of components are k and n-k respectively. The integrity of $K_{k,n-k}$ implies that $|A| = k$ and $\tau(K_{k,n-k} - A) = 1$. Hence both toughness and integrity attempt to describe the structure of the resulting graph after removing the cutset A from $K_{k,n-k}$. The tenacity of a bipartite graph shows us that $|A| = k$, $\tau(K_{k,n-k} - A) = 1, \omega(K_{k,n-k} - A) = n-k$. Hence we obtain the number of components, cardinality of cutset and, since $\tau(K_{k,n-k} - A) = 1$, all n-k components have order 1. Thus we have all of the necessary information for the repair and reconfiguration of the complete bipartite graphs. therefore, in this class, tenacity appears to be a better vulnerability measure.

In [2], the connectivity, binding number and toughness of C_n^k were determined. The integrity of C_n^k was calculated in [1]. By Theorem 4, we have the tenacity of C_n^k. Hence we have the following results for $G = C_n^k$:

$$\kappa(G) = 2k, 2 \leq k \leq n-2$$

$$bindG = \begin{cases} 1 & k = 1, 2 \mid n \\ \frac{n}{2} - 1 & 2k = n-2 \\ \frac{n-1}{n-2k} & otherwise \end{cases}$$

$$I(G) = k\lceil\sqrt{\frac{n}{k} - \frac{1}{4}} - \frac{3}{2}\rceil + \lceil\frac{n}{\sqrt{\frac{n}{k} + \frac{1}{4}} - \frac{1}{2}}\rceil, \text{where } \; 1 \leq k \leq \frac{n}{2}$$

$$t(G) = k$$

$$T(G) = k + \frac{1 + \lceil\frac{s}{r}\rceil}{r}$$

The value of $\kappa(C_n^k)$ shows us that it is necessary (and sufficient) to remove two disjoint nonadjacent subsets of k consecutive vertices each, along the circumference of the polygon. The toughness of C_n^k, uses the above fact and it was calculated and proved that the cardinality of the cutset is equal to 2k and the number of components is 2. But the enemy will selectively target more resources to break the network, since the resulting network with only two components is easily repaired. Also, toughness does not take into account the order of the components. Therefore, for breaking or reconstruction of C_n^k, tenacity and its minimal cutset seem to be better measures than connectivity and toughness in this class. If n is even and k = 1, the neighborhood of subset $A \subseteq V(C_n^k)$ and $\mid A \mid$, have the same order. When n is odd and k = 1, $bind(C_n^k) = 1 + min\frac{|N(A)|}{|A|}$. However, A is as large as possible when $\mid A \mid = n - 2$ and this maximum value of $\mid A \mid$ coincides with the minimum value of N(A), namly, N(A) = 1. In both of the above cases and when $k > 1$, the binding number does not show the order of the components, or number of components. By theorem 4 and lemma 2 tenacity gives us the number of components and the order of the largest component. If we compare connectivity, binding number and toughness with this class, integrity seems to be a better measure for the vulnerabilty of a network. But for repair and reconfiguration of C_n^k, we have a lack of information about the number of components. Thus in this class, for disruption and recontruction of network, tenacity appears to be a better measure of the vulnerability of a graph.

We now turn our discussion to the vulnerability of $G_{n,k}$. For $G = G_{n,k}$:

$$\kappa(G) = 1$$

$$t(G) = \frac{1}{2}$$

$$bind(G) = \begin{cases} 1 & k = 1 \\ \frac{n-1}{n-2} & k = 2 \\ \frac{n-k}{n-k-1} & k \geq 3 \end{cases}$$

$$I(G) = n - k$$

$$T(G) = \frac{n-k}{2}$$

The graphs $G_{n,k}$ perhaps best illustrate the inability of connectivity to provide a realistic measure of the vulnerability of graphs. Certainly disabling a station located at vertex v is less damaging to the operation of the remaining system when k = 1 than when $k = \lfloor \frac{n-1}{2} \rfloor$. Yet neither $\kappa(G_{n,k})$ nor $t(G_{n,k})$ reflect this. Also, $bind(G_{n,k})$ is quite insensitive to the value of k. On the other hand, $T(G_{n,k})$ provides a significant indication of the change in the nature of the structure of the system for $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. The integrity of $G_{n,k}$ implies that the cardinality of cutset $A \subseteq V(G_{n,k})$ is equal to 1 and $\tau(G_{n,k} - A) = n - k - 1$. Hence if we remove the vertex v, the integrity shows us the order of the largest component, but does not show the number of components. Therefore since $T(G_{n,k})$ has $\omega(G - A)$ in the denominator indicating the number of components, it then provides a more realistic measure of the vulnerability of the graphs. For instance, if a similar graph were constructed with three copies of K_m, the integrity would remain unchanged while tenacity would recognize this change.

CONCLUDING REMARKS

Deterministic measures tend to provide a worst case analysis of some aspects of the overall disconnection process. For example knowing only $\kappa(G)$, means that for a particular network, even if the enemy knows how the edges have been assigned to the vertices, at least $\kappa(G)$ vertices must be destroyed in order to break communications. Unfortunately, this measure does not indicate how many of these sets of vertices (called minimal cutsets) actually exist in the network, nor does it attempt to describe the resulting network.

Consider Figure 1. Both graphs have connectivity equal to 2. But the removal of a 2-vertex cutset in (a) leaves almost all of the graph intact, wherease the removal of a 2-vertex cutset in (b) cuts the graph in half, potentially a far more serious matter, for example, in a communication network.

Consider Figure 2. The graph has connectivity equal to 1, but removing 2 vertices can achieve considerably greater pairwise disconnection in this graph than can be achieved by the removal of a single vertex.

For both of these reasons we would like to attempt to quantify connectivity as a relative, as well as an absolute parameter.

Vertex integrity provides some information about the network after disconnection has taken place but, once again, it does not seem to provide the fine resolution that is often needed.

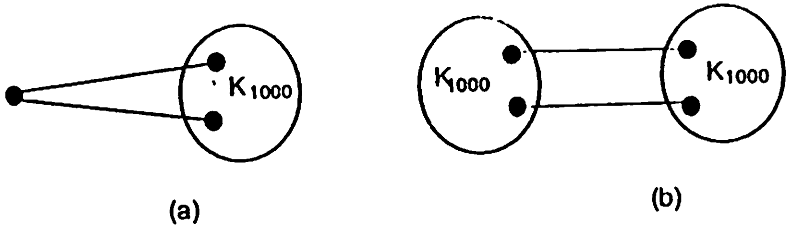

Figure 1

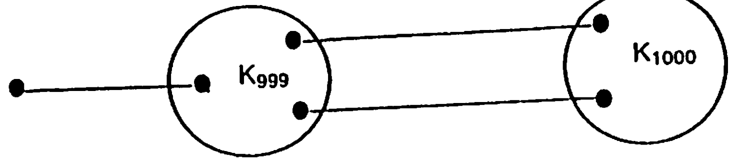

Figure 2

Deterministic measures are generally very difficult to compute. Eventhough $\kappa(G)$ can be computed quickly - linear in the number of vertices plus the number of edges - determination of vertex integrity of a graph is, in general, an NP-complete problem (see, for example [4]).

The effective design of a survivable communications network requires a means of accurately evaluating its structural vulnerability both as a whole and with respect to its individual resources. For a communications network operating in a tactical environment, this evaluation should be based on a worst-case assumption that the enemy will selectively target those resources most critical to its topological integrity. A critical concern of overall system survivability, therefore, must be the specific level of connectivity associated with the topological structure of the supporting communications network. In [9], Harary showed that in any graph or communications network, the connectivity of a graph with p vertices and q edges cannot exeed $\lfloor \frac{2q}{p} \rfloor$ if $q \geq p-1$ and is 0 otherwise. The power of a cycle, C_n^k, is an example of a graph with maximum connectivity. We would like to show the maximum tenacity relative to the maximum connectivity. We found this relation in Theorem 4. Since communication networks must be constructed to be as stable as possible, not only with respect to initial distruption, but also with respect to the possible reconstruction of the network, then C_n^k is a good example for network designers who are looking for a network with maximum connectivity relative to maximum tenacity.

Since it is not clear which networks constitute "optimal networks", the best we can do is to find some measure or measures that we believe do a reasonable job at measuring "goodness".

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