Conditional invariants and interpolation theorems for graphs

Sanming Zhou
Department of Mathematics
The University of Western Australia
NEDLANDS, Perth, WA 6907, Australia
email: *smzhou@maths.uwa.edu.au*

Abstract

Let $\mathcal{F}$ be a family of objects and φ an integer-valued function defined on $\mathcal{F}$. If for any $A, B \in \mathcal{F}$ and integer k between $\varphi(A)$ and $\varphi(B)$ there exists $C \in \mathcal{F}$ such that $\varphi(C) = k$, then φ is said to interpolate over $\mathcal{F}$. In this paper we first discuss some basic ideas used in proving interpolation theorems for graphs. By using this we then prove that a number of conditional invariants interpolate over some families of subgraphs of a given connected graph.

1 Introduction

Let $\mathcal{F}$ be a family of objects under consideration and $\varphi : \mathcal{F} \to \mathbb{Z}$ an integer-valued function. If for any $A, B \in \mathcal{F}$ and each integer k between $\varphi(A)$ and $\varphi(B)$ there exists $C \in \mathcal{F}$ such that $\varphi(C) = k$, then following [17] we say that φ *interpolates* over $\mathcal{F}$. Evidently, this is equivalent to saying that the image set $\varphi(\mathcal{F})$ consists of consecutive integers. The study on interpolation was initiated by Harary, Hedetniemi and Prins [14] when they proved the homomorphism interpolation theorem. Generalization of this theorem was made in [8]. In 1980, Chartrand [4] asked whether the number of pendant vertices interpolates over the family of

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spanning trees of a connected graph. This question was answered affirmatively in [22, 26] and a lot of interpolation results concerning some basic graphical invariants were obtained in the last decade. For example, it was proved [16] that the diameter interpolates over the family of spanning trees of a 2-connected graph. As a generalization to this result it was proved in [27] that both the diameter and the radius interpolate over the family of spanning trees of any tree-like graph (a graph G is tree-like if it can be obtained from a tree by identifying the centers of some multi-blocks, which are pairwise vertex-disjoint and are also vertex-disjoint with G, to some vertices of the tree, where a multi-block is a graph obtained from some vertex-disjoint blocks of order ≥ 3 by identifying one vertex of each block to a new vertex, called the center.). For other interpolation results the reader can consult, for example, [2, 15, 17, 28, 29, 30].

The purpose of this paper is two-fold. In the first part (Section 2) we discuss some basic ideas used in proving interpolation results. By using this we then in the second part (Section 3) prove a number of interpolation theorems for some conditional invariants of graphs. As we shall see, the concepts of transformation and continuity of an invariant with respect to the transformation play an important role in the research of interpolation.

The graphs considered in this paper are finite, undirected and simple. For a graph $G = (V(G), E(G))$, denote by $G[X]$ the subgraph of G induced by X if $X \subseteq V(G)$ or $X \subseteq E(G)$. We use K_n to denote the complete graph on n vertices and $\overline{G}$ to denote the complement graph of G. By $H \subseteq G$ we mean H is a subgraph of G. The union of two graphs G and H is denoted by $G \cup H$. The degree-one vertices of G are called the pendant vertices. For a vertex v of G, we use $G - v$ to denote the graph resulted from G by deleting v together with the edges incident with v. For an edge e of G, $G - e$ is the spanning subgraph of G with edge set $E(G) - \{e\}$. If f is an edge of $\overline{G}$, then $G + f$ is the graph defined by $G = (G+f) - f$. In general, if $E \subseteq E(G), F \subseteq E(\overline{G})$, let $G - E$ and $G + F$ be the graphs with vertex set $V(G)$ and edge sets $E(G) - E$ and $E(G) \cup F$, respectively. For other undefined terminology and notations the reader is referred to [3].

2 Basic ideas

First we have the following simple but useful fact.

Proposition 1 ([28]) *φ interpolates over $\mathcal{F}$ if and only if there exists a connected graph $G(\mathcal{F})$ with vertex set $\mathcal{F}$ such that $A, B \in \mathcal{F}$ is adjacent in $G(\mathcal{F})$ implies $|\varphi(A) - \varphi(B)| \leq 1$.*

This can be viewed as a refinement of some basic ideas used in the research of interpolation [16, 17, 18, 22, 26, 27, 28, 29, 30]. We say roughly that $G(\mathcal{F})$ is a *transformation graph* defined on $\mathcal{F}$. Indeed, in proving concrete interpolation theorems the graph $G(\mathcal{F})$ is often associated with some transformations among elements of $\mathcal{F}$ (see e.g. [16, 17, 18, 22, 26, 28]). For instance, the *simple edge-exchange*, *adjacent edge-exchange* and *leaf edge-exchange* (see [17] for definitions) were used in the research of interpolation. For two graphs G, H, we call the transformation $G \to H$ a *single edge deletion or addition* (EDA) if $H = G - e$ for an edge $e \in E(G)$ or $G = H - f$ for an edge $f \in E(H)$. If $H = G - e + f$ for some $e \in E(G) - E(H)$ and $f \in E(H) - E(G)$, then we say that $G \to H$ is a *simple edge transformation* (SET, or simple edge-exchange as used in [17]). These two types of transformation and the edge-exchanges above are useful in proving interpolation theorems for some families of spanning subgraphs of a given graph [17, 28]. Nevertheless, new transformations are needed for the study of interpolation for non-spanning subgraphs. So let us introduce the following transformations. We call $G \to H$ a *pendant vertex deletion or addition* (VDA) if $H = G - v$ for a pendant vertex v of G or $G = H - w$ for a pendant vertex w of H. If there exist respectively pendant vertices v, w (not necessarily distinct) of G, H such that $G - v = H - w$, then we call $G \to H$ a *pendant vertex transformation* (PVT). Obviously a leaf edge-exchange is a PVT, but not conversely. If $H = G - v + e$ for a pendant vertex v of G and an edge e of $\overline{G}$ or G can be expressed in terms of H in a similar way, then we call $G \to H$ a *mixed edge transformation* (MET). Let $L \subseteq$ {EDA, VDA, SET, PVT, MET} and let $\mathcal{F}$ be a family of graphs. A sequence $G_0, G_1, \ldots, G_n$ with all terms in $\mathcal{F}$ is called an *L-sequence* of $\mathcal{F}$ from G_0 to G_n if each $G_i \to G_{i+1}$ is one type of transformation in L. $\mathcal{F}$ is said to be *L-connectable* if for any distinct $G, H \in \mathcal{F}$ there exists an L-sequence of $\mathcal{F}$ from G to H. If we define $L(\mathcal{F})$ to be the graph with vertex set $\mathcal{F}$ and two members G, H being adjacent if and only if $G \to H$ is one type of transformation in L, then $\mathcal{F}$ is L-connectable if and only if $L(\mathcal{F})$ is connected. So the L-connectability is an alternative (but convenient) way of stating the connectedness of $L(\mathcal{F})$. Thus, the connectedness of the tree-graph of a connected graph is equivalent to saying that the family of spanning trees of the graph is SET-connectable.

Now we introduce another concept, namely the continuity of an integer-valued function φ with respect to a transformation. Let Λ be a transformation defined for graphs. We say that φ is *continuous* with respect to Λ if $\Lambda : G \to H$ implies

$$|\varphi(G) - \varphi(H)| \leq 1. \tag{1}$$

For a graphical invariant φ, if

$$\varphi(G) - 1 \leq \varphi(G - e) \leq \varphi(G) \tag{2}$$

for any graph G and $e \in E(G)$, then φ is said to be *positive* [17]. If

$$\varphi(G) \leq \varphi(G - e) \leq \varphi(G) + 1 \tag{3}$$

instead, then φ is *negative* [17]. If φ satisfies both (2) and

$$\varphi(G) - 1 \leq \varphi(G - v) \leq \varphi(G) \tag{4}$$

for any pendant vertex v of G, then we say that φ is *strongly positive.* The concept of *strongly negetive* invariant is understood in a similar way. Clearly any positive or negative invariant is continuous with respect to both EDA and SET. Also, we observe that any strongly positive or strongly negative invariant is continuous with respect to all types of transformation defined above. So Proposition 1 implies the following

Proposition 2 *Let $\mathcal{F}$ be a family of graphs and φ a graphical invariant. Let $L \subseteq \{EDA, VDA, SET, PVT, MET\}$. If φ is continuous with respect to each type of transformation in L and $\mathcal{F}$ is L-connectable, then φ interpolates over $\mathcal{F}$. In particular, we have*

(i) *If $\mathcal{F}$ is $\{EDA, SET\}$-connectable, then any positive or negative invariant interpolates over $\mathcal{F}$;*

(ii) *If $\mathcal{F}$ is $\{EDA, VDA, SET, PVT, MET\}$-connectable, then any strongly positive or strongly negative invariant interpolates over $\mathcal{F}$.*

3 Interpolation theorems

A property P associated with graphs is said to be *hereditary* [20] if whenever a graph G has property P and $H \subseteq G$ then H also has property P. Dually, a property Q is *cohereditary* [8] if whenever a graph G possesses Q and $G \subseteq H$ then H possesses Q as well. Although a number of interpolation results for some families of spanning subgraphs (particularly for the family of spanning trees) of a connected graph have been obtained ([2, 16, 17, 18, 22, 26, 28]), few results are known for non-spanning subgraphs. The only example of such kind known to the author is that the number of vertices with degree $\leq M$ interpolates over the family of connected m-edge subgraphs of a connected graph for any given M, m [28]. In this paper we shall consider both spanning and non-spanning subgraphs. For a given connected graph K with order $p = |V(K)|$ and size $q = |E(K)|$, we shall discuss interpolation problems for the following typical families of subgraphs of K.

$\mathcal{F}_1$: the family of connected spanning subgraphs with m edges for a given m, $p-1 \leq m \leq q$.

$\mathcal{F}_2$: the family of connected spanning subgraphs having at least m and at most n edges, where m, n are given integers with $p-1 \leq m \leq n \leq q$.

$\mathcal{F}_3$: the family of connected spanning subgraphs.

$\mathcal{F}_4$: the family of spanning subgraphs having m edges, $0 \leq m \leq q$.

$\mathcal{F}_5$: the family of spanning subgraphs having at least m and at most n edges, where m, n are given integers with $0 \leq m \leq n \leq q$.

$\mathcal{F}_6$: the family of spanning subgraphs.

$\mathcal{F}_7$: the family of spanning subgraphs with maximum degree $\leq M$ for a given integer M.

$\mathcal{F}_8$: the family of spanning subgraphs with minimum degree $\geq N$ for a given integer N.

$\mathcal{F}_9$: the family of spanning subgraphs having a given hereditary property P_0.

$\mathcal{F}_{10}$: the family of spanning forests having a given hereditary property P_0.

$\mathcal{F}_{11}$: the family of connected spanning subgraphs having a given cohereditary property Q_0.

$\mathcal{F}_{12}$: the family of spanning subgraphs having a given cohereditary property Q_0.

$\mathcal{F}_{13}$: the family of subtrees with m vertices for a given $m, 1 \leq m \leq p$.

$\mathcal{F}_{14}$: the family of connected subgraphs with m vertices for a given $m, 1 \leq m \leq p$.

$\mathcal{F}_{15}$: the family of connected subgraphs with m edges for a given $m, 1 \leq m \leq q$.

$\mathcal{F}_{16}$: the family of connected subgraphs that have at least m and at most n edges for given integers $m, n, 1 \leq m \leq n \leq q$.

$\mathcal{F}_{17}$: the family of connected subgraphs.

$\mathcal{F}_{18}$: the family of subgraphs with m edges for a given $m, 1 \leq m \leq q$.

$\mathcal{F}_{19}$: the family of subgraphs.

$\mathcal{F}_{20}$: the family of subgraphs having at least m and at most n edges, $1 \le m \le n \le q$.

$\mathcal{F}_{21}$: the family of subgraphs having a given hereditary property P_0.

$\mathcal{F}_{22}$: the family of subgraphs having a given cohereditary property Q_0.

Note that if $m = p - 1$ then $\mathcal{F}_1$ is just the family of spanning trees of K. We have

Lemma 1 (i) *Each $\mathcal{F}_i$ is {EDA, SET}-connectable,* $1 \le i \le 12$.

(ii) *Each $\mathcal{F}_i$ is {EDA, VDA, SET, PVT, MET}-connectable,* $13 \le i \le 22$.

Proof The EDA- or SET-connectability of $\mathcal{F}_i, i = 1,4,5,7,8,9,$ was actually proved in [17]. Let G, H be two graphs in $\mathcal{F}_2$. By deleting some edges from G one at a time we get a connected spanning subgraph G_1 of G with m edges. Similarly we can get a connected spanning m-edge subgraph H_1 of H. Since $\mathcal{F}_1$ is SET-connectable there is an SET-sequence from G_1 to H_1 of which all terms are connected spanning m-edge subgraphs. So $\mathcal{F}_2$ is {EDA, SET}-connectable. Let $G, H \in \mathcal{F}_3$. By adding the edges in $E(K) - E(G)$ to G one by one we get K. Then by deleting the edges in $E(K) - E(H)$ one at a time we get H. So we have an EDA-sequence of $\mathcal{F}_3$ from G to H and hence $\mathcal{F}_3$ is EDA-connectable. The similar argument can be used to prove the EDA-connectablity of $\mathcal{F}_6$. Let G, H be two members of $\mathcal{F}_{10}$. By deleting the edges of G one at a time we get the empty graph $\overline{K}_p$ with vertex set $V(G)$. Since P_0 is hereditary all the intermediate graphs in this process belong to $\mathcal{F}_{10}$. Oppositely, we can get H from $\overline{K}_p$ by adding the edges of H, one at a time. Hence $\mathcal{F}_{10}$ is EDA-connectable. By a similar discussion we know $\mathcal{F}_{11}$ and $\mathcal{F}_{12}$ are also EDA-connectable.

Now let us prove that $\mathcal{F}_{13}$ is {SET, PVT}-connectable. Obviously, this is true for $m = 1, 2$. If $m = p$, then $\mathcal{F}_{13}$ is the family of spanning trees of K and hence is SET-connectable as mentioned earlier. In the following we suppose $3 \leq m \leq p-1$. For $G, H \in \mathcal{F}_{13}$, denote $\delta(G,H) = (2m-1)-(|V(G) \cap V(H)|+|E(G) \cap E(H)|)$ and $V(G,H) = V(G) \cap V(H)$. Then $0 \leq \delta(G,H) \leq 2m-1$. If there is an {SET, PVT}-sequence of $\mathcal{F}_{13}$ from G to H, we write briefly $G \sim H$. Now we prove $G \sim H$ by induction on $\delta(G,H)$.

If $\delta(G,H) = 0$, then $G = H$. If $\delta(G,H) = 1$, then $G \to H$ is an SET. Now suppose $\delta(G,H) \geq 2$ and the result holds for smaller value of δ. For the case when $\delta(G,H) = 2m-1$ (i.e. G, H are vertex-disjoint), we execute the following algorithm:

Step 1 Let T_0 be a minimal subtree of K containing both G and H. Let $G_0 = G$ and $i = 0$.

Step 2 If $i = |E(T_0) - E(H)|$, stop; otherwise, take a pendant vertex v of T_i such that the edge incident with v lies in $E(T_i) - E(H)$, and then take an edge $e \in E(T_i) - E(G_i)$ which is incident with a vertex of G_i.

Step 3 Let $G_{i+1} = G_i + e - v, T_{i+1} = T_i - v$. Replace i by $i+1$ and return to Step 2.

It is not difficult to see that the sequence $G_0, G_1, \ldots$ generated by this algorithm is a PVT-sequence from G to H. Hence $G \sim H$. If $\delta(G,H) < 2m-1$, then $V(G,H) \neq \emptyset$. We distinguish two cases.

CASE 1 There is a pendant vertex of G or a pendant vertex of H which is not in $V(G,H)$.

Let, say, v be a pendant vertex of G which is not in H. Noting that $V(H) - V(G) \neq \emptyset$, we can take an edge e of H which has exactly one end-vertex in $V(G,H)$. Set $G_1 = G - v + e$. Then $G \to G_1$ is a PVT and $\delta(G_1, H) < \delta(G,H)$. By the induction hypothesis, we have $G_1 \sim H$ and hence $G \sim H$.

CASE 2 All vertices in $V(G)-V(H)$ and $V(H)-V(G)$ are non-pendant vertices of G and H, respectively.

Then $|V(G) \cap V(H)| \geq 2$ since any nontrivial tree has at least two pendant vertices. If there exists $f \in (E(G) - E(H)) \cup (E(H) - E(G))$

whose end-vertices are both in $V(G,H)$, let, say $f \in E(G)-E(H)$. Then there is an edge e which is in the unique cycle of $G+f$ but not in H. Let $G_1 = G-e+f$. Then $G \to G_1$ is an SET and $\delta(G_1,H) < \delta(G,H)$. Therefore we have $G_1 \sim H$, implying $G \sim H$. If no such f exists, then for each $e = uv \in E(G)$ with $u \in V(G)-V(H), v \in V(G,H)$ we define $l(e)$ to be the number of vertices of the component of $G-e$ containing v. Let $l(G)$ be the minimum of $l(e)$ taken over all such e. We have

CLAIM There exists a sequence $G = G_0, G_1, \ldots, G_l$ such that each $G_i \to G_{i+1}$ is a PVT and $\delta(G_l,H) < \delta(G,H)$, where $l = l(G)$.

We prove this by induction on l. If $l = 1$, then there is an edge $e = uv$ with $u \in V(G)-V(H)$ and $v \in V(G,H)$ a pendant vertex of G. Let f be an edge of H with exactly one end-vertex in $V(G,H)$ and let $G_1 = G-e+f$. Then $G \to G_1$ is a PVT and $\delta(G_1,H) < \delta(G,H)$. Suppose $l \geq 2$ and $e = uv$ is an edge which attains the minimum in $l(G)$, where $u \in V(G)-V(H), v \in V(G,H)$. Let $w \neq v$ be a pendant vertex of the component of $G-e$ containing v and f an edge of H with only one end-vertex in $V(G,H)$. Define $G_1 = G-w+f$. Then $G \to G_1$ is a PVT and $\delta(G_1,H) = \delta(G,H), l(G_1) = l(G)-1$. By the induction hypothesis we can find $G_2, \ldots, G_l$ with each $G_i \to G_{i+1}$ a PVT and $\delta(G_l,H) < \delta(G,H)$. This proves the claim.

By the claim above we have $G \sim G_1 \sim \cdots \sim G_l$ and $\delta(G_l,H) < \delta(G,H)$. By the induction hypothesis we have $G_l \sim H$ and hence $G \sim H$. This completes the proof of the {SET, PVT}-connectability of $\mathcal{F}_{13}$. From this it is clear that $\mathcal{F}_{14}$ is {EDA, SET, PVT}-connectable.

Now let us prove that $\mathcal{F}_{15}$ is {SET, PVT, MET}-connectable. We write $G \sim H$ if there is an {SET, PVT, MET}-sequence of $\mathcal{F}_{15}$ from G to H. Then it suffices to show that $G \sim H$ for any $G,H \in \mathcal{F}_{15}$. If K is a tree, then both G and H are subtrees with $m+1$ edges and hence from the {SET, PVT}-connectability of $\mathcal{F}_{13}$ we know $G \sim H$. Suppose K is not a tree and T, J are spanning trees of G, H, respectively. Extend respectively T and J to spanning trees T_0 and J_0 of K. If $m = p-1$, then the edges of $T_0 - E(T)$ can be ordered as $f_1, \ldots, f_t, t = |E(T_0) - E(T)|$, such that if we define $G_0 = G, G_{i+1} = G_i - e_{i+1} + f_{i+1}, 0 \le i \le t-1$, then each $G_i \to G_{i+1}$ is an MET, where $e_1, \ldots, e_t$ are the edges of $G - E(T)$. So we get $G \sim T_0$. Similarly we have $H \sim J_0$. But $T_0 \sim J_0$ by the connectedness of the tree-graph, so we have $G \sim H$. If $m > p-1$, let E and F be subsets of $E(K)$ each containing $m-p+1$ edges such that $E(T_0) \cap E = E(J_0) \cap F = \emptyset$ (actually we can take $E \subseteq E(G) - E(T), F \subseteq E(H) - E(J)$). By the similar method as above we get $G \sim T_0 + E, H \sim J_0 + F$. But $T_0 + E$ and $J_0 + F$ are connected spanning m-edge subgraphs of G. So by the SET-connectability of $\mathcal{F}_1$ we know $T_0 + E \sim J_0 + F$, implying $G \sim H$. If $m < p-1$, let E and F be respectively $(p-1-m)$-subsets of $E(T_0)$ and $E(J_0)$ such that $T_0 - E, J_0 - F$ are trees with $m+1$ vertices. From the {SET, PVT}-connectability of $\mathcal{F}_{13}$ we have $T_0 - E \sim J_0 - F$. Hence $G \sim T_0 - E \sim J_0 - F \sim H$, completing the proof of the {SET, PVT, MET}-connectability of $\mathcal{F}_{15}$.

For two members G, H of $\mathcal{F}_{16}$, there exists an {EDA, VDA}-sequence from G (respectively H) to a connected subgraph G_0 (respectively H_0) of K with n edges. By the conclusion we have just proved for $\mathcal{F}_{15}$, there is an {SET, PVT, MET}-sequence of connected n-edge subgraphs from G_0 to H_0. So $\mathcal{F}_{16}$ is {EDA, VDA, SET, PVT, MET}-connectable. It is clear that $\mathcal{F}_{17}$ and $\mathcal{F}_{19}$ are {EDA, VDA}-connectable. For $G, H \in \mathcal{F}_{18}$, one can check that there exists an {SET, PVT, MET}-sequence of $\mathcal{F}_{18}$ from G (respectively H) to a connected m-edge subgraph G_0 (respectively H_0) of K. Since $G_0 \sim H_0$ as we have shown and $\mathcal{F}_{15} \subseteq \mathcal{F}_{18}$, we know that $\mathcal{F}_{18}$ is {SET, PVT, MET}-connectable. This, together with a similar discussion as for $\mathcal{F}_{16}$, implies that $\mathcal{F}_{20}$ is {EDA, VDA, SET, PVT, MET}-connectable. Finally, one can check that both $\mathcal{F}_{21}$ and $\mathcal{F}_{22}$ are {EDA, VDA}-connectable. This completes the proof of Lemma 1.

Combining Proposition 2 with Lemma 1 we get the following

Theorem 1 *Let K be a connected graph and $\mathcal{F}_i, 1 \le i \le 22$, be families of subgraphs of K defined as above. Then any positive or negative invariants interpolates over $\mathcal{F}_i, 1 \le i \le 12$, and any strongly positive or strongly negative invariant interpolates over $\mathcal{F}_i, 1 \le i \le 22$.*

Interpolation properties for $\mathcal{F}_1$ were studied in [2, 17, 30]. In the sequel we shall use Theorem 1 to prove a number of interpolation results for some conditional invariants, which can be divided into the following three categories.

3.1 Invariants concerning conditional colorings

A property P associated with graphs is said to be *induced hereditary* [20] if whenever G has property P then every vertex induced subgraph of G has property P also. The *edge-induced hereditary property* is defined in a similar way. Obviously, a hereditary property is both induced and edge-induced hereditary and any induced hereditary property is possessed by K_1. For a property P, a partition $(V_1, \dots, V_n)$ of $V(G)$ is called a *P-n-coloring* of G if each $G[V_i]$ has property P. A partition $(E_1, \dots, E_n)$ is an *edge P-n-coloring* if each $G[E_i]$ has P. In a similar way, we call a partition $(X_1, \dots, X_n)$ of $V(G) \cup E(G)$ a *total P-n-coloring* of G if $G[V_i] \cup G[E_i]$ possesses P for each i, where $V_i = X_i \cap V(G), E_i = X_i \cap E(G)$. If $(V_1, \dots, V_n)$ is a P-n-coloring of G, then we also say that G is P-n-colorable and that the vertices in V_i are colored with color i. Similar terminology will be used for edge P-n-colorability and total P-n-colorability. If P is induced hereditary, then the *P-chromatic number* of G [11], denoted by $\chi_P(G)$, is defined to be the minimum n such that G is P-n-colorable. If P is edge-induced hereditary, then the *edge P-chromatic number* $\chi'_P(G)$ [11] is the minimum n such that G is edge P-n-colorable. If P is hereditary, we define the *total P-chromatic number* $\chi^T_P(G)$ to be the minimum n such that G is total P-n-colorable. Let P_C be the property of being complete graphs and P_N be the property of being edgeless graphs. We have

Lemma 2 (i) *If P is induced hereditary and $P \neq P_C$, then χ_P is continous with respect to EDA, VDA, SET, PVT and MET;*

(ii) *If P is edge-induced hereditary and $P \neq \dot{P}_N$, then χ'_P is strongly positive;*

(iii) *If P is hereditary and $P \neq P_N$, then χ^T_P is strongly positive.*

Proof (i) We first note that $P \neq P_C$ implies that $\overline{K}_2$ has property P. Let $k = \chi_P(G)$ and $l = \chi_P(G-e)$ for an edge e of G. Then from a P-k-coloring of G we can get a P-$(k+1)$-coloring of $G-e$ by assigning a new color to one end-vertex of e. Similarly one can obtain a P-$(l+1)$-coloring of G. So we have

$$\chi_P(G) - 1 \leq \chi_P(G-e) \leq \chi_P(G) + 1. \tag{5}$$

It is not difficult to prove that for a pendant vertex v of G,

$$\chi_P(G) - 1 \leq \chi_P(G-v) \leq \chi_P(G). \tag{6}$$

So χ_P is continuous with respect to EDA, VDA and PVT. Now we prove the continuity of χ_P with respect to SET and MET.

Let $G \to H$ be a SET, where $H = G - e + f, e = uv$ and $f = xy$. Let $k = \chi_P(G), l = \chi_P(H)$ and $(V_1, \ldots, V_k)$ be a P-k-coloring of G. We distinguish two cases.

Case 1 u, v are colored with the same color.

Let, say, $u, v \in V_1$. If x and y are colored distinctly, then $(\{u\}, V_1 - \{u\}, V_2, \ldots, V_k)$ is a P-$(k+1)$-coloring of H. If they are colored with the same color, then $(\{u, x\}, V_1 - \{u, x\}, \ldots, V_k - \{u, x\})$ is a P-$(k+1)$-coloring of H since both K_2 and $\overline{K}_2$ have property P. So we get $l \leq k+1$.

Case 2 u, v are colored with different colors.

In such case $(\{x\}, V_1 - \{x\}, \ldots, V_k - \{x\})$ is a P-$(k+1)$-coloring of H and hence we have $l \leq k+1$ as well.

Since $H \to G$ is an SET, we have symetrically that $k \leq l+1$. Therefore, χ_P is continuous with respect to SET.

Let $G \to H$ be an MET with $H = G - w + f$, where w is a pendant vertex of G and $f = xy$ an edge of the complement graph of $G-w$. Since $G-w = H-f$ we get immediately from (5)-(6) that $\chi_P(H) \leq \chi_P(G)+1$. Let $l = \chi_P(H)$ and $(V_1, \ldots, V_l)$ be a P-l-coloring of H. If x, y are colored distinctly, then $(\{w\}, V_1, \ldots, V_l)$ is a P-$(l+1)$-coloring of G. Otherwise, let, say, $x, y \in V_1$. Since w is a pendant vertex of G we may suppose without loss of generality that w is not adjacent to x in G. Since $\overline{K}_2$ has property P, $(\{w,x\}, V_1 - \{x\}, \ldots, V_l - \{x\})$ is a P-$(l+1)$-coloring of G. In both cases, we get $\chi_P(G) \leq \chi_P(H) + 1$ and hence χ_P is continuous with respect to MET.

(ii) Clearly we have $\chi'_P(G-e) \leq \chi'_P(G)$, $e \in E(G)$. On the other hand, $P \neq P_N$ implies that K_2 has property P and hence a P-$\chi'_P(G-e)$-coloring of $G-e$ can be extended to a P-$(\chi'_P(G-e)+1)$-coloring of G by assigning a new color to e. Thus, χ'_P is positive. For any pendant vertex w of G, we have $\chi'_P(G-w) = \chi'_P(G-e)$, where e is the edge incident with v. Hence χ'_P is in fact strongly positive.

(iii) Obviously, we have $\chi^T_P(G-e) \leq \chi^T_P(G)$. Suppose for an edge $e = uv \in E(G)$ that $l = \chi^T_P(G-e)$ and $(X_1, \ldots, X_l)$ is a total P-l-coloring of $G-e$. Let $Y_i = X_i - \{u\}$ for $1 \leq i \leq l$ and $Y_{l+1} = \{u, e\}$. Then $(Y_1, \ldots, Y_{l+1})$ is a total P-$(l+1)$- coloring of G. Hence $\chi^T_P(G) - 1 \leq \chi^T_P(G-e) \leq \chi^T_P(G)$. In a similar way, we can prove $\chi^T_P(G) - 1 \leq \chi^T_P(G-w) \leq \chi^T_P(G)$ for any pendant vertex w of G. So χ^T_P is strongly positive and the proof is completed.

Note 1 If $P = P_C$, one can prove that χ_P is continuous with respect to EDA, VDA, SET, PVT. But it is not continuous with respect to MET. Furthermore, χ_P is strongly positive if P is hereditary. But it is neither positive nor negative if P is induced-hereditary but not hereditary.

A property Q is said to be *coheriditary* [8] if whenever $G \subseteq H$ and G has property Q, then H has property Q as well. Q is *induced cohereditary* if whenever G is an induced subgraph of H and G has property Q, then H has property Q. The *edge-induced cohereditary property* can be defined similarly. For induced cohereditary Q, we define

$$\chi_Q(G) = \begin{cases} \max\{n\colon G \text{ is } Q\text{-}n\text{-colorable}\}, & \text{if } G \text{ has property } Q \\ 0, & \text{otherwise.} \end{cases}$$

For edge-induced cohereditary property Q, we define

$$\chi'_Q(G) = \begin{cases} \max\{n\colon G \text{ is edge } Q\text{-}n\text{-colorable}\}, & \text{if } G \text{ has property } Q \\ 0, & \text{otherwise.} \end{cases}$$

We call $\chi_Q(G)$ and $\chi'_Q(G)$, respectively, the *Q-cochromatic number* and the *edge Q-cochromatic number* of G. Similarly, we can define the *total Q-cochromatic number* $\chi_Q(G)$. We have

Lemma 3 (i) *If Q is cohereditary, then χ_Q is strongly positive and χ_Q^T is positive.*

(ii) *If Q is edge-induced cohereditary, then χ'_Q is strongly positive.*

Proof Let $e = uv \in E(G), k = \chi_Q(G)$ and $l = \chi_Q(G - e)$. Then evidently we have $l \le k$. In proving $k - 1 \le l$, we can assume $k > 1$. Let $(V_1, \dots, V_k)$ be a Q-k-coloring of G. If u, v are colored distinctly, then it is also a Q-k-coloring of $G - e$. If u, v are colored the same, let, say $u, v \in V_1$, then $(V_1 \cup V_2, V_3, \dots, V_k)$ is a Q-$(k-1)$-coloring of $G - e$. In either case we get $k - 1 \le l$. Similarly one can prove $\chi_Q(G) - 1 \le \chi_Q(G-v) \le \chi_Q(G)$ for any pendant vertex v of G. Hence χ_Q is strongly positive. Clearly we have $\chi_Q^T(G - e) \le \chi_Q^T(G)$. On the other hand, let $k = \chi_Q^T(G)$ and $(X_1, \dots, X_k)$ be a total Q-k-coloring of G. Suppose for example $e \in X_1$, then $(X_1 \cup X_2, X_3, \dots, X_k)$ is a total Q-$(k-1)$-coloring of $G - e$, implying that χ_Q^T is positive. In a similar way we can prove (ii).

Note 2 It can be proved that χ_Q is continuous with respect to EDA, VDA, PVT, MET and AEE (adjacent edge-exchange) if Q is induced cohereditary. But it is not continuous with respect to SET. For example, if Q is the property of being graphs containing either C_3 (the triangle) or P_4 (the path with four vertices) as an induced subgraph, then Q is induced cohereditary. If G is the vertex disjoint union of C_3 and P_4 and H is the vertex disjoint union of P_3 and C_4, then $G \to H$ is an SET but $\chi_Q(G) = 2, \chi_Q(H) = 0$.

Note 3 χ_Q^T is not strongly positive. For example, if Q is the property of being graphs containing P_4 as subgraph and G is the graph obtained from K_4 by deleting two adjacent edges, then $\chi_Q^T(G) = 2, \chi_Q^T(G-v) = 0$, where v is the unique pendant vertex of G.

Note that in the proof of Lemma 1 the MET's are used only for $\mathcal{F}_i, i = 15, 16, 18, 20$. From Lemmas 2 and 3, Proposition 2, Theorem 1 and Note 2 we get

Theorem 2 (i) *If $P \neq P_C$ is induced hereditary, then χ_P interpolates over $\mathcal{F}_i, 1 \leq i \leq 22$;*

(ii) *If $P \neq P_N$ is edge-induced hereditary, then χ'_P interpolates over $\mathcal{F}_i, 1 \leq i \leq 22$;*

(iii) *If $P \neq P_N$ is hereditary, then χ_P^T interpolates over $\mathcal{F}_i, 1 \leq i \leq 22$;*

(iv) *If Q is induced cohereditary, then χ_Q interpolates over $\mathcal{F}_i, 1 \leq i \leq 22, i \neq 15, 16, 18, 20$; if Q is cohereditary, then χ_Q interpolates over $\mathcal{F}_i, 1 \leq i \leq 22$;*

(v) *If Q is edge-induced cohereditary, then χ_Q interpolates over $\mathcal{F}_i, 1 \leq i \leq 22$.*

(vi) *If Q is cohereditary, then χ_Q^T interpolates over $\mathcal{F}_i, 1 \leq i \leq 12$.*

As noted in the literature, a large number of invariants for graphs can be expressed as χ_P, χ'_P, χ_Q or χ'_Q. Here we list some of them in

Tables 1 and 2. From Theorem 2 we have

Corollary 1 *All the invariants listed in Tables 1 and 2 interpolate over* $\mathcal{F}_i, 1 \leq i \leq 22$.

P	χ_P	χ'_P
Planarity		Thickness [12]
Acyclicity	Vertex arboricity [6]	Arboricity [11, 12]
With at most one cycle		Unicyclicity [12]
Without odd cycles		Biparticity [12]
Being linear forests	Vertex linear arboricity [23]	Linear arboricity [12]
Without paths of lenth n	n-Chromatic number [5]	
With chromatic number $\leq n$	Chromatic partition number [25]	
Being 1-regular graphs		Edge chromatic number
Being edgeless graphs	Chromatic number	
Being complete or edgeless graphs	Cochromatic number [21]	
Being disjoint union of complete graphs	Subchromatic number [1]	
Being complete r-partite graphs for any r	Partite chromatic number [9]	

Table 1

Q	χ_Q	χ'_Q
Nonplanarity		Coarseness [12]
Non-acyclicity	Vertex cycle multiplicity [6]	Anarboricity [12]
Being graphs other than paths		apathy [12]

Table 2

3.2 Generalized independence and covering numbers

A set X of vertices of G is called a P-set of G if $G[X]$ possesses property P. X is a P^*-set if it has non-empty intersection with every non-P-set of G. Let $\beta_P(G)$ and $\alpha_P(G)$ be respectively the maximum cardinality of a P-set and the minimum cardinality of a P^*-set of G. It was proved [20] that $\alpha_P(G) + \beta_P(G) = |V(G)|$ if P is hereditary. This equality becomes the well-known Gallai equality $\alpha(G)+\beta(G) = |V(G)|$ if $P = P_N$ because in such case α_P and β_P are just the ordinary vertex covering number and independence number, respectively.

Lemma 4 *If P is hereditary, then α_P is strongly positive whilst β_P is* *negative. Moreover, for any pendant vertex v of a graph G, $\beta_P(G) - 1 \leq \beta_P(G - v) \leq \beta_P(G)$.*

Proof Clearly we have $\beta_P(G) \leq \beta_P(G - e)$ since P is hereditary. On the other hand, let X be a maximum P-set of $G - e$ and v an end-vertex of e. Then $X - \{v\}$ is a P-set of G and hence $\beta_P(G-e) \leq \beta_P(G)+1$. For a pendant vertex v of G, any P-set of $G - v$ is a P-set of G and the set resulted from a P-set of G by possibly deleting v is a P-set of $G - v$. So $\beta_P(G) - 1 \leq \beta_P(G - v) \leq \beta_P(G)$. This, together with the negativeness of β_P we have just proved and the equality $\alpha_P(G) + \beta_P(G) = |V(G)|$, implies the strongly positiveness of α_P.

A property P is said to be *induced edge hereditary* (or induced line-hereditary as used in [20]) if (i) it is edge-induced hereditary, (ii) the edgeless graphs have P, and (iii) any graph obtained from a graph having property P by adding isolated vertices has property P as well. For such P, we call $X \subseteq E(G)$ a P-edge set if $G[X]$ has property P. $X \subseteq E(G)$ is a P^*-edge set if it intersects every non-P-edge set of G. We define $\alpha'_P(G)$ and $\beta'_P(G)$ to be respectively the maximum number of edges in a P-edge set and the minimum number of edges in a P^*-edge set of G. It was implied in [20] that $\alpha'_P(G) + \beta'_P(G) = |E(G)|$. From this we can prove

Lemma 5 *Both α'_P and β'_P are strongly positive.*

Let $\beta'_-(G)$ and $\alpha'_+(G)$ be, respectively, the minimum cardinality of a maximal edge independent set and the maximum cardinality of a minimal edge covering of G [7]. Then $\alpha'_+(G) + \beta'_-(G) = |E(G)|$ [7]. Let $\beta^-(G)$ be the *matchability number* [7] defined to be the minimum cardinality of a maximal set $X \subseteq V(G)$ such that there exist $|X|$ edges each incident with exactly one vertex in X. It is not difficult to prove

Lemma 6 *$\alpha'_+, \beta'_-, \beta^-$ are all strongly positive.*

Note that from Lemma 4 β_P is continuous with respect to EDA, VDA, SET and PVT. So we have

Theorem 3 (i) *If P is hereditary, then α_P interpolates over all $\mathcal{F}_i, 1 \leq i \leq 22$, and β_P interpolates over $\mathcal{F}_i, 1 \leq i \leq 22, i \neq 15, 16, 18, 20$.*

(ii) *If P is induced edge hereditary, then both α'_P and β'_P interpolate over $\mathcal{F}_i, 1 \leq i \leq 22$.*

(iii) *$\alpha'_+, \beta'_-, \beta^-$ interpolate over $\mathcal{F}_i, 1 \leq i \leq 22$.*

If $P = P_N$, then P is hereditary and α_P, β_P are just the ordinary vertex covering number α and independence number β, respectively. If P is the property of being graphs with maximum degree ≤ 1 then it is induced edge hereditary, and α'_P, β'_P are respectively the ordinary edge covering number α' and edge independence number β'. It was proved ([17], also partly contained in [28]) that α, β, α' and β' all interpolate over $\mathcal{F}_i$ for $i = 1, 4, 5, 7, 8, 9$. From Theorem 3 we know that the same result is true for α_P, β_P, α'_P and β'_P provided that P is hereditary. Also, Theorem 3 implies in particular that α, α', β' interpolates over all $\mathcal{F}_i$ and β interpolates over all these families except $\mathcal{F}_i, i = 15, 16, 18, 20$.

3.3 Conditional connectivities and subgraph invariants

The *P-connectivity* $\kappa_P(G)$ [13] is the minimum cardinality of a set $X \subseteq V(G)$ such that $G-X$ is disconnected with each component has property P. The *P-edge connectivity* [13] $\lambda_P(G)$ is defined similarly. For certain property P, $\kappa_P(G)$ or $\lambda_P(G)$ may not exist for some graph G [13]. We observe that if P is hereditary, then these two invariants are well-defined for all graphs.

Lemma 7 *If P is hereditary, then both κ_P and λ_P are positive.*

Proof We call $X \subseteq V(G)$ a P-seperating set of G if $G - X$ is disconnected with all components have property P. Obviously, we have

$\kappa_P(G-e) \leq \kappa_P(G)$ for each edge e of G. Suppose X is a minimum P-seperating set of $G-e$. If at least one of the two end-vertices of e is in X, then X is also a P-seperating set of G; otherwise one can check that X together with an end-vertex of e is a P-seperating set of G. In both cases we get $\kappa_P(G) - 1 \leq \kappa_P(G-e)$ and hence κ_P is positive. In a similar way, we can prove that λ_P is positive.

Neither κ_P nor λ_P is strongly positive. In fact they are not continuous with respect to VDA. For example, if P is the planarity and if G is a graph which contains a unique pendant vertex v and the deletion of v from G results in a planar graph with both connectivity and edge connectivity three, then $\kappa_P(G) = \lambda_P(G) = 1$ but $\kappa_P(G-v) \geq 3, \lambda_P(G-v) \geq 3$. Nevertheless, if we define, as suggested in [13], $\bar{\kappa}_P(G)$ (respectively $\bar{\lambda}_P(G)$) to be the minimum cardinality of a set $X \subseteq V(G)$ (respectively $X \subseteq E(G)$) such that each component of $G-X$ possesses P, then we have

Lemma 8 *Both $\bar{\kappa}_P$ and $\bar{\lambda}_P$ are strongly positive if P is hereditary.*

If P is the universal property (that is, the property possessed by all graphs), then $\kappa_P(G)$ is the ordinary connectivity $\kappa(G)$ (provided that G is not the complete graph) and $\lambda_P(G)$ is the ordinary edge connectivity $\lambda(G)$. So Lemma 8 implies that both κ and λ are positive [28]. Let $\beta_\kappa(G)$ be the maximum number of vertices in a set $X \subseteq V(G)$ such that $G[X]$ is disconnected or K_1. Let $\beta_\lambda(G)$ be the maximum number of edges in a set $X \subseteq E(G)$ such that $G[X]$ is disconnected. It was proved [19, 20] that $\kappa(G) + \beta_\kappa(G) = |V(G)|, \lambda(G) + \beta_\lambda(G) = |E(G)|$. From this and the positiveness of both κ and λ we get

Lemma 9 *β_κ is negative whilst β_λ is positive.*

The following subgraph invariants were introduced in [24]:

$$\hat{\kappa}(G) = \max\{\kappa(H) : H \subseteq G\}$$

$$\hat{\lambda}(G) = \max\{\lambda(H) : H \subseteq G\}$$
$$\hat{\delta}(G) = \max\{\delta(H) : H \subseteq G\}.$$

For background and results regarding these invariants the reader can consult [24] (a well-known inequality is $\chi(G) \leq 1 + \hat{\delta}(G)$). We can prove

Lemma 10 *$\hat{\kappa}, \hat{\lambda}$ and $\hat{\delta}$ are strongly positive.*

Combining Lemmas 7, 8, 9 and 10 with Theorem 1 we have

Theorem 4 *Suppose P is hereditary. Then*
(i) *$\kappa_P, \lambda_P, \kappa, \lambda, \beta_\kappa$ and β_λ interpolate over $\mathcal{F}_i, 1 \leq i \leq 12$.*
(ii) *$\bar{\kappa}_P, \bar{\lambda}_P, \hat{\kappa}, \hat{\lambda}$ and $\hat{\delta}$ interpolate over $\mathcal{F}_i, 1 \leq i \leq 22$.*

Finally, we point out that the *minimum degree* is positive and hence interpolates over $\mathcal{F}_i, 1 \leq i \leq 12$, and that the *maximum degree* is strongly positive and so interpolates over all $\mathcal{F}_i, 1 \leq i \leq 22$ (the interpolation of maximum degree with respect to $\mathcal{F}_i, i = 1, 4, 5, 7, 8, 9$ was known in [17] and partly known in [28]).

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