The 1-rotational (52,4,1)-RBIBD's

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Abstract. Up to isomorphisms, there are exactly 22 1-rotational resolved (52,4,1)-BIBD's.

By (v,k,λ)-RBIBD, here we mean a *resolved* (v,k,λ)-BIBD, i.e. a *resolvable* (v,k,λ)-BIBD together with a specific resolution of it. We say that a (v,k,λ)-RBIBD is *1-rotational* if admits an automorphism (i.e., a bijection on its points leaving invariant its resolution) consisting of a single cycle of length v-1.
Concerning the enumeration of 1-rotational (v,k,λ)-RBIBD's for a given admissible triple (v,k,λ), we recall that Jimbo and Vanstone [5] enumerated the 1-rotational (27,3,1)-RBIBD's (they are 47), while Buratti and Zuanni [4] enumerated the 1-rotational (33,3,1)-RBIBD's (they are 436).
The aim of this paper is to enumerate the 1-rotational (52,4,1)-RBIBD's. It is already known that there exists at least one 1-rotational (52,4,1)-RBIBD since Moore [6] proved the existence of a 1-rotational (3q+1,3,1)-RBIBD for any prime power $q \equiv 1 \pmod 4$.
By [3], a 1-rotational (52,4,1)-RBIBD is equivalent to a 4-set $\mathcal{F} = \{A_1, A_2, A_3, A_4\}$ of 4-subsets of $\mathbb{Z}_{51}$ satisfying the following conditions:
(1) any element of $\mathbb{Z}_{51} - \{0,17,34\}$ is representable as a difference of two elements of a same A_i.
(2) $A_1 \cup A_2 \cup A_3 \cup A_4 = \mathbb{Z}_{17} - \{0\} \pmod{17}$.
condition (1) means that $\mathcal{F}$ is a (51,3,4,1) difference family (DF, see [1]) while conditions (2) means that $\mathcal{F}$ is resolvable (see [3]).
Explicitely, the (52,4,1)-RBIBD associated with a given (51,3,4,1) resolvable DF $\{A_1, A_2, A_3, A_4\}$, is the one with point-set $\mathbb{Z}_{51} \cup \{\infty\}$ and parallel classes obtainable developing (mod 51) the *starter parallel class* given by

$$\{\{0,17,34,\infty\}\} \cup \{A_i+17j \mid 1 \le i \le 4;\ 0 \le j < 3\}.$$

Let $\mathcal{F} = \{A_1, A_2, A_3, A_4\}$ be a resolvable (51,3,4,1) difference family and, for $i = 1, \ldots, 4$, set $A_i = \{a_{i1}, a_{i2}, a_{i3}, a_{i4}\}$. For each pair $(i,j) \in \{1,2,3,4\}^2$ let s_{ij} be

the reduction of a_{ij} modulo 3 and note that for any element z of $\mathbb{Z}_3$ the equation $s_{ij}-s_{ih} = z$ must have exactly 16 solution triples (i,j,h) with $j \neq h$.
Using the terminology of [2], this means that the multiset S of 4-multisets $S = ((s_{11},s_{12},s_{13},s_{14}), (s_{21},s_{22},s_{23},s_{24}), (s_{31},s_{32},s_{33},s_{34}), (s_{41},s_{42},s_{43},s_{44}))$ is a (3,4,16) *strong difference family* (briefly, SDF).
We say that S is the underlying SDF of F.
Let Φ be the set of resolvable (51,3,4,1)-DF's and consider the following equivalence relation ρ defined on Φ.
For given $F = \{A_1, A_2, A_3, A_4\}$ and $F' = \{A'_1, A'_2, A'_3, A'_4\}$ of Φ, we set
$F \rho F' \Leftrightarrow \exists\, m \in \bigcup \mathbb{Z}_{51}$ and $\exists\, (t_1, t_2\, t_3, t_4) \in \{0,17,34\}^4$
such that, up to the order,
$\{mA_1, mA_2, mA_3, mA_4\} = \{A'_1+t_1, A'_2+t_2, A'_3+t_3, A'_4+t_4\} \pmod{51}$.

It is clear that the RBIBD's associated with ρ-equivalent resolvable DF's are isomorphic.
Now, let Σ be the set of all the (3,4,16)-SDF's and consider the following equivalence relation σ defined on Σ:
For given $S = (S_1,S_2,S_3,S_4)$ and $S' = (S'_1,S'_2,S'_3,S'_4)$ of Σ set:
$S \sigma S' \Leftrightarrow \exists\, m \in \{1,-1\}$ and $\exists\, (t_1, t_2\, t_3, t_4) \in \mathbb{Z}_3^4$
such that, up to the order,
$\{mS_1, mS_2, mS_3, mS_4\} = \{S'_1+t_1, S'_2+t_2, S'_3+t_3, S'_4+t_4\} \pmod{3}$.

We point out that if S and S' are σ-equivalent (3,4,16)-SDF's, then each (51,3,4,1)-DF admitting S as underlying SDF is ρ-equivalent to some (51,3,4,1)-DF whose underlying SDF is S'. Also, DF's with inequivalent underlying SDF's are not ρ-equivalent.
So, in order to enumerate the 1-rotational (52,4,1)-RBIBDs it is helpful to proceed as follows:
1st step: determine a complete set $\{S_1, ..., S_m\}$ of representatives for the σ-equivalence classes of (3,4,16)-SDF's.
2nd step: determine for each i = 1, ..., m a complete set $\{F_{i1}, ..., F_{im_i}\}$ of representatives for the ρ-equivalence classes of resolvable (51,3,4,1)-DF's admitting S_i as underlying SDF.
3rd step: check whether there exists some isomorphism among the RBIBD's associated with the F_{ij}'s.

1st step.

By hand, we have easily checked that a complete set of representatives for the σ-equivalence classes of (3,4,16)-SDF's consists of the following four SDF's:

S_1 = ((0,0,0,1) (0,0,1,1), (0,0,1,1), (0,0,1,2))
S_2 = ((0,0,0,1), (0,0,0,1), (0,0,1,2), (0,0,1,2))
S_3 = ((0,0,1,1), (0,0,1,1), (0,0,1,1), (0,0,1,1))
S_4 = ((0,0,0,1), (0,0,0,2), (0,0,1,2), (0,0,1,2)).

2nd step. This has been done by computer.

A complete set of representatives for the ρ-equivalence classes of (51,3,4,1) resolvable DF's admitting S_1 as underlying SDF:

F_1 = {{36,24,33,1}, {42,9,40,13}, {27,48,37,22}, {21,15,28,29}}
F_2 = {{36,9,15,1}, {6,48,7,46}, {42,27,4,22}, {30,33,37,11}}
F_3 = {{36,12,33,1}, {24,30,22,31}, {9,27,4,40}, {3,42,28,32}}
F_4 = {{3,6,48,1}, {36,21,43,13}, {27,45,7,46}, {39,15,25,50}}
F_5 = {{39,6,42,1}, {36,15,37,43}, {24,48,28,16}, {21,12,10,47}}.

A complete set of representatives for the ρ-equivalence classes of (51,3,4,1) resolvable DF's admitting S_2 as underlying SDF:

F_6 = {{36,21,42,1}, {3,27,45,40}, {9,48,16,5}, {12,15,13,41}}
F_7 = {{36,9,15,1}, {42,27,30,31}, {3,21,28,23}, {24,33,46,5}}
F_8 = {{3,39,45,1}, {36,6,24,16}, {21,48,25,47}, {9,12,49,44}}
F_9 = {{3,42,45,1}, {36,9,30,22}, {21,6,46,41}, {48,15,16,44}}
F_{10} = {{39,42,33,1}, {3,30,15,19}, {24,45,31,23}, {9,27,4,29}}
F_{11} = {{39,48,15,1}, {3,24,9,19}, {6,45,46,38}, {30,33,10,8}}.

A complete set of representatives for the ρ-equivalence classes of (51,3,4,1) resolvable DF's admitting S_3 as underlying SDF:

F_{12} = {{18,33,4,13}, {36,15,25,43}, {3,48,22,46}, {6,45,7,10}}
F_{13} = {{18,33,4,13}, {36,15,25,43}, {3,48,22,46}, {24,27,40,28}}
F_{14} = {{18,33,22,46}, {36,15,7,10}, {3,48,4,13}, {6,45,25,43}}
F_{15} = {{18,33,22,46}, {36,15,7,10}, {3,48,4,13}, {42,9,40,28}}
F_{16} = {{18,33,22,46}, {36,15,7,10}, {21,30,37,31}, {42,9,40,28}}
F_{17} = {{18,33,22,46}, {21,30,37,31}, {6,45,25,43}, {24,27,19,49}}

A complete set of representatives for the ρ-equivalence classes of (51,3,4,1) resolvable DF's admitting S_4 as underlying SDF:

F_{18} = {{36,21,12,1}, {27,45,15,41},{3,48,16,26}, {39,42,40,47}}
F_{19} = {{36,6,9,1}, {24,42,30,44}, {3,45,4,32}, {48,33,22,29}}
F_{20} = {{36,42,15,1}, {3,6,45,5}, {9,27,4,47}, {12,48,16,41}}
F_{21} = {{36,42,33,1}, {3,30,15,26}, {21,39,46,41}, {6,27,28,14}}
$F22$ = {{21,6,48,1}, {9,12,30,41}, {36,42,49,50}, {39,27,37,11}}.

3rd step.

Let $F_1, F_2, ..., F_{22}$ be the (51,3,4,1) resolvable DF's listed in the 2nd step and let $D_1, D_2, ..., D_{22}$ be their respective associated RBIBD's. Given any two distinct blocks A, B of D_i through 0, let Γ_i (A,B) be the isomorphism class of the graph whose vertices are the 9 blocks of D_i joining a point of A-{0} with a point of B-{0}, and in which two vertices are adjacent if and only if the corresponding blocks have a common point out of A ∪ B.
It is clear that the multiset

$$M_i = (\Gamma_i\ (A,B) \mid A, B \text{ distinct blocks of } D_i \text{ through } 0)$$

is an isomorphism invariant of D_i. Then, since by computer we have found that $M_i \neq M_j$ each time that $i \neq j$, we conclude that, up to isomorphisms, there are exactly 22 1-rotational (52,4,1)-RBIBD's.

References

[1] M. Buratti, Recursive constructions for difference matrices and relative difference families, *J. of Combin. Designs* 6 (1998), 165-182.
[2] M. Buratti, Old and new designs via difference multisets and strong difference families, *J. of Combin. Designs*, to appear.
[3] M. Buratti and F. Zuanni, G-invariantly resolvable Steiner 2-designs which are 1-rotational over G, *Bull. Belg. Math. Soc.* 5 (1998), 221-235.
[4] M. Buratti and F. Zuanni, The 1-rotational Kirkman triple systems of order 33, *J. Statist. Plann. Inf.*, to appear.
[5] M. Jimbo and S. Vanstone, Recursive constructions for resolvable and doubly resolvable 1-rotational Steiner 2-designs, *Utilitas Math.* 26 (1984), 45-61.
[6] E.H. Moore, Tactical Memoranda I-III, *Amer. J. Math.* 18 (1896), 264-303.