On intersection numbers, total clique covers and regular graphs

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ABSTRACT. Let $\mathcal{A} = \{A_1, .., A_l\}$ be a partition of $[n]$ and $\mathcal{F} = \{S_1, .., S_m\}$ be a intersecting family of distinct nonempty subsets of $[n]$ such that $\mathcal{A}$ and $\mathcal{F}$ are pairwise intersecting families. Then $|\mathcal{F}| \leq \frac{1}{2}\prod_{i=1}^{l}(2^{|A_i|}-2) + \sum_{S \subsetneqq [l]}(\prod_{i \in S}(2^{|A_i|}-2))$. From this result and some properties of intersection graphs on multifamilies, we determine the intersection numbers of 3, 4, 5-regular graphs and some special graphs.

1 Introduction

In this paper, we consider finite undirected simple graphs. For a vertex set C, $N(C)$ denotes the set of vertices which are adjacent to vertices of C and do not belong to C and is called the *neighborhood* of C, and denote $N(\{u\})$ by $N(u)$. We denote the set $\{1, 2, .., n\}$ by $[n]$. Let $\mathcal{F} = \{S_1, S_2, .., S_p\}$ be a family of distinct nonempty subsets of a set X. Then $S(\mathcal{F})$ denotes the union of sets in $\mathcal{F}$. The *intersection graph* of $\mathcal{F}$ is denoted by $\Omega(\mathcal{F})$ and defined by $V(\Omega(\mathcal{F})) = \mathcal{F}$, with S_i and S_j adjacent whenever $i \neq j$ and $S_i \cap S_j \neq \emptyset$. The *intersection number* $\omega(G)$ of a given graph G is the minimum cardinality of a set $S(\mathcal{F})$ such that G is an intersection graph on $\mathcal{F}$. We have some results on the relations between graphs and families ([1],[4],[5],[8],[11]). In these papers, we consider the minimal families whose intersection graphs are special graphs. For example, the families on complete graphs are intersecting families, the families on complete bipartite graphs are Latin squares, and so on ([3],[9]). These results depend on the quality of the graphs. However these results do not contribute to determining the intersection numbers of other graphs, for example, triangle-free graphs, K_4-free graphs, regular graphs and so on. Therefore we consider

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another class of graphs in this paper. That is, graphs $K_m + N_l$ correspond to families $\{S_i \subseteq [n](i = 1,..,m) | S_i \cap S_j \neq \emptyset, S_i \cap A_j \neq \emptyset\} \cup \{A_i \subseteq [n](i = 1,..,l) | A_i \cap A_j = \emptyset, A_1 \cup .. \cup A_l = [n]\}$. This family is the key family of this paper. The intersection numbers of $K_m + N_l$ are used for estimating the intersection numbers of many graphs. That is, we classify graphs by the configuration of subgraphs isomorphic to $K_m + N_l$ to determine the intersection numbers. And we obtain the intersection numbers of K_4-free graphs, 3,4,5-regular graphs and so on. Terminology and notation of combinatorics and graph theory can be found in [2], [5] and [10].

2 Intersection numbers of multifamilies

First we consider multifamilies of nonempty subsets of a set X, namely its elements need not be distinct. As is the case with families, we can define *intersection graphs with respect to multifamilies* and *intersection number with respect to multifamilies* $\omega_m(G)$. For a graph G, $\mathcal{Q} = \{Q_1, .., Q_n\}$ is a *total clique cover* (*tcc*) if and only if every Q_i is a complete subgraph of G, $\bigcup_{i=1}^n V(Q_i) = V(G)$ and $\bigcup_{i=1}^n E(Q_i) = E(G)$.

Theorem 2.1 *For a graph G, $\omega_m(G) = min_{\mathcal{Q}:tcc\ of\ G}|\mathcal{Q}|$.*

Proof: Let $\mathcal{F} = \{S_1, S_2, .., S_p\}$ be a multifamily of nonempty subsets of a set X such that $\Omega(\mathcal{F}) \cong G$, $S(\mathcal{F}) = \{a_1, .., a_n\}$ and $n = \omega_m(G)$. Let $A(a_i) = \{S_j | a_i \in S_j\}$, $Q(a_i) = \Omega(A(a_i))$ and $\mathcal{Q}(\mathcal{F}) = \{Q(a_1), .., Q(a_n)\}$. Then $\mathcal{Q}(\mathcal{F})$ is a total clique cover of G and $\omega_m(G) = |\mathcal{Q}(\mathcal{F})| \geq min_{\mathcal{Q}:tcc\ of\ G}|\mathcal{Q}|$.

Conversely, let $\mathcal{Q} = \{Q_1, .., Q_n\}$ be a total clique cover of G such that $|\mathcal{Q}| = min_{\mathcal{R}:tcc\ of\ G}|\mathcal{R}|$. We define that $S(v) = \{Q_i | v \in V(Q_i)\}$ for each $v \in V(G)$ and $\mathcal{F}(\mathcal{Q}) = \{S(v) | v \in V(G)\}$. Then $\Omega(\mathcal{F}(\mathcal{Q})) \cong G$ and $\omega_m(G) = min_{\Omega(\mathcal{F}) \cong G}|S(\mathcal{F})| \leq |\bigcup_{S(v) \in \mathcal{F}(\mathcal{Q})} S(v)| = |\mathcal{Q}| = min_{\mathcal{R}:tcc\ of\ G}|\mathcal{R}|$. □

This result is a well-known duality between a representation of a graph by a family of subsets and a clique cover ([6], [7]). We will routinely identify the elements a_i of $S(\mathcal{F})$ with cliques $Q(a_i)$ of G as was done in this proof. By Theorem 2.1, we are now in a position to give some examples of intersection numbers.

Example 2.2 *Let G be a 2-cell embedable graph whose regions are triangles. $\omega_m(G) \leq \omega(G) \leq \alpha_0(G^*) = |V(G^*)| - \beta_0(G^*) \leq (1 - \frac{1}{\chi(G^*)}) \times |V(G^*)|$, where G^* is the dual graph of G, $\alpha_0(G^*)$ is the vertex covering number of G^*, $\beta_0(G^*)$ is the vertex independence number of G^* and $\chi(G^*)$ is the chromatic number of G^*.* □

Example 2.3 *For a connected maximal plane graph G except K_4, $\beta_0(G^*) \geq \frac{|V(G^*)|}{3}$ and $|V(G^*)| = 2 \times |V(G)| - 4$. Hence by Example 2.2, $\omega_m(G) \leq$*

$\omega(G) \leq \frac{4}{3} \times (|V(G)| - 2)$. *If a maximal plane graph G is an Euler graph, then G^* is* 2-colorable, *and Example 2.2 then implies* $\omega_m(G) \leq \omega(G) \leq |V(G)| - 2$. □

The following result is a well-known result on structural matters which lead to differences between $\omega(G)$ and $\omega_m(G)$.

Proposition 2.4 **([6])** *Let G be a connected graph and $\mathcal{F} = \{S(v)|v \in V(G)\}$ be a multifamily such that $\Omega(\mathcal{F}) \cong G$, $S(\mathcal{F}) = \{a_1, .., a_n\}$ and $n = \omega_m(G)$. For vertices u and v with $S(u) = S(v)$, u and v are adjacent and $N(u) - \{v\} = N(v) - \{u\}$.* □

In general, the converse of Proposition 2.4 is false. However adding a condition to the consequence of the proposition, its converse becomes true. The next results give a class of graphs satisfying the condition.

Proposition 2.5 *Let G be a connected graph and $\mathcal{F} = \{S(v)|v \in V(G)\}$ such that $\Omega(\mathcal{F}) \cong G$, $S(\mathcal{F}) = \{a_1, .., a_n\}$ and $n = \omega_m(G)$. For vertices u and v such that $N(u) - \{v\}$ and $N(v) - \{u\}$ are both independent sets, u and v are adjacent and $N(u) - \{v\} = N(v) - \{u\}$ if and only if $S(u) = S(v)$.*

Proof: Since the sufficiency is true by Proposition 2.4, we show the necessity. If $S(u) \neq S(v)$, then there exists a clique Q which belongs to $S(u) - S(v)$ (or $S(v) - S(u)$). Since $N(u) - \{v\}$ is an independent set, $Q = \{u, w\}$ and $w \in N(u) - \{v\} = N(v) - \{u\}$. Thus there exists either a clique $A_1 = \{v, w\}$ or a clique $A_2 = \{u, v, w\}$ in $\{a_1, .., a_n\}$. Then in the former case $(\{a_1, ..a_n\} - \{Q, A_1\}) \cup \{A_2\}$ is a *tcc* of G and in the latter $\{a_1, .., a_n\} - \{Q\}$ is a *tcc* of G, which contradicts the minimality of the multifamily $\mathcal{F}$. □

The relation $\sim$ on the vertices of graph G is defined by $u \sim v$ iff $\{u, v\} \in E(G)$ and $N(u) - \{v\} = N(v) - \{u\}$. This relation is an equivalence relation. So we obtain the following result. For a graph G, we denote the edge set $\{\{u, v\} \in E(G)|N(u) - \{v\} = N(v) - \{u\}\}$ by $E_N(G)$.

Proposition 2.6 **([6])** *For any graph G, the subgraph generated by $E_N(G)$ is a union of complete graphs.* □

3 General results

The results in Section 2 lead to the following facts. The subgraph of G whose vertex set consists of two classes one of which consists of the vertices corresponding to the same set $S(v)$ and the other is its neighborhood is isomorphic to $K_m + N_l$, where K_m is a complete graph with m vertices and N_l is a totally disconnected graph with l vertices. Thus we need values of

$\omega(K_m + N_l)$ and $\omega_m(K_m + N_l)$. We can see easily that $\omega_m(K_m + N_l) = l$. But it is not easy to get the value of $\omega(K_m + N_l)$. So we consider the family: $\{S_1, .., S_m, A_1, .., A_l\}$ such that each $S_i, A_i \subseteq [n]$, $S_i \cap S_j \neq \emptyset$, $S_i \cap A_j \neq \emptyset$, $A_i \cap A_j = \emptyset$, and $\bigcup_{i=1}^{l} A_i = [n]$. Then $K_m + N_l$ is the intersection graph of this family, where S_i corresponds to a vertex of K_m and A_i corresponds to a vertex of N_l. We have the following result.

Theorem 3.1 *Let $\mathcal{A} = \{A_1, .., A_l\}$ be a partition of $[n]$ and $\mathcal{F} = \{S_1, .., S_m\}$ be an intersecting family of distinct nonempty subsets of $[n]$ such that $\mathcal{A}$ and $\mathcal{F}$ are pairwise intersecting families. Then $|\mathcal{F}| \leq \frac{1}{2}\prod_{i=1}^{l}(2^{|A_i|} - 2) + \sum_{S \subsetneq [l]}(\prod_{i \in S}(2^{|A_i|} - 2))$.*

Proof: For each element S_i of $\mathcal{F}$, we can partition S_i into $S_{i,1} \cup ... \cup S_{i,l}$, where $\emptyset \neq S_{i,j} \subseteq A_j (j = 1, .., l)$ and so $S_{i,j} \cap A_k = \emptyset (j \neq k)$. First we consider the case such that S_i contains no A_j. If $S_i \in \mathcal{F}$, then $\overline{S_i} = (A_1 - S_{i,1}) \cup ... \cup (A_l - S_{i,l})$ does not belong to $\mathcal{F}$, since $S_i \cap \overline{S_i} = \emptyset$. So the number of sets S_i which contain no A_j is less than or equal to $\frac{1}{2}\prod_{i=1}^{l}(2^{|A_i|} - 2)$. Next we consider the case such that for some $k, S_{i,k} = A_k$. Then S_i satisfies the condition such that $S_{i,j}$ is nonempty for each $j \neq k$. Thus the number of sets S_i which contain some A_j is less than or equal to $\sum_{\emptyset \neq S \subseteq [l]}(\prod_{i \in [l]-S}(2^{|A_i|} - 2))$. Therefore $|\mathcal{F}| \leq \frac{1}{2}\prod_{i=1}^{l}(2^{|A_i|} - 2) + \sum_{S \subsetneq [l]}(\prod_{i \in S}(2^{|A_i|} - 2))$. □

For $a_1 \in A_1$ and a partition $\{A_1, .., A_l\}$ of $[n]$ such that $|A_1| \geq, ..., \geq |A_l| > 0$, $\mathcal{F} = \{(\{a_1\} \cup S_1) \cup (\bigcup_{i=2}^{l} S_i) | S_1 \subsetneq A_1 - \{a_1\}, \emptyset \neq S_i \subset A_i (i \neq 1)\} \cup (\bigcup_{\emptyset \neq S \subseteq [l]}\{(\bigcup_{i \in S} A_i) \cup (\bigcup_{j \in [l]-S} S_j) | \emptyset \neq S_j \subsetneq A_j\})$. Then $\mathcal{F}$ satisfies the conditions of Theorem 3.1 and $|\mathcal{F}| = (2^{|A_1|-1} - 1) \times \prod_{i=2}^{l}(2^{|A_i|} - 2) + \sum_{\emptyset \neq S \subseteq [l]}(\prod_{i \in [l]-S}(2^{|A_i|} - 2)) = \frac{1}{2}\prod_{i=1}^{l}(2^{|A_i|} - 2) + \sum_{S \subsetneq [l]}(\prod_{i \in S}(2^{|A_i|} - 2))$. Thus Theorem 3.1 is best possible. We call the above family *the standard family*. For natural numbers $a_1 \geq ... \geq a_l > 0$, where $a_1 + ... + a_l = n$, we denote the maximal value of $\frac{1}{2}\prod_{i=1}^{l}(2^{a_i} - 2) + \sum_{S \subsetneq [l]}(\prod_{i \in S}(2^{a_i} - 2))$ by $m(l, n)$.

Facts. (1) $m(l, n) < m(l, n + 1)$.
(2) Let $m = \lceil \frac{n}{l} \rceil$ and $n = k \times m + (l - k) \times (m - 1)$. Then $m(l, n) = (2^m - 1)^k \times (2^{m-1} - 1)^{l-k} - \frac{1}{2}(2^m - 2)^k \times (2^{m-1} - 2)^{l-k}$ and $a_1 =, .., = a_k = m$ and $a_{k+1} =, .., = a_l = m - 1$. □

We have the following result by the above facts. For an edge subset $S \subseteq E(G)$, $< S >_E$ denotes the subgraph generated by S and for a vertex subset $S \subseteq V(G)$, $< S >_V$ denotes the subgraph generated by S.

Proposition 3.2 *$\omega(K_m + N_l) = n$ if $m(l, n - 1) < m \leq m(l, n)$.* □

Proposition 3.3 *Let G be a connected graph and $C_1, .., C_k$ be components of $< E_N(G) >_E$. If there exist no edges joining vertices of C_i and vertices* *of C_j for each C_i and $C_j (i \neq j)$ and $< N(C_i) >_V$ is a union of l_i complete graphs $\alpha_{i,1},..,\alpha_{i,l_i}$ for each C_i, then $\omega(G) = \omega_m(G) + \sum_{i=1}^{k} s_i$, where $m(l_i, n_i - 1) < |V(C_i)| \leq m(l_i, n_i)$ and $s_i = n_i - l_i$.*

Proof: Let $\mathcal{Q} = \{Q_1,..,Q_n\}$ be a total clique cover of G such that $|\mathcal{Q}| = min_{\mathcal{R}:tcc\ of\ G}|\mathcal{R}|$, $S(v) = \{Q_i | v \in V(Q_i)\}$ for each $v \in V(G)$, and $\mathcal{F}(\mathcal{Q}) = \{S(v) | v \in V(G)\}$. We replace $S(v)$ by

$$S(v) = \begin{cases} (S(v) - \{Q(C_{i_t}, v)(t = 1, ..., r)\}) \cup (\bigcup_{t=1}^{r} A(\alpha_{i_t,j_t}, v)) \\ \qquad \text{if } v \in N(C_{i_1}),..,N(C_{i_r}) \text{ and } v \in V(\alpha_{i_1,j_1}),..,V(\alpha_{i_r,j_r}) \\ S(C_i, v) \quad \text{if } v \in V(C_i) \\ S(v) \qquad \text{otherwise,} \end{cases}$$

where $Q(C_i, v)$ are cliques of $\mathcal{Q}$ which contain v and vertices of C_i, $A(\alpha_{i_t,j_t}, v)$ correspond to a set A_{i_t,j_t}, and $S(C_i, v)$ correspond to a set of the standard family whose cardinality is $m(l_i, n_i)$. Then $\mathcal{F} = \{S(v) | v \in V(G)\}$ is a family whose intersection graph is G. Thus $\omega(G) \leq \omega_m(G) + \sum_{i=1}^{k}(n_i - l_i) = \omega_m(G) + \sum_{i=1}^{k} s_i$ by Proposition 3.2 .

Let $\mathcal{F} = \{S(v) | v \in V(G)\}$ be a family such that $\Omega(\mathcal{F}) \cong G$, $S(\mathcal{F}) = \{a_1,..,a_n\}$ and $n = \omega(G)$. Noting that each $< C_i \cup N(C_i) >_V$ contains $K_{|V(C_i)|} + N_{l_i}$ as an induced subgraph, we need n_i elements for each $< C_i \cup N(C_i) >_V$ by Proposition 3.2. We replace $S(v)$ by

$$S(v) = \begin{cases} \{a_{i,1},..,a_{i,l_i}\} \qquad \text{if } v \in V(C_i) \\ (S(v) - \bigcup\{S(u) | u \in V(C_{i_1}) \cup ... \cup V(C_{i_r})\}) \cup \{a_{i_1,j_1},..,a_{i_r,j_r}\} \\ \qquad \text{if } v \in N(C_{i_1}),..,N(C_{i_r}) \text{ and } v \in V(\alpha_{i_1,j_1}),..,V(\alpha_{i_r,j_r}) \\ S(v) \qquad \text{otherwise,} \end{cases}$$

where each $a_{i,j}$ is a complete subgraph $< C_i \cup \alpha_{i,j} >_V$. Then $\mathcal{F} = \{S(v) | v \in V(G)\}$ is a multifamily and its intersection graph is G. As is the case with the proof of Theorem 2.1, we can define a total clique cover $\mathcal{Q}(\mathcal{F})$. $\omega(G) - \sum_{i=1}^{k} l_i + \sum_{i=1}^{k} n_i = |\mathcal{Q}(\mathcal{F})| \geq \omega_m(G)$. Thus $\omega(G) \geq \omega_m(G) + \sum_{i=1}^{k} s_i$. □

Corollary 3.4 *Let G be a connected graph with $|V(G)| \geq 3$. If all components of $< E_N(G) >_E$ are K_2, and $N(u) - \{v\} = N(v) - \{u\}$ is an independent set for each $\{u, v\} \in E_N(G)$, then $\omega(G) = \omega_m(G) + |E_N(G)|$.*

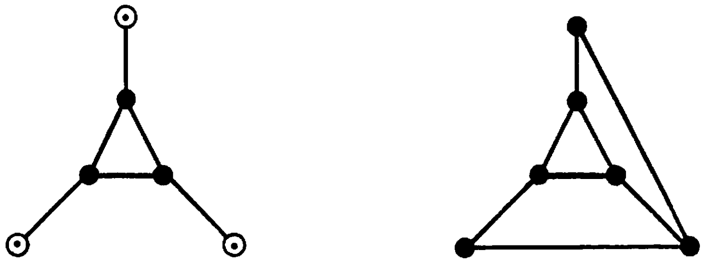
Proof: By the hypothesis of the Corollary, for each component of $E_N(G)$, s_i is 1. Thus $\omega(G) = \omega_m(G) + \sum_{i=1}^{|E_N(G)|} 1 = \omega_m(G) + |E_N(G)|$, by Proposition 3.3. □

Corollary 3.5 *For a connected K_4-free graph G with $|V(G)| \geq 4$, $\omega(G) = \omega_m(G) + |E_N(G)|$.*

Proof: Since G satisfies the conditions of Corollary 3.4 , $\omega(G) = \omega_m(G) + |E_N(G)|$. □

4 Regular graphs.

In this section, we consider the intersection number of regular graphs. Connected 1- and 2-regular graphs are K_2 and cycles, respectively. So we can easily obtain their intersection numbers and intersection numbers on multifamilies. Namely, for a 1-regular connected graph G, $\omega(G) = 3$ and $\omega_m(G) = 1$. For a 2-regular graph G other than K_3, both $\omega(G)$ and $\omega_m(G)$ are equal to the number of its edges. Next we consider 3-, 4- and 5-regular graphs. For two disjoint subsets V_1, V_2 of $V(G)$, the subgraph of G with vertex set $V_1 \cup V_2$ whose edge set is the set of those edges of G that have both ends in V_1 or one end in V_1 and other end in V_2 is called a *quasi-induced subgraph,* and V_1, V_2 are called the *base set* and the *neighborhood set,* respectively. For example, Figure 4.1(a) is a quasi-induced subgraph of the graph shown in Figure 4.1(b), where black circles are vertices of the base set and double circles are vertices of the neighborhood set. Hereafter, for a quasi-induced subgraph, black circles denote vertices of the base set and double circles denote vertices of the neighborhood set. For a graph, E_0 denotes the number of edges which are contained no triangles.

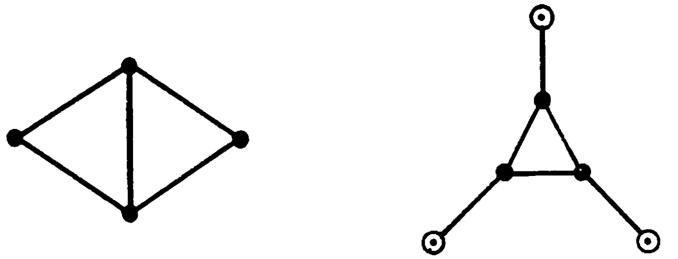


(a) (b)

Figure 4.1

Theorem 4.1 *For a connected 3-regular graph G other than K_4, $\omega_m(G) = E_0 + T$, where T is the number of triangles of G.*

Proof: Since $\{e \in E(G) | e$ is contained in no triangles $\} \cup \{\tau | \tau$ is a triangle of $G\}$ is a total clique cover of G, $\omega_m(G) \leq E_0 + T$. Since G is a 3-regular graph other than K_4, each triangle is contained in graphs of type Figure 4.2 (a) or (b) which are quasi-induced subgraphs of G. Thus we need T triangles and all edges contained in no triangles for total clique covering. So $\omega_m(G) \geq E_0 + T$. □



(a) (b)

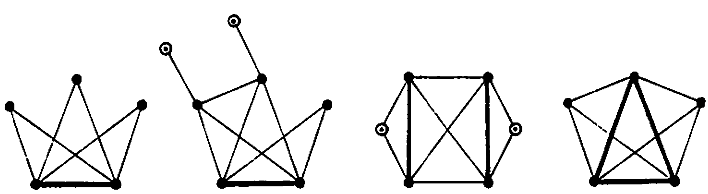
Figure 4.2

Theorem 4.2 *For a connected 3-regular graph G other than K_4, $\omega(G) = E_0 + T + F$, where T is the number of triangles of G and F is the number of subgraphs isomorphic to $K_4 - e$ of G.*

Proof: Since G is a 3-regular graph other than K_4, G is K_4-free with $|V(G)| \geq 4$. By Corollary 3.5,$\omega(G) = \omega_m(G) + |E_N(G)|$. For each component C of $< E_N(G) >_E$, $< C \cup N(C) >_V$ is $K_4 - e$. So $|E_N(G)|$ is F. By Theorem 4.1, $\omega_m(G) = E_0 + T$ and $\omega(G) = E_0 + T + F$. □

Combining Theorem 4.1 and Theorem 4.2, we can completely calculate intersection numbers of 3-regular graphs. And furthermore, $\omega(K_4) = 3$, and $\omega_m(K_4) = 1$. Next we consider 4-regular graphs.

Theorem 4.3 *For a connected 4-regular graph G other than K_5, $\omega(G) = \omega_m(G) + E_2 + 2 \times E_3 - F_c$, where E_2 is the number of components of $< E_N(G) >_E$ isomorphic to K_2, E_3 is the number of components of $< E_N(G) >_E$ isomorphic to K_3 and F_c is the number of quasi-induced subgraphs isomorphic to the graph in Figure 4.3(c).*



(a) G_a (b) G_b (c) G_c (d) G_d

Figure 4.3

Proof: Based on complete subgraphs of $< E_N(G) >_E$, we classify the quasi-induced subgraphs which contain a component of $< E_N(G) >_E$. Since G is not K_5, only the quasi-induced subgraphs of Figure 4.3 are possible.

The graphs in Figure 4.3(a),(b),(c) contain components of $< E_N(G) >_E$ isomorphic to K_2 and the graph in Figure 4.3(d) contains a component of $< E_N(G) >_E$ isomorphic to K_3.

For the graphs in Figure 4.3, we have intersection numbers and intersection numbers with respect to multifamilies as follows:

$\omega(G_a) = 4, \omega_m(G_a) = 3,$

$\omega(G_b) = 5, \omega_m(G_b) = 4,$

$\omega(G_c) = 4, \omega_m(G_c) = 3,$

$\omega(G_d) = 4, \omega_m(G_d) = 2.$

Since each graph in Figure 4.3 is a quasi-induced subgraph of G, we have $\omega(G) = \omega_m(G) + E_2 + 2 \times E_3 - F_c$. □

The next result gives $\omega_m(G)$ for all 4-regular graphs G except K_5 and the octahedron graph. We can easily see that ω_m (octahedron) = 4, so Theorem 4.3 gives intersection numbers of all 4-regular graphs other than K_5. We already know that $\omega_m(K_5) = 1$ and $\omega(K_5) = 4$.

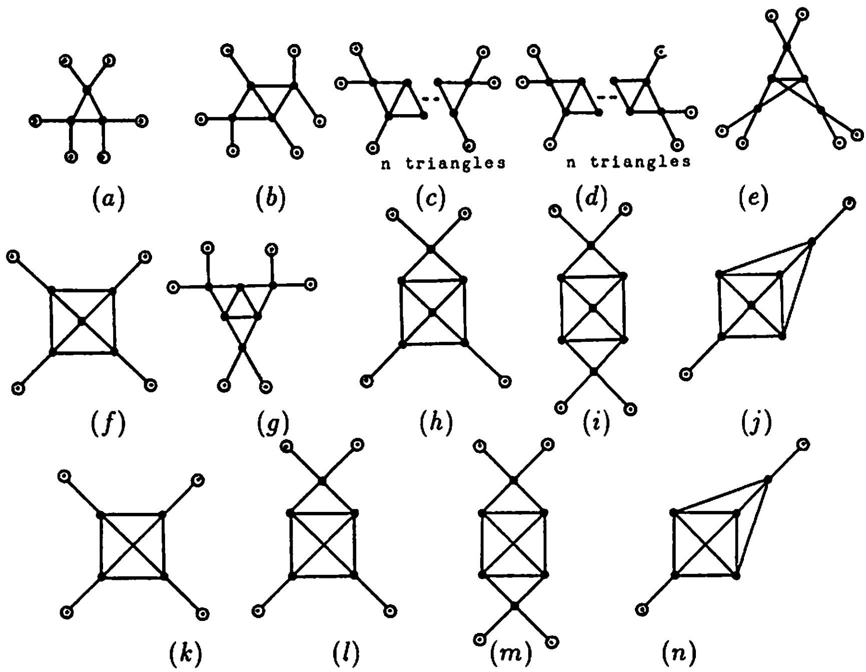
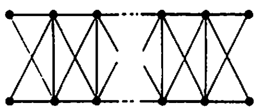


Figure 4.4

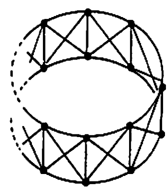
Theorem 4.4 *For a connected 4-regular graph G other than K_5 and the octahedron graph, $\omega_m(G) = E_0 + |\{\tau|\tau$ is a triangle contained no $K_4\}|$ $+|\{\sigma|\sigma$ is a $K_4\}|$ $-(F_g + F_h + F_j + 2 \times F_i)$, where $F_g, .., F_j$ are the number of quasi-induced subgraphs isomorphic to graphs of Figure 4.4 (g) ,.., (j), respectively.*

Proof: In terms of complete subgraphs, we classify the quasi-induced subgraphs of 4-regular graphs. Since G is neither K_5 nor the octahedron graph, only quasi-induced subgraphs of Figure 4.4 are possible. For each graph of Figure 4.4 except (g), (h), (i) and (j), its total clique cover is the set of all its maximal complete subgraphs. For each graph G of Figure 4.4(g), (h) and (j), $\omega_m(G) = |\{Q|Q$ is a maximal complete subgraph of $G\}| - 1$. And we also easily get that ω_m (the graph of Figure 4.4(i)) $= |\{Q|Q$ is a maximal complete subgraph of the graph of Figure 4.4(i) $\}| - 2$. Thus $\omega_m(G) = E_0 + |\{\tau|\tau$ is a triangle contained no $K_4\}| + |\{\sigma|\sigma$ is a $K_4\}| -(F_g + F_h + F_j + 2 \times F_i)$. □

Next we consider 5-regular graphs. As the situation is getting more complicated, we only obtain the intersection numbers. Let R_n and S_n denote the graphs shown in Figure 4.5 (a) and (b), respectively. We can easily obtain that $\omega_m(R_n) = n + 4$, $\omega(R_{2n}) = 3n + 5, \omega(R_{2n+1}) = 3n + 6, \omega_m(S_n) = n, \omega(S_{2n}) = 3n$ and $\omega(S_{2n+1}) = 3n + 2$.



(a) R_n



(b) S_n

Figure 4.5

Theorem 4.5 *For a 5-regular graph G except K_6 and S_n, $\omega(G) = \omega_m(G) + E_2 - (F_d + F_h + F_i + F_k + F_n + \sum_{R_n \in \mathcal{H}_{e,f}} \lceil \frac{n}{2} \rceil) + 2 \times (E_3 + E_4)$, where E_2, E_3 and E_4 are the number of components on $< E_N(G) >_E$ such that it is isomorphic to K_2, K_3 and K_4, respectively. F_d, F_h, F_i, F_k and F_n are the number of quasi-induced subgraphs isomorphic to graphs in Figure 4.6 (d),(h),(i),(k) and (n), respectively. $\mathcal{H}_{e,f}$ is the family of quasi-induced subgraphs in G isomorphic to graphs in Figure 4.6(e),(f).*

Proof: Based on complete subgraphs of $< E_N(G) >_E$, we classify the quasi-induced subgraphs which contained a component of $< E_N(G) >_E$. Since G is not K_6, only the quasi-induced subgraphs of Figure 4.6 are possible. The graphs in Figure 4.6(a),..,(k) contain a component of $< E_N(G) >_E$ isomorphic to K_2, the graphs in Figure 4.6(l),..,(n) contain a component of $< E_N(G) >_E$ isomorphic to K_3 and the graph in Figure 4.6(o) contains a component of $< E_N(G) >_E$ isomorphic to K_4. For the graphs in Figure 4.6, we have intersection numbers and intersection numbers on multifamilies as follows:

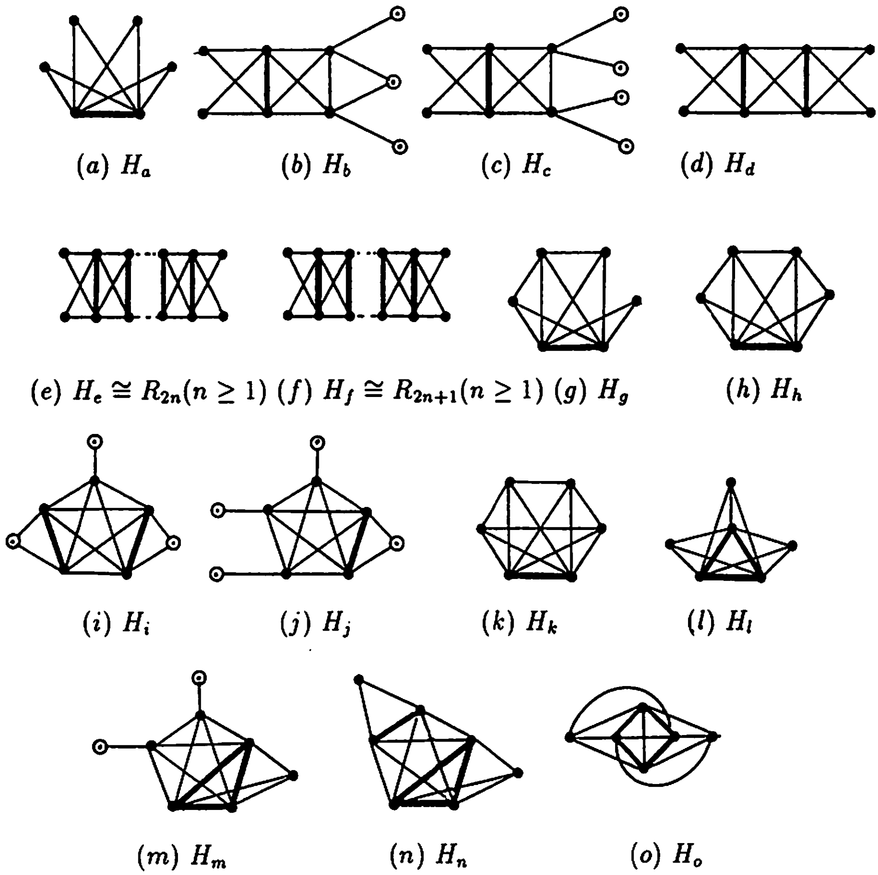


Figure 4.6

$\omega(H_a) = 5,\ \omega_m(H_a) = 4,$
$\omega(H_b) = 7,\ \omega_m(H_b) = 6,$
$\omega(H_c) = 8,\ \omega_m(H_c) = 7,$
$\omega(H_d) = 6,\ \omega_m(H_d) = 5,$
$\omega(H_e) = 3n + 5,\ \omega_m(H_e) = 2n + 4,$
$\omega(H_f) = 3n + 6,\ \omega_m(H_f) = 2n + 5,$
$\omega(H_g) = 4,\ \omega_m(H_g) = 3,$
$\omega(H_h) = 3,\ \omega_m(H_h) = 3,$
$\omega(H_i) = 5,\ \omega_m(H_i) = 4,$
$\omega(H_j) = 6,\ \omega_m(H_j) = 5,$

$\omega(H_k) = 4,\ \omega_m(H_k) = 4,$
$\omega(H_l) = 5,\ \omega_m(H_l) = 3,$
$\omega(H_m) = 6,\ \omega_m(H_m) = 4,$
$\omega(H_n) = 5,\ \omega_m(H_n) = 3,$
$\omega(H_o) = 4,\ \omega_m(H_o) = 2.$

Since each graphs in Figure 4.6 is a quasi-induced subgraph, we have $\omega(G) = \omega_m(G) + E_2 - (F_d + F_h + F_i + F_k + F_n + \sum_{R_n \in \mathcal{H}_{e,f}} \lceil \frac{n}{2} \rceil) + 2 \times (E_3 + E_4)$. □

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