Generalized Irredundance in Graphs: Hereditary Properties and Ramsey Numbers

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Abstract

For each vertex s of the subset S of vertices of a graph G, we define Boolean variables p, q, r which measure existence of three kinds of S–private neighbours (S–pns) of s. A 3–variable Boolean function $f = f(p,q,r)$ may be considered as a compound existence property of S–pns. The set S is called an *f–set* of G if $f = 1$ for all $s \in S$ and the class of all f–sets of G is denoted by Ω_f. Special cases of Ω_f include the independent sets, irredundant sets, open irredundant sets and CO–irredundant sets of G.

There are 62 non–trivial families Ω_f which include the 7 families of a framework proposed earlier by Fellows, Fricke, Hedetniemi and Jacobs.

The functions f for which Ω_f is hereditary for any graph G are determined, the existence and properties of f–Ramsey numbers (analogues of the elusive classical Ramsey numbers) are investigated and future directions for the theory of the classes Ω_f are considered.

1 Introduction

The open (closed) neighbourhood of the vertex subset S of a simple graph $G = (V, E)$ is denoted by $N(S)$ ($N[S]$) and as usual, for $s \in V$, $N(\{s\})$ and $N[\{s\}]$ are abbreviated to $N(s)$ and $N[s]$.

The basic ingredients of this work are three properties which make a vertex s (informally) important in a vertex subset S of a graph G. It will also help the intuition to replace the word "important" by "essential" or

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"non–redundant". Each property depends on the existence of one of the three types of *S–private neighbour* (*S–pn*) t for s, which we now formally define.

For $s \in S$, vertex t is an:

(i) *S–self private neighbour* (*S–spn*) of s
if $t = s$ and s is an isolated vertex of $G[S]$

(ii) *S–internal private neighbour* (*S–ipn*) of s
if $t \in S - \{s\}$ and $N(t) \cap S = \{s\}$

(iii) *S–external private neighbour* (*S–epn*) of s
if $t \in V - S$ and $N(t) \cap S = \{s\}$.

Observe that each such t is an element of $N[s] - N(S - \{s\})$ and that no $s \in S$ may have S–pns of both type (i) and type (ii).

For additional motivation suppose that transmitters are to be placed at the vertices S of a graph which models some sort of communications network where information may pass along edges. If for example, some $s \in S$ has an S–epn t, then removal of s from S would reduce the set of vertices who could receive information directly from transmitters, because $S - \{s\}$ cannot transmit to t. Thus s is an essential (or non–redundant) vertex of s.

Several classes of vertex subsets S have been studied whose definition involves existence of S–pns for *every* $s \in S$. Firstly, the sets S in which each vertex is an S–spn, are precisely the independent sets of G. These are possibly the most well–studied sets in graph theory. The second obvious example is the class of *CC–irredundant sets* of G containing these sets S for which each $s \in S$ has an S–spn or an S–epn.

The CC (which is usually omitted) in the name is due to the fact that there are two Closed neighbourhoods in the following well–known characterisation of CC–irredundant sets.

Proposition 1. *S is CC–irredundant if and only if for each $s \in S$ $N[s] - N[S - \{s\}] \neq \emptyset$.*

In the remainder of the paper, we will drop the CC. Irredundant sets were first defined and studied by Cockayne, Hedetniemi and Miller [2] due to an important connection with the theory of dominating sets which we now state.

Proposition 2. [2]

(i) *The dominating set S is minimal dominating if and only if S is irredundant.*

(ii) *If S is minimal dominating, then S is maximal irredundant.*

The theory of irredundant sets is a most active research area currently (circa 120 published papers since 1981). The reader is referred to the extensive bibliography of the book by Haynes, Hedetniemi and Slater [10].

Various authors have studied other classes of vertex sets of this type such as open irredundant and closed–open irredundant. These classes, about which very little is known, will be detailed and referenced below. The four aforementioned classes of sets will be seen to be special cases of the class Ω_f of f–sets for a suitable function f in a set $\mathcal{F}$ of 62 Boolean functions defined in Section 2. Thus we greatly extend the work of Fellows, Fricke, Hedetniemi and Jacobs [7], who considered a "private neighbour cube" of 8 classes of sets which are detailed in Section 3. It is anticipated that each Ω_f has a rich theory analogous to that of the irredundant sets and further, unifying results of the type: If f is in the subset $\mathcal{F}'$ of $\mathcal{F}$, then Ω_f has property X, will abound. A result of the latter type is given in Section 4 where we determine the functions $f \in \mathcal{F}$ for which the class Ω_f is hereditary for all graphs G. The existence and properties of f–Ramsey numbers (analogues of the elusive, well–known classical Ramsey numbers) are discussed in Section 5. Finally in Section 6 we define two parameters involving Ω_f and list some areas in which research on these parameters should prove fruitful.

2 The General Framework

The negation of a Boolean variable x will be denoted by $\overline{x}$. For $s \in S$ let $p(s,S)$, $q(s,S)$, $r(s,S)$ be Boolean variables which take the value 1 if and only if s has an S–pn of type (i), (ii), (iii) respectively. Wherever possible we use the abbreviations p, q, r for these variables. Observe that for each $s \in S$, $p(s,S) \wedge q(s,S) = 0$, i.e., p, q, r are not independent variables.

Let $S(s) = (p,q,r)$. We will also (imprecisely) say that $S(s) = i$, where $i \in \{0,1,\ldots,7\}$ is the integer having binary representation pqr and will call this the *integer form* of $S(s)$. The condition $p \wedge q = 0$, however, implies that $S(s)$ is never $(1,1,0)$ or $(1,1,1)$, i.e., (in integer form) $S(s) \in \{0,\ldots,5\}$.

Example 1. Consider the vertex subset $S = \{a, b, c, d\}$ of the graph G depicted in Figure 1. The S–pns of vertices of S are tabulated in Table 1 and we observe

$$S(a) = (0,1,1), \quad S(b) = (0,0,0), \quad S(c) = (0,0,1) \quad \text{and} \quad S(d) = (1,0,1)$$

(or using integer form, $S(a)$, $S(b)$, $S(c)$ and $S(d)$ are equal to 3, 0, 1 and 5 respectively).

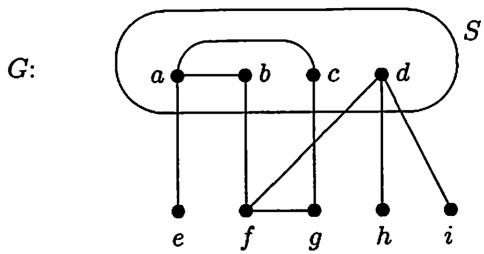

FIG. 1. Graph G for Example 1

	type (i)	type (ii)	type (iii)
a		b, c	e
b			
c			g
d	d		h, i

TABLE 1. S–pns of vertices of S for graph G

We now define the most important concept of this work.

Let f be a Boolean function of the three variables p, q, r. The vertex subset S of G is an *f–set of G* if for all $s \in S$, $f(p,q,r) = 1$ (i.e., $f(S(s)) = 1$). The function f may be viewed as a compoud existence/non–existence property of the three types of S–pn. For S to be an f–set, each $s \in S$ has the property f. The class of all f–sets of G will be denoted by $\Omega_f(G)$ and abbreviated to Ω_f whenever possible.

The rows of the truth table of f will be labelled $0, ..., 7$, so that the entry in row i is $f(p, q, r)$, where pqr is the binary representation of the integer i (e.g., $f(1, 0, 1)$ is the fifth entry in the table). Recall that for each $s \in S$, $S(s)$ is never equal to $(1, 1, 0)$ or $(1, 1, 1)$. We deduce:

(a) If the only 1's in the truth table for f occur in rows 6 or 7, then $\Omega_f = \emptyset$.

(b) If f' is formed from f by replacing the values in rows 6 and 7 by 0's, then $\Omega_{f'} = \Omega_f$.

In addition $f = 0$ gives $\Omega_f = \emptyset$ and the function g with 1's in all rows $0, 1, ..., 5$ is uninteresting since for any $G = (V, E)$, Ω_g contains every subset of V. Thus we are only concerned with the set $\mathcal{F}$ of the 62 non–zero functions f with 0's in rows 6, 7 and in at least one other row.

For each $f \in \mathcal{F}$, let A_f be the set of rows of the truth table of f in which $f = 1$. This sets up a 1–1 correspondence between $\mathcal{F}$ and the non–empty proper subsets of $\{0, 1, ..., 5\}$. Further, we observe that logical implication in $\mathcal{F}$ and set containment are related by

Proposition 3. *Let $f, g \in \mathcal{F}$.*

$$f \Rightarrow g \quad \textit{if and only if} \quad A_f \subseteq A_g \; .$$

Example 2. Let $f = (p \vee q) \wedge \overline{r}$.
The subset S is an f–set of G if and only if each $s \in S$ has an S–pn of type (i) or (ii) but no S–pn of type (iii).

In disjunctive normal form

$$f = (p \wedge q \wedge \overline{r}) \vee (\overline{p} \wedge q \wedge \overline{r}) \vee (p \wedge \overline{q} \wedge \overline{r}) \; .$$

However, since $p \wedge q = 0$, the first term is false, hence f has 1's in rows 2 and 4, i.e., $A_f = \{2, 4\}$. If a, b, c, d is the vertex sequence of C_4, then $\Omega_f(C_4) = \{\{a, c\}, \{b, d\}\}$. For the graph G with $V = \{a, b, c, d\}$ and $E = \{ab, bc, ca, da\}$, $\Omega_f(G) = \{\{b, c\}, \{b, c, d\}\}$.

3 The Private Neighbour Cube

In this section we consider the seven non–trivial classes of sets given in the cube structure [7] and observe that each is the class Ω_f for some function

f formed from p, q, r and the disjunction operator. In each case we specify the function f and the set A_f. It should be emphasized that our Boolean variables p, q, r differ from those used in [7].

1. $f = p$ ($A_f = \{4, 5\}$).
   Each $s \in S \in \Omega_f$ is isolated in $G[S]$. Thus Ω_f is the class of independent sets of G.

2. $f = q$ ($A_f = \{2, 3\}$).
   Each $s \in S \in \Omega_f$ has an S–ipn. Thus Ω_f is the class of sets S such that $G[S] \cong pK_2$. The edge set of such induced subgraphs has been called an *induced matching* by Cameron [1] and a *strong matching* by Golumbic and Laskar [9]. In [7], the authors call S a *strong matching set.*

3. $f = r$ ($A_f = \{1, 3, 5\}$).
   Each $s \in S \in \Omega_f$ has an S–epn. Thus S is an f–set if and only if for each $s \in S$, $N(s) - N[S - \{s\}] \neq \emptyset$. Since there is an Open and Closed neighbourhood involved in this characterisation, such sets are called OC–*irredundant.* They were also named *open irredundant* by Farley and Shacham [5] and have also been studied by Favaron [6] and Hedetniemi, Jacobs and Laskar [11].

4. $f = p \vee q$ ($A_f = \{2, 3, 4, 5\}$).
   Each $s \in S \in \Omega_f$ has an S–ipn or is isolated in $G[S]$. Thus Ω_f is the class of sets S such that $G[S] \simeq pK_1 \cup qK_2$. Fink and Jacobson [8] have called these sets 1–*dependent.*

5. $f = p \vee r$ ($A_f = \{1, 3, 4, 5\}$).
   Each $s \in S \in \Omega_f$ is an S–spn or has an S–epn. Thus Ω_f is the class of irredundant sets of G.

6. $f = q \vee r$ ($A_f = \{1, 2, 3, 5\}$).
   Each $s \in S \in \Omega_f$ has an S–ipn or an S–epn. Thus S is an f–set if and only if for each $s \in S$, $N(s) - N(S - \{s\}) \neq \emptyset$. Due to the neighbourhoods involved in this characterisation, such sets are called OO–irredundant. Such sets have been studied in [11] and also by Farley and Proskurowski [4].

7. $f = p \vee q \vee r$ ($A_f = \{1, 2, 3, 4, 5\}$).
   Each $s \in S \in \Omega_f$ has at least one S–pn (of any type) and so these sets

are characterised by $N[s] - N(S - \{s\}) \neq \emptyset$. Thus S is called CO–irredundant. Such sets were mentioned in [7, 9] and Ramsey properties of CO–irredundant sets were studied by Cockayne, MacGillivray and Simmons [3].

4 When is Ω_f Hereditary?

In this section we determine the subset $\mathcal{H}$ of $\mathcal{F}$ which contains those functions $f \in \mathcal{F}$ for which $\Omega_f(G)$ is hereditary for all graphs G. The motivation for this work is the fact that both p and $p \vee r$ are in $\mathcal{H}$.

Theorem 4. *If both f and g are in $\mathcal{H}$, then $f \wedge g \in \mathcal{H}$.*

Proof. For any graph G, let $S \in \Omega_{f\wedge g}$ and $T \subseteq S$. Then for each $s \in S$,

$$\begin{aligned} & f \wedge g(S(s)) = 1 \\ \Rightarrow\ & f(S(s)) \wedge g(S(S)) = 1 \\ \Rightarrow\ & S \in \Omega_f \quad \text{and} \quad S \in \Omega_g \\ \Rightarrow\ & T \in \Omega_f \quad \text{and} \quad T \in \Omega_g \end{aligned}$$

(since Ω_f, Ω_g are hereditary). Hence for each $t \in T$,

$$\begin{aligned} & f(T(t)) \wedge g(T(t)) = 1 \\ \Rightarrow\ & f \wedge g(T(t)) = 1 \\ \Rightarrow\ & T \in \Omega_{f\wedge g}\ . \end{aligned}$$

□

Corollary 5. *Let $f, g \in \mathcal{H}$ and h be the function in $\mathcal{F}$ such that $A_f \cap A_g = A_h$. Then $h \in \mathcal{H}$.*

Proof. Since $A_{f\wedge g} = A_f \cap A_g$, the result follows immediately from Theorem 4. □

Theorem 6. *If $f \in \mathcal{H}$, then $5 \in A_f$.*

Proof. Let $A_f \subseteq \{0, 1, 2, 3, 4\}$. It is easy to construct G and $S \in \Omega_f(G)$ with $|S| \geq 2$ and some $s \in S$ not isolated in G. Consider the subset $B = \{s\} \subseteq S$. Observe that $B(s) = (1, 0, 1)$ which implies that $f(B(s)) = 0$. Hence $B \notin \Omega_f(G)$, contrary to hypothesis. □

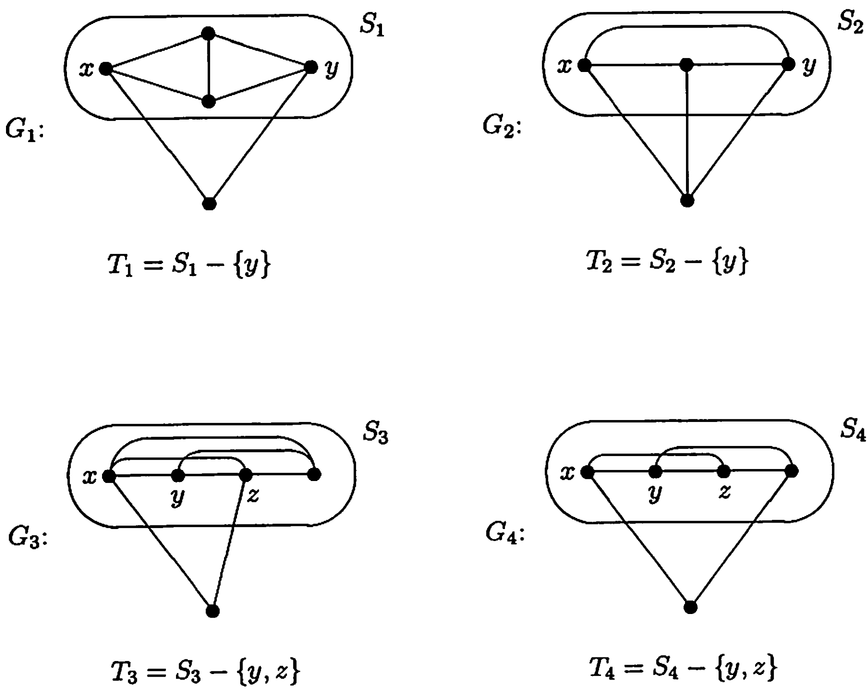

FIG. 2. Graphs for the proof of Theorem 7

Theorem 7. *If Ω_f is hereditary for all G and $0 \in A_f$, then* $A_f = \{0, 1, ..., 5\}$.

Proof. By Theorem 6, $5 \in A_f$. We show by contradiction that each $i = 1, 2, 3, 4$ is in A_f. Specifically, in Figure 2 we exhibit a graph G_i together with vertex subsets T_i, S_i with $T_i \subseteq S_i$ such that

(a) for all $s \in S_i$, $S_i(s) = 0$ (integer form) and

(b) there exists $x \in T_i$ such that $T_i(x) = i$ (integer form).

Note that hypothesis and (a) show that $S_i \in \Omega_f$ while, if $i \notin A_f$, (b) gives $T_i \notin \Omega_f$ contrary to the definition of $\mathcal{H}$. □

In the statement of the next three results, for each function f mentioned, the set A_f follows in parentheses.

Theorem 8. *Each of the functions* $p(\{4,5\})$, $p \vee r(\{1,3,4,5\})$, $p \vee q(\{2,3,4,5\})$, $q \vee r(\{1,2,3,5\})$, $p \vee q \vee r(\{1,2,3,4,5\})$ *and* $p \vee (q \wedge \bar{r})(\{2,4,5\})$ *is in* $\mathcal{H}$.

Proof. For any graph G, $\Omega_p(G)$ and $\Omega_{p\vee r}(G)$ are precisely the classes of independent and irredundant sets of G respectively and are known to be hereditary. If $T \subseteq S \in \Omega_{p\vee q}(G)$, then $G[S]$ and consequently $G[T]$ have the form $\lambda K_1 \cup \mu K_2$ (Section 3). Therefore $T \in \Omega_{p\vee q}(G)$ and so $p \vee q \in \mathcal{H}$.

Let $s \in S \in \Omega_{q\vee r}(G)$. ($S$ is OO–irredundant in G (Section 3)). We show that $S - \{s\}$ is also in $\Omega_{q\vee r}(G)$. Let $x \in S - \{s\}$. Since $x \in S$, x has an S–epn t or an S–ipn t'. In the former case t is also an $(S - \{s\})$–epn for x, while in the latter case either $t' \neq s$ and t' is an $(S - \{s\})$–ipn for x or $t' = s$ and t' is an $(S - \{s\})$–epn for x. Hence $S - \{s\}$ is OO–irredundant in G which shows that $q \vee r \in \mathcal{H}$.

Let $s \in S \in \Omega_{p\vee q\vee r}$ (S is CO–irredundant in G (Section 3)). Let $x \in S - \{s\}$. If x is isolated in $G[S]$, then x is also isolated in $G[S - \{s\}]$. Any S–epn of x is also an $(S - \{s\})$–epn for x and if x has an S–ipn t', then, as in the preceding paragraph t' is either an $(S - \{s\})$–ipn or an $(S - \{s\})$–epn for x. Thus $S - \{s\}$ is CO–irredundant in G which implies that $p \vee q \vee r \in \mathcal{H}$.

Finally let $s \in S \subseteq \Omega_{p\vee(q\wedge\bar{r})}$ and $x \in S - \{s\}$. Then x is isolated in $G[S]$ or x has an S–ipn but no S–epn. In the former case x is also isolated in $G[S - \{s\}]$. In the latter case since t is not isolated in $G[S]$, x is the S–ipn for t and the single edge xt forms a component of $G[S]$. If $t \neq s$, then t is an $(S - \{s\})$–ipn for x and removal of s from S does not create any $(S - \{s\})$–epn for x. If $t = s$, then x is isolated in $G[S - \{s\}]$. Hence in all cases either x is isolated in $G[S - \{s\}]$ or has an $(S - \{s\})$–ipn but no $(S - \{s\})$–epn. Thus $S - \{s\} \in \Omega_{p\vee(q\wedge\bar{r})}$, i.e., $p \vee (q \wedge \bar{r}) \in \mathcal{H}$. □

Theorem 9. *Each of the functions* $p \wedge r(\{5\})$, $r(\{1,3,5\})$, $(p \wedge r) \vee q(\{2,3,5\})$, $p \vee (q \wedge r)(\{3,4,5\})$, $(p \vee q) \wedge r(\{3,5\})$, $(p \wedge r) \vee (q \wedge \bar{r})(\{2,5\})$ *is in* $\mathcal{H}$.

Proof. For each function f of the statement, we indicate that $A_f = A_g \cap A_h$ where membership of g and h in $\mathcal{H}$, has already been established. Then Corollary 5 implies that $f \in \mathcal{H}$.

$$\begin{array}{rcccc} p \wedge r: & \{5\} & = & \{1,2,3,5\} & \cap & \{4,5\} \\ r: & \{1,3,5\} & = & \{1,2,3,5\} & \cap & \{1,3,4,5\} \\ (p \wedge r) \vee q: & \{2,3,5\} & = & \{1,2,3,5\} & \cap & \{2,3,4,5\} \\ p \vee (q \wedge r): & \{3,4,5\} & = & \{1,3,4,5\} & \cap & \{2,3,4,5\} \\ (p \vee q) \wedge r: & \{3,5\} & = & \{1,3,5\} & \cap & \{2,3,5\} \\ (p \wedge r) \vee (q \wedge \overline{r}): & \{2,5\} & = & \{2,4,5\} & \cap & \{2,3,5\} \end{array}$$

□

Theorem 10. *None of the functions* $\overline{q} \wedge r(\{1,5\})$, $p \vee (q \wedge \overline{r}) \vee (\overline{q} \wedge r)(\{1,2,4,5\})$, $p \vee (\overline{q} \wedge r)(\{1,4,5\})$, $(q \wedge \overline{r}) \vee (\overline{q} \wedge r)(\{1,2,5\})$ *is in* $\mathcal{H}$.

Proof. Consider the graph G and set S of Figure 3. For each $s \in S$, $S(s) = 1$ (strictly $(0,0,1)$). Hence $S \in \Omega_{\overline{q} \wedge r}(G)$. However if $B = S - \{x\}$, then $B(y) = 3$ (i.e., $(0,1,1)$) which implies that $B \notin \Omega_{\overline{q} \wedge r}(G)$. Therefore $\overline{q} \wedge r$ is not in $\mathcal{H}$.

By Theorem 9, the function $r(\{1,3,5\}) \in \mathcal{H}$. Observe that the intersection of $\{1,3,5\}$ and each of the sets $\{1,2,4,5\}$, $\{1,2,5\}$, $\{1,4,5\}$ is equal to $\{1,5\}$. If any of the functions $p \vee (q \wedge \overline{r}) \vee (\overline{q} \wedge r)(\{1,2,4,5\})$, $p \vee (\overline{q} \wedge r)(\{1,4,5\})$, $(q \wedge \overline{r}) \vee (\overline{q} \wedge r)(\{1,2,5\})$, were in $\mathcal{H}$, then we could deduce (by Corollary 5) that $\overline{q} \wedge r(\{1,5\}) \in \mathcal{H}$, a contradiction which completes the proof. □

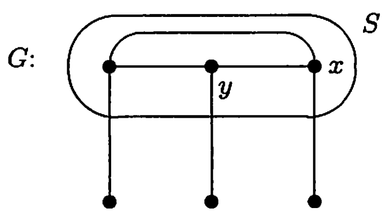

FIG. 3. Graph G for proof of Theorem 10

The class $\mathcal{H}$ is now completely determined by the above theorems and the 12 functions of $\mathcal{H}$ are detailed in Table 2.

label	function f	A_f	name
h_1	$p \wedge r$	$\{5\}$	
h_2	$(p \wedge r) \vee (q \wedge \overline{r})$	$\{2,5\}$	
h_3	$(p \vee q) \wedge r$	$\{3,5\}$	
h_4	$(p \wedge r) \vee q$	$\{2,3,5\}$	
h_5	r	$\{1,3,5\}$	OC–irredundant
h_6	$q \vee r$	$\{1,2,3,5\}$	OO–irredundant
h_7	p	$\{4,5\}$	independent
h_8	$p \vee (q \wedge \overline{r})$	$\{2,4,5\}$	
h_9	$p \vee (q \wedge r)$	$\{3,4,5\}$	
h_{10}	$p \vee q$	$\{2,3,4,5\}$	1–dependent
h_{11}	$p \vee r$	$\{1,3,4,5\}$	(CC)–irredundant
h_{12}	$p \vee q \vee r$	$\{1,2,3,4,5\}$	CO–irredundant

TABLE 2. The class $\mathcal{H}$

The Hasse diagram of the PO–set $\mathcal{H}$ is given in Figure 4 and has an interesting structure. Let D be the graph consisting of the cycle C_6 together with one major chord (joining the first and fourth vertices in the sequence). The diagram consists of two disjoint copies D_1, D_2 of D with vertex sets $\{h_1, \ldots, h_6\}$ and $\{h_7, \ldots, h_{12}\}$ together with a matching $\{h_i h_{i+6} | i = 1, \ldots, 6\}$. In Figure 4, the matching is drawn with broken lines to emphasize this structure. We observe that

(i) for each $i = 1, \ldots, 6$, h_i, h_{i+6} are similarly situated on D_1, D_2 respectively and h_i is obtained from h_{i+6} by the replacement of p by $p \wedge r$ (equivalently $A_{h_{i+6}} = A_{h_i} \cup \{4\}$).

(ii) $V(D_1) = \{f \in \mathcal{H} | p \implies f\}$ and $V(D_2) = \{f \in \mathcal{H} | f \implies q \vee r\}$ (equivalently $V(D_1) = \{f \in \mathcal{H} | \{4,5\} \subseteq A_f\}$ and $V(D_2) = \{f \in \mathcal{H} | A_f \subseteq \{1,2,3,5\}\}$).

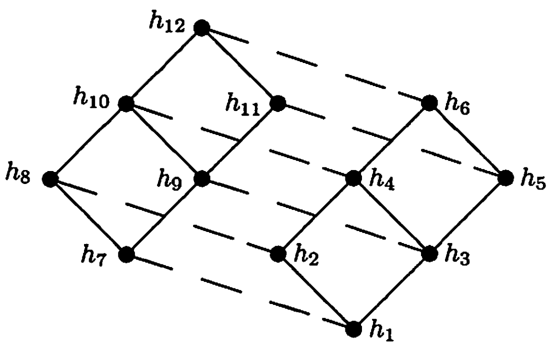

FIG. 4. Hasse diagram for $\mathcal{H}$

5 f–Ramsey Numbers

In this section we consider analogues of the classical Ramsey graph numbers which involve f–sets. Suppose that each edge of K_n is coloured with one of t (≥ 2) colours $\alpha_1, ..., \alpha_t$. For $i = 1, ..., t$, let G_i be the spanning subgraph containing the edges coloured α_i. Then $(G_1, \ldots, G_t)$ is called a t–edge colouring of K_n.

Throughout this section let $n_1, ..., n_t \geq 2$. *The classical graph Ramsey number* $R_p(n_1, ..., n_t)$ is the smallest n such that in every t–edge colouring $(G_1, ..., G_k)$ of K_n, for some $i \in \{1, ..., t\}$, $\overline{G}_i$ has a p–set (i.e., an independent set) of cardinality n_i. These numbers exist by Ramsey's celebrated Theorem [13].

They are usually defined in terms of complete graphs in G_i rather than p–sets in $\overline{G}_i$. We need the above equivalent definition for purposes of generalisation.

We now consider the question: For $f \in \mathcal{F}$ can we analogously define f–Ramsey numbers? i.e., Do the numbers given by the following "definition" exist?

The *f–Ramsey number* $R_f(n_1, ..., n_t)$ is the smallest n such that in every t–edge colouring $(G_1, ..., G_t)$ of K_n, for some $i \in \{1, ..., t\}$, $\overline{G}_i$ contains an f–set of cardinality n_i.

Theorem 11. *Suppose $f, g \in \mathcal{F}$, $f \Longrightarrow g$ and the number $R_f(n_1, \ldots, n_t)$ exists. Then $R_g(n_1, \ldots, n_t)$ exists and satisfies*

$$R_g(n_1, ..., n_t) \leq R_f(n_1, ..., n_t) \ .$$

Proof. Let $N = R_f(n_1, ..., n_t)$. Then for each t–edge colouring $(G_1, ..., G_k)$ of K_N, some $\overline{G}_i$ has an f–set X_i of size n_i. But X_i is also a g–set. Hence $R_g(n_1, ..., n_t)$ exists and is at most N, as required. □

Corollary 12. *Let $p \Rightarrow f \in \mathcal{F}$. Then $R_f(n_1, ..., n_t) \leq R_p(n_1, ..., n_k)$.*

Proof. Immediate from Theorem 11 and Ramsey's theorem. □

Theorem 13. *Let $f \in \mathcal{F}$ satisfy $f \Rightarrow q \vee r$ and $n_1, ..., n_t \geq 3$. Then $R_f(n_1, ..., n_y)$ does not exist.*

Proof. Consider the colouring $(G_1, ..., G_t)$ of K_n $(n \geq 3)$ where G_1 contains all edges incident with a fixed vertex (i.e., $G_1 \cong K_{1,n-1}$), G_2 contains all remaining edges (i.e., $G_2 \cong K_{n-1} \cup K_1$) and for $i = 3, ..., t$, $G_i \cong \overline{K}_n$. It is easily seen that no $\overline{G}_i$ contains a $(q \vee r)$–set of size n_i and so $R_{q\vee r}(n_1, ..., n_t)$ does not exist. If $f \Rightarrow q \vee r$ and $R_f(n_1, ...n_t)$ exists, then by Theorem 11 $R_{q\vee r}(n_1, ..., n_t)$ exists, a contradiction and the result follows. □

It is interesting to note (using Corollary 12 and Theorem 13) that f–Ramsey numbers do (do not) exist for f in the left hand side (right hand side) of the Hasse diagram for $\mathcal{H}$ (Figure 4). Since $A_p = \{4, 5\}$ and $A_{q\vee r} = \{1, 2, 3, 5\}$, there are sixteen functions covered by each of Corollary 12 and Theorem 13. The existence of f–Ramsey numbers for the remaining thirty values of f will be determined in later work.

In order to establish the final result about f–Ramsey numbers, we require a preliminary result. Let

$$\mathcal{T} = \{f \in \mathcal{F} | 2i \in A_f \Longrightarrow 2i + 1 \in A_f, \text{ for } i = 0, 1 \text{ and } 2\} \ .$$

Theorem 14. *S is an f-set of $G[U]$ implies S is an f-set of G for all graphs G and all $U \subseteq V(G)$ if and only if $f \in \mathcal{T}$.*

Proof. Since more than one graph is involved, we need to use different notations for Boolean variables and for $S(s)$. For example $q(s, S, H)$ will denote the Boolean variable which measures the existence of an S–ipn for s in the graph H. Furthermore the notation $S(s, H)$ will denote $(p(s, S, H)$,

$q(s,S,H), r(s,S,H))$.
For any graph G and any $U \subseteq V(G)$

$$\left.\begin{array}{rrcl} & p(s,S,G[U]) & = & p(s,S,G) \\ & q(s,S,G[U]) & = & q(s,S,G) \\ \text{and} & r(s,S,G[U]) & = & 1 \Longrightarrow r(s,S,G)=1 \end{array}\right\} \quad (1)$$

However $r(s,S,G[U])=0$ need not imply that $r(s,S,G)=0$, since s might have an S–epn in G but not in $G[U]$.

Let $f \in \mathcal{T}$, suppose that S is an f–set of $G[U]$ where $U \subseteq V(G)$ and let $s \in S$. If $r(s,S,G[U]) \neq 0$, then by (1) $f(S(s,G)) = f(S(s,G[U])) = 1$. If $r(s,S,G[U]) = 0$, then (using the integer form) $S(s,G[U]) = 2i$ for some $i = 0,1$ or 2 and so $2i \in A_f$. Then $S(s,G) = 2i$ or $2i+1$ both of which are in A_f, again $f(S(s,G)) = 1$. Therefore S is an f–set of G.

Conversely suppose $f \notin \mathcal{T}$. Then for some $i \in \{0,1,2\}$ $2i \in A_f$ but $2i+1 \notin A_f$. Choose any graph H with vertex set U, an f–set S of H and $s \in S$ such that $S(s,H) = 2i$. (Note: this implies $r(s,S,H)=0$) Form the graph G from H by adding a single new vertex x and the single edge xs. Then $S(s,G) = 2i+1$ and so $f(S(s,G)) = 0$ by hypothesis. Hence S is not an f–set of G. □

We now show that the well-known recurrence inequality for classical Ramsey graph numbers may be generalised to certain f–Ramsey numbers. Only two colour numbers are considered here but (as in the classical case) there is an obvious extension to t (> 2) colours.

Theorem 15. *If $p \Longrightarrow f$ and $f \in \mathcal{T}$, then*

(a) $R_f(l,m) \leq R_f(l-1,m) + R_f(l,m-1)$

(b) *If both terms on the right of (a) are even, then the inequality is strict.*

Proof. (a) Let $L = R_f(l-1,m)$, $M = R_f(l,m-1)$ and (B,Y) be any 2–edge colouring of $G = K_{L+M}$ where the edges of B, Y are coloured blue, yellow respectively. Among the edges joining a fixed vertex x at least L are blue or at least M are yellow. Suppose (without loss of generality) that X is a set of vertices which join x with blue edges. By definition of L, the 2–edge coloured graph $G[X]$ has a blue f–set S_B of size m or a yellow f–set S_Y of size $l-1$. In the former case, since $f \in \mathcal{T}$, S_B is an f–set of B of size m. In the latter case, since $f \in \mathcal{T}$, S_Y is an f–set in Y of size $l-1$. Consider $T = S_Y \cup \{x\}$. Since all edges between x and S_Y are blue, for each $s \in S_Y$, $T(s,Y) = S_Y(s,Y)$ and so $f(T(s,Y)) = f(S_Y(s,Y)) = 1$.

Moreover $T(x, Y) = (1,0,0)$ or $(1,0,1)$ and this implies (since $p \implies f$) that $f(T(x, Y)) = 1$. We conclude that T is an f–set of Y of size l which completes the proof.

(b) This proof is identical to that used to establish the result for $f = p$ and is omitted.
□

The Hasse diagram of the functions of Theorem 15 is drawn in Figure 5.

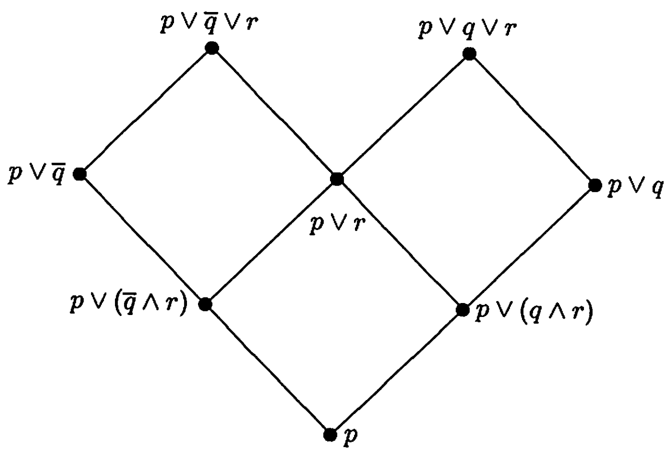

FIG. 5. Hasse diagram for $\{f \in T | p \implies f\}$

The inequalities of Theorem 14 have been used in the calculation of values of $R_f(n_1, \ldots, n_t)$ for $f = p \vee r$ (i.e., irredundant Ramsey numbers) (see [12]) and for $f = p \vee q \vee r$ (i.e., CO–irredundant Ramsey numbers) (see [3]). Calculation of these numbers has proved to be at least as difficult as computations of the classical Ramsey numbers.

6 Parameters: future research

Following the theories of the independent sets (Ω_p) and irredundant sets ($\Omega_{p \vee r}$), we define the parameters $c_f(G)$ and $C_f(G)$ (abbreviated to c_f, C_f wherever possible) to be the smallest and largest cardinalities of a maximal f–set of G. The word "maximal" is of course superfluous in the definition of C_f. For $f = p$ ($f = p \vee r$) the parameters c_f, C_f are the well–studied upper and lower independence numbers (irredundance numbers) usually denoted by i, β (ir, IR).

We finally list a few directions for further research.

1. Let $f \Rightarrow g$. It is obvious that $C_f \leq C_g$. Since a maximal f–set need not be a maximal g–set, we cannot deduce a relationship between c_f and c_g. However we know that any maximal independent set of a graph is also maximal irredundant (see e.g. [10]) and so $c_{p \vee r} \leq c_p$ (i.e., $ir \leq i$). Are there other f, g for which c_f, c_g may be related?

2. Calculation of c_f, C_f for various functions f and classes of graphs.

3. Complexity of computation of c_f, C_f for various functions and classes of graphs. Are there results of the form: If $f \in \mathcal{F}' \subseteq \mathcal{F}$ and G is in a class $\mathcal{G}$ of graphs, then the complexity of computing c_f, C_f for G is X?

4. Determination of the class of (c_f, C_f)*–graphs* i.e., these graphs for which $c_f = C_f$.

5. Find bounds for c_f, C_f involving other parameters. Simpler results might involve minimum (maximum) degree, number of vertices and edges etc.

6. Consider the obvious analogues of colourings and chromatic numbers. Define the *f–chromatic number of G* as the smallest order of a *proper f–colouring of G*, i.e., a partition of $V(G)$ into f–sets.

7. Further calculations of f–Ramsey numbers.

8. Behaviour of c_f, C_f under various graph operations such as edge addition or removal, products etc. Notions of criticality.

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